# On the regularity of solutions to the Poisson equation in Orlicz-spaces

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**Abstract**

We prove in this paper some regularity results of solutions of the Poisson equation $\Delta u = f$, in Orlicz spaces.

## 1 Introduction

Let $\Omega$ be an open subset of $\mathbb{R}^N$ with $N \geq 2$ and let $f$ be a distribution on $\Omega$. We consider the Poisson equation

$$\Delta u = f \qquad in\ \Omega. \tag{$E$}$$

The regularity of solutions $u$ of equation $(E)$, in relation to the regularity of the second member $f$ is one of the classical questions concerning this equation (cf.[4-5]). In particular, it's well be known that:

$(R_1)$ If $f$ is a distribution of order 1 (resp. 0), then the solution $u \in L^p_{loc}(\Omega)$ for all $p < \frac{N}{N-1}$ (resp. $u \in L^p_{loc}(\Omega)$ for all $p < \frac{N}{N-2}$ and $\frac{\partial u}{\partial x_i} \in L^q_{loc}(\Omega)$ for all $q < \frac{N}{N-1}$).

$(R_2)$ If $f \in L^r_{loc}(\Omega)$ with $r > \frac{N}{2}$ (resp. $r > N$), then $u$ is continuous (resp. continuously differentiable) on $\Omega$.

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$(R_3)$ If $f \in L^p(\Omega)$, then $\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(\Omega)$ for all $1 < p < \infty$.

When $p = 1$ (resp. $p = \infty$), the second derivatives of $u$ are in general not integrable (resp. bounded) in $\Omega$. In the case where $f$ lies in an Orlicz space $L_A(\Omega)$, the result $(R_3)$ is proved in [2] under the assumption that the N-function $A$ and its conjugate $\overline{A}$ satisfy the $\Delta_2$ condition.

It's our purpose in the first part of this paper, to prove the analogous regularity results of $(R_1)$ and $(R_2)$ in the general setting of Orlicz-spaces. The results obtained here constitute in particular a refinement of the $L^p$ case. The second part is devoted to obtain in the radial case some regularity results on the second derivatives $u$, without using any assumptions on the N-function $A$.

## 2 Preliminaries

In this section we list some definitions and well-known facts about N-functions, Orlicz spaces and Newtonian potential. For more details, we refer the reader to [1-4-5-9-10].

**2.1-** Let $A : \mathbb{R}^+ := [0, \infty) \to \mathbb{R}^+$ be an N-function, i.e. $A$ is continuous, convex, with $A(t) > 0$ for $t > 0$, $\frac{A(t)}{t} \to 0$ (resp. $+\infty$) as $t \to 0^+$ (resp. $t \to +\infty$). Equivalently, $A$ admits the representation:

$$A(t) = \int_0^t a(s)ds,$$

where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t) \to +\infty$ as $t \to +\infty$. The N-function $\overline{A}$ conjugate to $A$ is defined by:

$$\overline{A}(t) = \int_0^t \overline{a}(s)ds,$$

where, $\overline{a}(t) = sup\{s, a(s) \le t\}$. Clearly $\overline{\overline{A}} = A$ and one has Young's inequality:

$$t.s \le A(t) + \overline{A}(s),$$

for all $s, t > 0$. It's well known that we can assume that $a$ and $\bar{a}$ are continuous and strictly increasing. We will extend the N-functions into even functions on all $\mathbb{R}$. The N-function $A$ is said to satisfy the $\Delta_2$ condition (resp. near infinity) if there exists $k > 0$ (resp. $t_0 > 0$) such that

$$A(2t) \le kA(t),$$

for all $t \ge 0$ (resp. $t \ge t_0$).

