An Interesting Example for a Three-Point Boundary Value Problem.

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Abstract

Let α, $A \in R$, $\eta \in (0,1)$, and $e(t) \in L^1[0,1]$ be given. Further, let $p(t)$, $q(t)$ be given functions such that $p(t) \geq 0$, $q(t) \geq 0$ for $t \in [0,1]$. This paper concerns the three point boundary-value problem

$$x''(t) = p(t)x(t) + Aq(t)x'(t) + e(t),\ 0 < t < 1, \tag{1}$$
$$x(0) = 0,\ x(1) = \alpha x(\eta). \tag{2}$$

This problem of existence of a solution for this boundary value problem was studied earlier by Gupta, Gupta-Trofimchuk with $p(t) = q(t) = t^{-\frac{1}{4}}$ for various values of α and η. Existence of a solution for this boundary value problem were given for A near zero. When $\alpha = 2$ and $\eta = .6$ Gupta-Trofimchuk were not able to show in [6] that a solution to this boundary value problem exists for any A. In this paper we show that given α, η, there exists an A_1, such that for $A_1 < A < \infty$, the three-point boundary value problem (1)-(2) has a unique solution. Further if $\alpha \leq 1$ then the three-point boundary value problem (1)-(2) has a unique solution for all $A \in R$. This is done as an application of a sharpened existence condition given by the authors earlier for such three-point boundary value problems. The authors made extensive use of computer algebra systems like Maple and MathCad.

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1 Introduction.

Let $f : [0,1] \times R^2 \to R$ be a function satisfying Caratheodory's conditions and $e : [0,1] \to R$ be a function in $L^1[0,1]$. We study the problem of existence of solutions for the three-point boundary value problem

$$x''(t) = f(t, x(t), x'(t)) + e(t),\ 0 < t < 1,$$
$$x(0) = 0,\ x(1) = \alpha x(\eta), \tag{3}$$

where $\alpha \in R$, $\eta \in (0,1)$ with $\alpha\eta \neq 1$ are given. The authors had given conditions for the existence of a solution for the three-point boundary value problem (3), in [6], using the spectral radius of a related linear operator. The purpose of this paper is to obtain sharper conditions for the solvability of the three-point boundary value problem (3) when the function $f(t, x(t), x'(t))$ in (3) is independent of $x'(t)$. We then apply this existence theorem to obtain the existence of a solution for the three-point boundary value problem

$$x''(t) = p(t)x(t) + Aq(t)x'(t) + e(t),\ 0 < t < 1, \tag{4}$$
$$x(0) = 0,\ x(1) = \alpha x(\eta), \tag{5}$$

where α, $A \in R$, $p(t) \geq 0$, $q(t) \geq 0$ for $t \in [0,1]$, $\eta \in (0,1)$ are given. We prove that there exists an $A_* \in R$, such that for $A_* < A < \infty$, the three-point boundary value problem (4)-(5) has a unique solution. We apply the results for the three-point boundary value problem (4)-(5) to the following example:

$$x''(t) = t^{-\frac{1}{4}}x(t) + At^{-\frac{1}{4}}x'(t) + e(t),\ 0 < t < 1, \tag{6}$$
$$x(0) = 0,\ x(1) = \alpha x(\eta), \tag{7}$$

where α, $A \in R$, and $\eta \in (0,1)$ are given. We show that there exists an $A_1 \in R$, such that for $A_1 < A < \infty$, the three-point boundary value problem (6)-(7) has a unique solution. We accordingly obtain existence of a solution for the three-point boundary value problem (6)-(7) for A belonging to an infinite interval in R, for any given $\alpha \in R, \eta \in (0,1)$, whereas earlier results could not decide if a solution exists for certain $\alpha \in R$, $\eta \in (0,1)$ and $\alpha\eta \neq 1$ (see [6] for an example, where for $\alpha = 2$ and $\eta = .6$ the problem of existence of a solution for the three-point boundary value problem (6)-(7) remained unsolved.)

The study of multi-point boundary value problems for linear second order ordinary differential equations was initiated by V. A. Il'in and E. A. Moiseev in [7], [8] motivated by the work of Bitsadze and Samarskiĭ on non-local linear elliptic boundary problems, [1], [2], [3].

We use the classical spaces C[0,1], C^k[0,1], L^k[0,1], and L^∞[0,1] of continuous, k-times continuously differentiable, measurable real-valued functions whose k-th power of the absolute value is Lebesgue integrable on [0,1], or measurable functions that are essentially bounded on [0,1]. We denote the norm in L^∞[0,1] by $|| \,.\, ||_\infty$.