**2.2-** Let $\Omega$ be an open subset of $\mathbb{R}^N$ and $A$ an N-function. The Orlicz class $K_A(\Omega)$ is defined as the set of real-valued measurable functions $u$ on $\Omega$ such that,

$$\int_\Omega A(u(x))\, dx < \infty.$$

The Orlicz-space $L_A(\Omega)$ is the set of (equivalence classes of) real valued measurable functions $u$ such that $\frac{u}{\lambda} \in K_A(\Omega)$, for some $\lambda = \lambda(u) > 0$. It's a Banach space under the norm

$$\|u\|_{A,\Omega} = inf\{\lambda > 0 : \int_\Omega A(\frac{u(x)}{\lambda})dx \leq 1\}.$$

The closure in $L_A(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_A(\Omega)$. The equality $E_A(\Omega)$=$L_A(\Omega)$ holds if and only if $A$ satisfies the $\Delta_2$ condition, for all $t$ or for $t$ large according to whether $\Omega$ has a finite measure or not. The dual space of $E_A(\Omega)$ can be identified with $L_{\bar{A}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$ and the dual norm on $L_{\bar{A}}(\Omega)$ is equivalent to $\|u\|_{\bar{A},\Omega}$. We recall the Hlder's inequality

$$\int_\Omega u(x)v(x)\,dx \leq 2\|u\|_{A,\Omega}\|v\|_{\overline{A},\Omega},$$

for all $u \in L_A(\Omega)$ and all $v \in L_{\overline{A}}(\Omega)$. We say that $u_n$ converges to $u$ in modular sense (denoted by $u_n \to u$ (mod)) in $L_A(\Omega)$, if for some constant $\lambda > 0$,

$$\int_\Omega A(\frac{u_n - u}{\lambda})\,dx \to 0,\ as\, n \to +\infty.$$

**Lemma 2.1.** (see [6])- *Let $u$ be an element of $L_A(\mathbb{R}^N)$ with $2u \in K_A(\mathbb{R}^N)$ and $T_y u$ the translation of $u$ ,.i.e.,$T_y u(x) = u(x-y)$. Then:*

$$\int_{\mathbb{R}^N} A(T_y u - u)dx \to 0,\ as\ \mid y \mid \to 0.$$

**2.3-** We define the Newtonian function $P_N(x)$ on $\mathbb{R}^N \setminus \{0\}$ by:

$$P_N(x) = P_N(\mid x \mid) = \begin{cases} \frac{|x|}{2} & \text{if} \quad N = 1; \\ \frac{log(|x|)}{2\pi} & \text{if} \quad N = 2; \\ \frac{1}{k_N |x|^{N-2}} & \text{if} \quad N \geq 3, \end{cases}$$

$k_N = (2-N)\sigma_N$, where $\sigma_N$ is the measure of unit sphere of $\mathbb{R}^N$. The function $P_N(x)$ which is locally integrable, is the elementary solution of the Poisson equation

$$\Delta P_N = \delta \ \ on\ \mathbb{R}^N,$$

where $\delta$ is the Dirac measure. It's easy to prove that:

$$\frac{\partial P_N(x)}{\partial x_i} = \frac{x_i}{\sigma_N \mid x \mid^N},\ (1 \leq i \leq N)$$

and that:
for $N = 1$, $P_1(x)$ and $\frac{dP_1(x)}{dx}$ lie in $L^p_{loc}(\mathbb{R})$, for $1 \leq p \leq \infty$;
for $N = 2$, $P_2(x) \in L^p_{loc}(\mathbb{R}^2)$ (resp. $\frac{\partial P_2(x)}{\partial x_i} \in L^p_{loc}(\mathbb{R}^2)$), for all $p < \infty$ (resp. $p < 2$);
for $N \geq 3$, $P_N(x) \in L^p_{loc}(\mathbb{R}^N)$ (resp $\frac{\partial P_N(x)}{\partial x_i} \in L^p_{loc}(\mathbb{R}^N)$), for all $p < \frac{N}{N-2}$ (resp. $p < \frac{N}{N-1}$).

**Definition 2.2.** (see.[4])- *Let $f$ be a distribution on $\mathbb{R}^N$ with compact support. The distribution $P.N(f) = P_N * f$ is called the Newtonian potential of $f$.*

In particular, $P_N$ is the Newtonian potential of $\delta$ and each distribution $u$ with compact support is the Newtonian potential of its laplacian.

## 3 Regularity of the solution and its first derivatives

The following lemmas will be used below. The first can be found in [11], we give its proof for more convenience.

**Lemma 3.1.** *Let $\Omega$ be an open subset of $\mathbb{R}^N$ and let $A$ be an N-function. If $u_n \to u(mod)$ in $L_A(\Omega)$, then $u_n.f \to u.f$ in $L^1(\Omega)$, for all $f \in L_{\overline{A}}(\Omega)$.*

*Proof*- Let $f \in L_{\overline{A}}(\Omega)$, i.e. there exists a constant $\lambda_1 > 0$ such that

$$\overline{A}(\frac{f}{\lambda_1}) \in L^1(\Omega).$$

Also, there exists a constant $\lambda_2 > 0$ such that

$$M(\frac{2(u_n - u)}{\lambda_2}) \to 0 \, in \, L^1(\Omega),$$

which implies that there exists a subsequence $(u_{n_k})$ of $(u_n)$ satisfying:

$$u_{n_k} \to u \ a.e \ in \ \Omega$$

and

$$A(\frac{2(u_{n_k} - u)}{\lambda_2}) \leq h_1(x) \ a.e \ in \ \Omega,$$

for some function $h_1(x) \in L^1(\Omega)$.
On the other hand, by convexity of $A$ we get

$$A(\frac{u_{n_k}}{\lambda}) \leq \frac{1}{2}A(\frac{2(u_{n_k} - u)}{\lambda}) + \frac{1}{2}A(\frac{2u}{\lambda}).$$

For $\lambda > \lambda_2$ such that $A(\frac{2u}{\lambda}) \in L^1(\Omega)$, we have

$$u_{n_k}(x)\frac{\mid f(x) \mid}{\lambda_2} \leq \lambda A^{-1}(h(x))\frac{\mid f(x) \mid}{\lambda_2}$$

$$\leq \lambda[h(x) + \overline{A}(\frac{\mid f(x) \mid}{\lambda_2})],$$

where $h(x) = \frac{1}{2}[h_1(x) + A(\frac{2u}{\lambda})] \in L^1(\Omega)$. By virtue of Lebesgue theorem we get

$$u_{n_k}.f \to u.f \ in \ L^1(\Omega).$$

Finally, we conclude for the original sequence $(u_n)$ by a standard contradiction argument.

**Lemma 3.2.** *Let $A$ be an N-function and let $g \in L_A^{loc}(\mathbb{R}^N)$.*
*1)- If $f$ is a Radon measure on $\mathbb{R}^N$ with compact support, then the convolution $g * f$ is a function which lies in $L_A^{loc}(\mathbb{R}^N)$.*
*2)- If $f \in L_{\overline{A}}(\mathbb{R}^N)$ with compact support, then $g * f$ is continuous on $\mathbb{R}^N$.*

*Proof-* Let $R > 0$ such that supp$(f) \subset B(0,R)$. Using the Lebesgue decomposition theorem we can write,

$$f = v.m,$$

where $v$ is a function taking the values 1 or -1 and $m$ is the variation of $f$.
Giving a real $r > 0$ and a continuous function $\xi$ on $\mathbb{R}^N$ with a support in $\overline{B}(0,r)$, we have

$$\mid \langle g * f, \xi \rangle \mid \leq \int_{B(0,r)} \left( \int_{B(0,R)} \mid g(x-y)v(y) \mid \, dm(y) \right) \mid \xi(x) \mid \, dx =$$

$$\int_{B(0,R)} \left( \int_{B(y,r)} \mid g(x) \mid . \mid \xi(x) \mid \, dx \right) dm(y)$$

$$\leq \left( \int dm \right) \parallel g \parallel_{A, \overline{B}(0,R+r)} \cdot \parallel \xi \parallel_{\overline{A}, B(0,r)},$$

where $\langle u, v \rangle = \int u(x) v(x) \, dx$.

For 2), there exists a real $\lambda^* > 0$ such that

$$\frac{g}{\lambda^*} \in K_A(B(0,R)).$$

Moreover,

$$\mid \frac{g}{2\lambda^*} * f(x_1) - \frac{g}{2\lambda^*} * f(x_2) \mid \leq$$

$$\int_{B(0,R)} \mid T_{x_2 - x_1}\left(\frac{g}{2\lambda^*}\right)(z) - \frac{g}{2\lambda^*}(z) \mid . \mid f(x_2 - z) \mid \, dz.$$

Since,

$$\mid T_{x_2 - x_1}\left(\frac{g}{2\lambda^*}\right) - \frac{g}{2\lambda^*} \mid \to 0 \, (\text{mod}) \ \text{ in } \ L_A(B(0,R)), \ as \ x_2 \to x_1,$$

(due to lemma 2.1) then the conclusion follows directly from lemma 3.1.