2 Main results.

Definition 1. *A function $f : [0,1] \times R^2 \longmapsto R$ satisfies Caratheodory's conditions if (i) for each $(x,y) \in R^2$, the function $t \in [0,1] \longmapsto f(t,x,y) \in R$ is measurable on [0,1], (ii) for a.e. $t \in [0,1]$, the function $(x,y) \in R^2 \longmapsto f(t,x,y) \in R$ is continuous on R^2, and (iii) for each $r > 0$, there exists $\alpha_r(t) \in L^1[0,1]$ such that $| f(t,x,y) | \leq \alpha_r(t)$ for a.e. $t \in [0,1]$ and all $(x,y) \in R^2$ with $\sqrt{x^2+y^2} \leq r$.*

Theorem 2. *Let $f : [0,1] \times R^2 \longmapsto R$ be a function satisfying Caratheodory's conditions. Assume that there exist functions $p(t)$, $r(t)$ in $L^1[0,1]$ such that*

$$|f(t,x_1,x_2)| \leq p(t)\,|x_1| + r(t) \tag{8}$$

for a.e. $t \in [0,1]$ and all $(x_1,x_2) \in R^2$. Let $\alpha \in R, \eta \in (0,1)$ with $\alpha\eta \neq 1$ be such that

$$\max\{\mu(\alpha)H(\alpha,\eta,p), \| \sqrt{2t}P(t) \|_2\} < 1, \tag{9}$$

where $P(t) = \int_t^1 p(s)ds$, $H(\alpha,\eta,p) = \dfrac{\alpha}{| 1-\alpha\eta |}[(1-\eta)\int_0^\eta sp(s)ds + \eta\int_\eta^1 (1-s)p(s)ds]$ and $\mu(\alpha) = \{ \begin{array}{ll} 0, & if\ \alpha \leq 1, \\ 1, & if\ \alpha > 1. \end{array}$ Then the three-point boundary value problem (3) has at least one solution in $C^1[0,1]$.

Proof:- It suffices to prove that the set of solutions of the three-point boundary value problem (3) is uniformly bounded in $C^1[0,1]$, in view of Leray Schauder Continuation theorem. Let now $x(t)$, $0 < t < 1$, be a solution of the three-point boundary value problem (3). Two cases arise:

Case1:- there exists an $s \in (0,1)$ such that $x'(s) = 0$. In this case, it follows from the proof of Theorem 5 and Corollary 6 of [6] that there exists a constant C (independent of $x(t)$) such that $\| x(t) \|_{C^1[0,1]} \leq C$, in view of (8) and the assumption $\| \sqrt{2t}P(t) \|_2 < 1$.

Case 2:- $x'(s) \neq 0$ for all $s \in [0,1]$. We note that in this case $\alpha > 1$ since $x(0) = 0$, $x(1) = \alpha x(\eta)$. Now, in this case, $x(t)$ is a strictly monotonic function on $[0,1]$ and $\max | x(t) | = \| x(t) \|_\infty = | x(1) |$. Next, we see from the equation (3) that

$$x(t) = \int_0^t (t-s)f(s,x(s),x'(s))ds + At, \text{ and}$$
$$A + \int_0^1 (1-s)f(s,x(s),x'(s))ds = \alpha(A\eta + \int_0^\eta (\eta-s)f(s,x(s),x'(s))ds).$$

It follows that

$$A = \frac{1}{1-\alpha\eta}[\alpha\int_0^\eta (\eta-s)f(s,x(s),x'(s))ds - \int_0^1 (1-s)f(s,x(s),x'(s))ds],$$

$$\begin{aligned} x(1) &= A + \int_0^1 (1-s)f(s,x(s),x'(s))ds \\ &= \tfrac{1}{1-\alpha\eta}[\alpha\int_0^\eta (\eta-s)f(s,x(s),x'(s))ds - \alpha\eta\int_0^1 (1-s)f(s,x(s),x'(s))ds] \\ &= -\tfrac{\alpha}{1-\alpha\eta}[(1-\eta)\int_0^\eta sf(s,x(s),x'(s))ds + \eta\int_\eta^1 (1-s)f(s,x(s),x'(s))ds]. \end{aligned}$$