**Definition 3.3.** *Let $A$ be an N-function. We say that $A$ satisfies the property (P1) (resp. (P2)) if*

$$\int_1^{+\infty} \frac{A(t)}{t^{1+\frac{N}{N-2}}} \, dt < \infty$$

*(resp.*

$$\int_1^{+\infty} \frac{A(t)}{t^{1+\frac{N}{N-1}}} \, dt < \infty).$$

**Remarks 3.4.** *Note that: (P2) $\Rightarrow$ (P1).*

**Lemma 3.5.** *Let $A$ be an N-function and $\alpha > 1$. The following conditions are equivalent*

$$\int_1^{\infty} \frac{A(s)}{s^{1+\alpha}} ds < \infty \tag{3.1}$$

$$\int_1^\infty (\frac{s}{\bar{A}(s)})^{\alpha-1} ds < \infty. \tag{3.2}$$

*Proof.* For the first implication, we have

$$\lim_{c\to\infty} \int_c^{2c} \frac{A(s)}{s^{1+\alpha}} ds = 0,$$

which implies that

$$\lim_{c\to\infty} \frac{A(c)}{c^\alpha} = 0,$$

which is equivalent to

$$\lim_{c\to\infty} \frac{c}{A^{-1}(c)^\alpha} = 0$$

and also to

$$\lim_{c\to\infty} \frac{c^\alpha}{\bar{A}(c)^{\alpha-1}} = 0,$$

where we have used the following inequalities

$$t \leq A^{-1}(t)\bar{A}^{-1}(t) \leq 2t. \tag{3.3}$$

The change of the integration variable and the integration by parts allow to have

$$\int_1^\infty \frac{A(s)}{s^{1+\alpha}} ds = \int_{c_1}^\infty \frac{t}{(A^{-1}(t))^{1+\alpha}} dA^{-1}(t)$$

$$= c_2 + \frac{1}{\alpha} \int_{c_1}^\infty \frac{dt}{(A^{-1}(t))^\alpha},$$

for some constants $c_1$ and $c_2$. Hence by (3.3), we can write

$$\int_1^\infty \frac{A(s)}{s^{1+\alpha}} ds \geq \frac{1}{\alpha} \int_{c_1}^\infty (\frac{\bar{A}^{-1}(t)}{2t})^\alpha dt = \frac{1}{\alpha 2^\alpha} \int_{c_3}^\infty (\frac{s}{\bar{A}(s)})^\alpha d\bar{A}(s)$$

$$= c_4 + c_5 \int_{c_3}^\infty (\frac{s}{\bar{A}(s)})^{\alpha-1} ds,$$

for some constants $c_3$, $c_4$ and $c_5$. Then, (3.1) implies (3.2).
For the reverse implication, doing the above steps in reverse order we can deduce analogously.

In the following, we denote by $\mathrm{ord}(f)$ the order of the distribution $f$.

**Theorem 3.6.** *Let $\Omega$ be an open subset of $\mathbb{R}^N$ $(N \geq 3)$, $f$ be a distribution on $\Omega$ and $u$ the solution of the equation $(E)$. Then*
*1)- If ord($f$) =0, then $u \in L_A^{loc}(\Omega)$ (resp. $\frac{\partial u}{\partial x_i} \in L_A^{loc}(\Omega)$) for all N-function $A$ satisfying (P1) (resp. all N-function $A$ satisfying (P2));*
*2)- if ord($f$)=1, then $u \in L_A^{loc}(\Omega)$ for all N-function $A$ satisfying (P2);*
*3)- if $f \in L_A^{loc}(\Omega)$ and $\int_1^\infty (\frac{t}{A(t)})^{\frac{2}{N-2}} dt < \infty$, then $u$ is continuous;*
*4)- if $f \in L_A^{loc}(\Omega)$ and $\int_1^\infty (\frac{t}{A(t)})^{\frac{1}{N-1}} dt < \infty$, then $u$ is continuously differentiable.*