We then get from (8) that

$$\begin{aligned} \| x(t) \|_\infty = | x(1) | &\leq \tfrac{\alpha}{|1-\alpha\eta|}[(1-\eta)\int_0^\eta sp(s)ds + \eta\int_\eta^1 (1-s)p(s)ds] \| x(t) \|_\infty + C_0, \\ &= H(\alpha,\eta,p) \| x(t) \|_\infty + C_0, \end{aligned}$$

where C_0 is a constant independent of $x(t)$. We, next, use (9) to conclude that there exists a constant C_1, independent of $x(t)$, such that

$$\| x(t) \|_\infty \leq C_1.$$

Finally, it is easy to see from the equation $x'(t) = A + \int_0^t f(s, x(s), x'(s))ds$, (8), and $\| x(t) \|_\infty \leq C_1$ that there exists a constant C_2, independent of $x(t)$, such that

$$\| x'(t) \|_\infty \leq C_2.$$

This completes the proof of the theorem. ■

The following Theorem will help us prove that in the case of the linear three-point boundary value problem (4), (5) a solution always exists and is unique when $\alpha\eta < 1$ and $A \geq 0$. So for the linear three-point boundary value problem (4), (5) the existence condition, (9) in Theorem 2 is needed only when $\alpha\eta > 1$. In fact we shall see even in the case $\alpha\eta > 1$ the existence condition, (9) in Theorem 2 is needed only when $\alpha\phi(\eta) > 1$, where $\phi(t)$ is a suitably defined function on $[0, 1]$ and is such that $\phi(t) \leq t$ for all $t \in [0, 1]$.

Theorem 3. *Let us suppose that $p(t)$, $e(t) \in L^1[0, 1]$ and $p(t) \geq 0$, for $t \in [0, 1]$. Also suppose that $\alpha \in R$, $\eta \in (0, 1)$ with $\alpha\eta < 1$ be given. Then the linear three-point boundary value problem*

$$x''(t) = p(t)x(t) + e(t),\ 0 < t < 1,$$
$$x(0) = 0,\ x(1) = \alpha x(\eta), \tag{10}$$

has exactly one solution.

Proof:- It suffices to show, in view of the Fredholm Alternative, that the set of solutions of the homogeneous linear three-point boundary value problem

$$x''(t) = p(t)x(t),\ 0 < t < 1,$$
$$x(0) = 0,\ x(1) = \alpha x(\eta), \tag{11}$$

consists of the trivial solution. Let Φ denote the set of all solutions $x(t)$ of the linear three-point boundary value problem (11) such that $x'(\theta) = 0$ for some $\theta \in [0, 1]$. We observe that if $\alpha \leq 1$ and $x(t)$ is a solution of the linear three-point boundary value problem (11) then there exists a $\theta \in [0, 1]$ such that $x'(\theta) = 0$. Also, suppose that Ψ denote the set of all solutions $x(t)$ of the linear three-point boundary value problem (11) such that $x'(t) \neq 0$ for all $t \in [0, 1]$. Let $x(t) \in \Psi$ be such that $x'(t) > 0$, for all $t \in [0, 1]$. In this case, we must have $\alpha > 1$, $x(t) \geq 0$ for all $t \in [0, 1]$ and $x''(t) = p(t)x(t) \geq 0$ for all $t \in [0, 1]$. However, by the Mean Value Theorem there are points $\lambda \in (0, \eta)$ and $\mu \in (\eta, 1)$ such that

$$x'(\lambda) = \frac{1}{\eta}x(\eta),\ x'(\mu) = \frac{\alpha - 1}{1 - \eta}x(\eta).$$

Since, now, $\alpha\eta < 1$ it is easy to see from above that $x'(\lambda) > x'(\mu)$. But this contradicts the following:

$$x'(\mu) = x'(\lambda) + \int_\lambda^\mu x''(s)ds \geq x'(\lambda).$$

Similarly, $x(t) \in \Psi$ be such that $x'(t) < 0$, for all $t \in [0,1]$ leads to a contradiction. Thus Ψ is an empty set and the set of solutions of the homogeneous linear three-point boundary value problem (11) consists of the set Φ of all solutions $x(t)$ of the linear three-point boundary value problem (11) such that $x'(\theta) = 0$ for some $\theta \in [0,1]$. Claim:- $x(t) \in \Phi$ implies that $x(t) \equiv 0$ for $t \in [0,1]$. Indeed, let $x(t) \in \Phi$. If, now, $x'(0) = 0$ then $x(t) \equiv 0$ for $t \in [0,1]$ by the standard uniqueness theorem for linear initial value problems. (See, for example, Theorem 3, p. 5, [4].) Suppose, now, $x'(\mu) = 0$ for $\mu \in (0,1]$. It follows that

$$\begin{aligned} \textstyle\int_0^\mu x''(t)x(t)dt &= -\textstyle\int_0^\mu (x'(t))^2 dt \\ &= \textstyle\int_0^\mu p(t)(x(t))^2 dt \geq 0. \end{aligned}$$

Accordingly, $x'(t) \equiv 0$ for $t \in [0,\mu]$ and $x'(0) = 0$. Again it follows, as above, that $x(t) \equiv 0$ for $t \in [0,1]$. This proves the claim and completes the proof of the theorem. ■

We shall apply Theorem 2 to the study of the three-point boundary value problem (4)-(5).