*Proof.* Given an open bounded $U$ such that $\overline{U} \subset \Omega$ and a function $\varphi \in D(\Omega)$ such that $\varphi \equiv 1$ on $U$, we consider the functions $F_0 = \varphi.f$ and $F_1 = \Delta(\varphi.u) - F_0$. Let $u_0$ and $u_1$ be the Newtonian potentials of $F_0$ and $F_1$ respectively. We have $u_0 = P_N * F_0 = P_N * (\varphi.f)$ and $u_1 = P_N * F_1 = P_N * (\Delta(\varphi.u)) - P_N * F_0$.
This implies that

$$u_0 + u_1 = P_N * (\Delta(\varphi.u)) = u \text{ on } U.$$

Moreover, $u_1$ is harmonic on $U$ (because $F_1 = 0$ on $U$), and so $u$ and $u_0$ have similar regularity (locally). Hence, one can assume that $\Omega = \mathbb{R}^N$, $f$ is a distribution with compact support and $u = P.N(f)$.
Note that if $\text{ord}(f) \leq 1$, i.e.

$$f = f_0 + \sum_{k=1}^{N} \frac{\partial f_k}{\partial x_k},$$

where $f_k (0 \leq k \leq N)$ are measures with compact support, then we have

$$u = \sum_{k=0}^{N} u_k,$$

where $u_0 = P.N(f_0)$ and $u_k = P.N(\frac{\partial f_k}{\partial x_k})$.

1) Suppose that ord($f$)=0, i.e. $f = f_0$.
If $A$ satisfies (P1), then we claim that

$$P_N \in L_A^{loc}(\mathbb{R}^N).$$

Indeed, for all real $R > 0$ we have

$$\begin{aligned}
\int_{|x|<R} A(\mid k_N.P_N(x) \mid)\, dx &= \sigma_N \int_0^R r^{N-1} A(\frac{1}{r^{N-2}})\, dr \\
&= \frac{\sigma_N}{N-2} \int_{\frac{1}{R^{N-2}}}^{+\infty} \frac{A(t)}{t^{1+\frac{N}{N-2}}}\, dt < \infty,
\end{aligned}$$

. This implies by using lemma 3.2 that $u = P_N * f \in L_A^{loc}(\mathbb{R}^N)$.
Similarly, if $A$ satisfies (P2), then

$$\frac{\partial P_N}{\partial x_i}(x) \in L_A^{loc}(\mathbb{R}^N),$$

(for $1 \leq i \leq N$). Indeed, for all real $R > 0$ we have

$$\begin{aligned}
&\int_{|x|<R} A(\sigma_N \frac{\partial P_N(x)}{\partial x_i})\, dx \\
\leq\ & \int_{|x|<R} A(\frac{1}{\mid x \mid^{N-1}})\, dx \\
=\ & \sigma_N \int_0^R r^{N-1} A(\frac{1}{r^{N-1}})\, dr \\
\leq\ & \frac{\sigma_N}{N-1} \int_{\frac{1}{R^{N-1}}}^{+\infty} \frac{A(t)}{t^{1+\frac{N}{N-1}}}\, dt < \infty,
\end{aligned}$$

then, the conclusion follows from lemma 3.2.
2) Assume that ord($f$)=1 and $A$ satisfies (P2), then $\frac{\partial P_N}{\partial x_i}(x) \in L_A^{loc}(\mathbb{R}^N)$ for $1 \leq i \leq N$ and (since (P2) implies (P1)) $P_N \in L_A^{loc}(\mathbb{R}^N)$, hence $u \in L_A^{loc}(\mathbb{R}^N)$.
For 3) and 4) Taking $\alpha = \frac{N}{N-2}$ (resp. $\alpha = \frac{N}{N-1}$) in lemma 3.5, one deduce that $\bar{A}$ satisfies $(P_1)$ (resp. $(P_2)$), which implies as above that $P_N \in L_{\bar{A}}^{loc}(\mathbb{R}^N)$ (resp. $\frac{\partial P_N}{\partial x_i} \in L_{\bar{A}}^{loc}(\mathbb{R}^N)$).
Then by using the lemma 3.2 for $f$ and $g = P_N$ (resp. $f$ and $g = \frac{\partial P_N}{\partial x_i}$), we conclude.

**Remark 3.7.** *In the $L^p$ case, the property (P1) (resp. (P2)) is equivalent to $p < \frac{N}{N-2}$ (resp. $p < \frac{N}{N-1}$).*

**Example 3.8.** *We consider the N-functions:*

$$A(t) = \frac{t^{\frac{N}{N-2}}}{(log(1+t))^\alpha} \text{ and } B(t) = \frac{t^{\frac{N}{N-1}}}{(log(1+t))^\alpha},$$

*where $\alpha > 1$. The N-function $A$ (resp. $B$) satisfies the property (P1) (resp. (P2)).*
Indeed,

$$\int_1^\infty \frac{A(t)}{t^{1+\frac{N}{N-2}}})\, dt = \int_1^\infty \frac{1}{t(log(1+t))^\alpha}\, dt < \infty,$$

and

$$\int_1^\infty \frac{B(t)}{t^{1+\frac{N}{N-1}}})\, dt = \int_1^\infty \frac{1}{t(log(1+t))^\alpha}\, dt < \infty.$$

**Theorem 3.9.** *Let $\Omega$ be an open subset of $\mathbb{R}^2$, and $A$ be an N-function such that*

$$\int_1^{+\infty} \frac{A(t)}{e^{2\lambda t}}\, dt < \infty,$$

*for some $\lambda > 0$. If $f$ is a measure on $\Omega$ (resp. $\in L_{\bar{A}}^{loc}(\Omega)$), then every solution $u$ of the equation $(E)$ lies in $L_A^{loc}(\Omega)$ (resp. is continuous on $\Omega$).*

*Proof* - As in the proof of theorem 3.6, we can assume that $\Omega = \mathbb{R}^2$, supp($f$) is compact and $u = P.N(f)$.
First, we have $P_2 \in L_A^{loc}(\mathbb{R}^2)$. Indeed, for all real $R > 0$ we have

$$\int_{|x|<R} A(\frac{2\pi}{\lambda} P_2(x))\, dx$$

$$= \sigma_2 \int_0^R r A(\frac{log(r)}{\lambda})\, dr$$

$$= \sigma_2 \int_{\frac{1}{R}}^{+\infty} \frac{1}{r^3} A(\frac{log(r)}{\lambda})\, dr$$

$$= \sigma_2 \lambda \int_c^{+\infty} \frac{A(t)}{e^{2\lambda t}}\, dt < \infty,$$

where $c = \frac{-log(R)}{\lambda}$, then we conclude by lemma 3.2.

**Example 3.10.** *We consider the N-function* $A(t) = e^t - t - 1$*, its conjuguate is* $\overline{A}(t) = (1+ \mid t \mid) log(1+ \mid t \mid) - \mid t \mid$. *The N-function* $A$ *satisfies the condition of theorem 3.9.*

Indeed, for $R > 0$ and $\lambda > \frac{1}{2}$ we have

$$\int_1^{+\infty} \frac{A(t)}{e^{2\lambda t}} dt \leq \int_1^{+\infty} \frac{1}{e^{(2\lambda-1)t}} dt < \infty.$$

## 4 Maximal regularity of radial solutions

In the following, we note by $B(0, R_0)$ an open ball of $\mathbb{R}^N$ with radius $R_0 > 0$ which is equal to $\mathbb{R}^N$ for $R_0 = +\infty$.

**Theorem 4.1.** *Let* $A$ *be an N-function and* $f \in L_A(B(0, R_0))$*. Then every radial solution* $u$ *of the equation*

$$\Delta u = f \ \ in \, B(0, R_0), \tag{4.1}$$

*satisfies*

$$\frac{\partial^2 u}{\partial x_i \partial x_j} \in L_A^{loc}(B(0, R_0) - \{0\}).$$

*If in addition* $A(\mid f(x) \mid) log(\mid x \mid)$ *is integrable then,*

$$\frac{\partial^2 u}{\partial x_i \partial x_j} \in L_A^{loc}(B(0, R_0)). \tag{4.2}$$

The proof uses the following lemma (see. [4]).

**Lemma 4.2.** *Let* $f$ *be a radial and integrable function on* $\mathbb{R}^N$ *with compact support. The Newtonian potential* $P.N(f)$ *is continuously differentiable on* $\mathbb{R}^N - \{0\}$ *and*

$$P.N(f)(x) = P_N(x) \int_{B(0,|x|)} f(y)dy + \int_{\mathbb{R}^N - B(0,|x|)} P_N(y) f(y) dy$$

$$\triangledown P.N(f)(x) = \frac{x}{\sigma_N \mid x \mid^N} \int_{B(0,|x|)} f(y)dy.$$

*The second derivatives of* $P.N(f)$ *are locally integrable on* $\mathbb{R}^N - \{0\}$ *and*

$$\frac{\partial^2 P.N(f)}{\partial x_i \partial x_j}(x) = \frac{x_i x_j}{\mid x \mid^2} f(x) + (\frac{\delta_{ij}}{N} - \frac{x_i x_j}{\mid x \mid^2}) \frac{N}{\sigma_N \mid x \mid^N} \int_{B(0,|x|)} f(y)dy.$$

*Proof of Theorem 4.1.* Let $B_1$ be an open ball such that $\overline{B}_1 \subset B(0, R_0)$ and let $\Phi$ be a radial smooth function with compact support in $B(0, R_0)$ such that $\Phi \equiv 1$ in $B_1$. We consider the function$f_0 = \Phi.f$ and $f_1 = \Delta(\Phi u) - f_0$. We have

$$P.N(f_0) + P.N(f_1) = P_N * (\Delta(\Phi u)) = u \text{ on } B_1.$$

Since $P.N(f_1)$ is harmonic on $B_1$, then $P.N(f_0)$ and $u$ have the same regularity on $B_1$. Hence, we can assume that $f$ is radial with compact support and $u = P.N(f)$. Let $0 < R < R_0$ and let $\varepsilon > 0$. We shall show that

$$\frac{\partial^2 u}{\partial x_i \partial x_j} \in L_A(B(\varepsilon, R)), \tag{4.3}$$

where

$$B(\varepsilon, R) = \{x \in \mathbb{R}^N : \varepsilon <| x |< R\}.$$

It follows directly from lemma 4.2 that,

$$| \frac{\partial^2 u}{\partial x_i \partial x_j} | \leq | f(x) | + \frac{C_1}{| x |^N} \int_{B(0,|x|)} | f(y) | \, dy, \tag{4.4}$$

for some positive constant $C_1$. Hence, by virtue of convexity of $A$ we have

$$\int_{B(\varepsilon,R)} A(\frac{1}{2\lambda} | \frac{\partial^2 u}{\partial x_i \partial x_j} |) dx \leq \frac{1}{2} I_1 + \frac{1}{2} I_2,$$

where

$$I_1 = \int_{B(\varepsilon,R)} A(\frac{| f(x) |}{\lambda}) dx$$

and

$$I_2 = \int_{B(\varepsilon,R)} A(\frac{C_1}{| x |^N} \int_{B(0,|x|)} \frac{| f(y) |}{\lambda} dy) dx.$$

Since $I_1 < \infty$ it's sufficient to show that $I_2 < \infty$. For that we have by Jensen's inequality and Fubini's theorem,

$$\begin{aligned} I_2 &\leq \int_{B(\varepsilon,R)} (\frac{C_2}{| x |^N} \int_{B(0,|x|)} A(\frac{| f(y) |}{\lambda}) dy) dx \\ &= \int_{B(0,R)} (A(\frac{| f(y) |}{\lambda}) \int_{max(\varepsilon,|y|)<|x|<R} \frac{C_2}{| x |^N} dx) dy \\ &= C_3 \int_{B(0,R)} A(\frac{| f(y) |}{\lambda}) log(\frac{R}{max(\varepsilon, | y |)}) dy, \end{aligned}$$

for some positive constants $C_2$ and $C_3$, which implies (4.3). For the second part of theorem, assumming now that $A(| f(x) |) log(| x |)$ is integrable. As above by Jensen's inequality and Fubini's theorem we get for $0 < R < R_0$,

$$\begin{aligned} &\int_{B(0,R)} A(\frac{1}{2\lambda} | \frac{\partial^2 u}{\partial x_i \partial x_j} |) dx \\ &\leq \frac{1}{2} \int_{B(0,R)} A(\frac{| f(x) |}{\lambda}) dx + \frac{1}{2} \int_{B(0,R)} (\frac{C_4}{| x |^N} \int_{B(0,|x|)} A(\frac{| f(y) |}{\lambda}) dy) dx \\ &= \frac{1}{2} \int_{B(0,R)} A(\frac{| f(x) |}{\lambda}) dx + \frac{1}{2} \int_{B(0,R)} (A(\frac{| f(y) |}{\lambda}) \int_{|y|<|x|<R} \frac{C_4}{| x |^N} dx) dy \\ &= \frac{1}{2} \int_{B(0,R)} A(\frac{| f(x) |}{\lambda}) dx + C_5 \int_{B(0,R)} A(\frac{| f(y) |}{\lambda}) log(\frac{R}{| y |}) dy, \end{aligned}$$

for some positive constants $C_4$ and $C_5$, which achieves the proof of theorem 4.1.

**Theorem 4.3.** *Let $A$ be an N-function. If $f \in L_A(B(0,R_0))$ (with $R_0 < \infty$), then every radial solution $u$ of the equation (4.1) satisfies:*

$$\frac{\partial^2 u}{\partial x_i \partial x_j} \in L_A(B(\varepsilon, R_0)),$$

*for all $\varepsilon > 0$. If in addition $A(\mid f(x) \mid) log(\mid x \mid)$ is integrable, then*

$$\frac{\partial^2 u}{\partial x_i \partial x_j} \in L_A(B(0, R_0)).$$

*Proof.* Replacing $R$ by $R_0$ in the proof of theorem 4.1, we obtain the desired statement.

**Theorem 4.4.** *Let $A$ be an N-function such that $\overline{A}$ satisfies the $\Delta_2$ condition and let $f \in L_A(B(0,R_0))$. Then every radial solution $u$ of the Poisson equation (4.1) satisfies*

$$\frac{\partial^2 u}{\partial x_i \partial x_j} \in L_A(B(0, R_0)).$$

*Proof.* Using lemma 4.2, we have

$$\mid \frac{\partial^2 u}{\partial x_i \partial x_j} \mid \leq \mid f(x) \mid + \frac{C_1}{\mid x \mid^N} \int_{B(0,|x|)} \mid f(y) \mid dy$$

$$= \mid f(x) \mid + C_1 T f(x),$$

where the mapping $T$ is defined by the formula

$$Tf(x) = \frac{1}{\mid x \mid^N} \int_{B(0,|x|)} \mid f(y) \mid dy.$$

Since $T$ is of weak type (1,1) and of type $(\infty, \infty)$, then the interpolation theorem III.1 of [2] implies that $Tf \in L_A(B(0,R_0))$,which gives the conclusion.

**Remark 4.5.** *The previous argument cannot be generalized for arbitrary N-function (compare Remark 5. p.290 in [4]).*
*In fact, let $B = B(0, \frac{1}{2}) \subset \mathbb{R}^3$ and consider the N-function $A(t) = (1+t)\log(1+t) - t$. It's easy to verify that the function $f(x) = \frac{1}{|x|^3 (\log|x|)^2} \in L_A(B)$, but $Tf(x) = \frac{1}{|x|^3} \int_{B(0,|x|)} \mid f(y) \mid dy$ is not in $L_A(B)$.*

## References

[1] R.Adams, Sobolev spaces, Acad. press (1975).

[2] A.Benkirane, Potentiel de Riesz et Problmes elliptiques dans les espaces d'Orlicz, Thse de Doctorat, Universit Libre de Bruxelles, 1988.

[3] A.Benkirane and J.P.Gossez, An approximation theorem in higher Orlicz-Sobolev spaces and application, Studia Math. 92 (1989) pp.231-255.

[4] R.Dautray et J.L.Lions, Analyse Mathmatique et calcul numrique, Volume 2, Masson (1987).

[5] D.Gilbarg and N.S.Trudinger, Elliptic partial differential equations of second ordre, Springer Verlag (1983).

[6] J.P.Gossez, Some approximation properties in Orlicz-Sobolev spaces, Studia Math. 74 (1982) pp. 17-24.

[7] J.P.Gossez, Non linear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients, Trans. Amer. Math. Soc. 190 (1974), pp. 163-205.

[8] Grter and Wildman, Manuscripa Math 37 (1982) 303-342.

[9] M.Krasnosel'skii and Ya.Rutickii, Convex functions and Orlicz spaces, Noordhoff, 1961.

[10] A.Kufner, O.John and S.fucik, Function spaces, Academia, 1977.

[11] M.Tienari, A degree theory for a class of mappings of monotone type in Orlicz-Sobolev spaces, Ann. Acad. Scientiarum Fennice Helsinki (1994).

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