Set $\omega(t) = \int_0^t q(u)du$, $t \in [0,1]$. Let us make the following change of independent variable in the equations (4)-(5):

$$s = \phi_A(t) = [\int_0^1 \exp(A\omega(t))dt]^{-1} \int_0^t \exp(A\omega(u))du, \tag{12}$$

and define $y(s) = x(\phi_A^{-1}(s))$, $f(s) = e(\phi_A^{-1}(s))(\phi_A'(t))^{-2}$ for $s \in [0,1]$ or equivalently $y(s) = x(t)$, $f(s) = e(t)(\phi_A'(t))^{-2}$ where $s = \phi_A(t)$. With this change of variable, equations (4)-(5) become

$$y''(s) = (\phi_A'(t))^{-2} p(t) y(s) + f(s),\ 0 < s < 1 \tag{13}$$

$$y(0) = 0,\ y(1) = \alpha y(\zeta),\ \text{where } \zeta = \phi_A(\eta) \tag{14}$$

Now, in this case, we have $\mathbf{p}(s) = (\phi_A'(t))^{-2}p(t)$, where $s = \phi_A(t)$ and $P(s) = \int_s^1 \mathbf{p}(u)du$ for $0 < s < 1$. We, next, calculate $P(s)$ below:

$P(s) = \int_s^1 \mathbf{p}(u)du = \int_{u=s}^{u=1} (\phi_A'(v))^{-2} p(v) \frac{du}{dv} dv$, where $u = \phi_A(v)$.

Now $\phi_A'(v) = [\int_0^1 \exp(A\omega(t))dt]^{-1} \exp(A\omega(v))$ and $\frac{du}{dv} = \phi_A'(v)$.

So $P(s) = \int_{u=s}^{u=1} (\phi_A'(v))^{-1} p(v) dv = [\int_0^1 \exp(A\omega(t))dt] \int_{u=s}^{u=1} \exp(-A\omega(v)) p(v) dv$, where $s = \phi_A(t)$.

Now, the three-point boundary value problem (4)-(5) is equivalent to the three-point boundary value problem (13)-(14). To apply Theorem 3 to the three-point boundary value problem (13)-(14) we need to determine values of A for which $\alpha\zeta = \alpha\phi_A(\eta) < 1$. When $\alpha\zeta = \alpha\phi_A(\eta) > 1$ to obtain the existence of a solution of the three-point boundary value problem (13)-(14) we need to apply Theorem 2. Accordingly, we need to calculate $\| \sqrt{2s}P(s) \|_2$, $H(\alpha, \zeta, \mathbf{p})$, where $\zeta = \phi_A(\eta)$ to determine values of A for which each one of them is less than 1. The following lemmas determine range of values of A for which $\alpha\zeta = \alpha\phi_A(\eta) < 1$.

Lemma 4. *$\phi_A(t) \le t$ for $t \in (0,1)$ if $A \ge 0$.*

Proof:- We have for $A \ge 0$, and $t \in (0,1)$ that

$$\begin{aligned}\phi_A(t) &= [\textstyle\int_0^1 \exp(A\omega(s))ds]^{-1} \int_0^t \exp(A\omega(u))du \\ &= t[\textstyle\int_0^1 \exp(A\omega(s))ds]^{-1} \int_0^1 \exp(A\omega(ts))ds \\ &\le t.\end{aligned}$$

This proves the lemma. ∎

Lemma 5. *If $q(t)$ is not identically equal to zero for $t \in [\eta, 1]$ there exists an $A_1 \in R$ such that $\alpha\phi_A(\eta) = \alpha\zeta < 1$ for $A_1 < A < \infty$, where $\phi_A(t)$ is defined in equation (12). Further, $A_1 \le 0$ if $\alpha\eta < 1$.*

Proof:- Indeed,

$$\begin{aligned}\lim_{A\to\infty} \phi_A(\eta) &= \lim_{A\to\infty}[\textstyle\int_0^1 \exp(A\omega(t))dt]^{-1} \int_0^\eta \exp(A\omega(u))du \\ &= \lim_{A\to\infty}[\textstyle\int_0^\eta \exp(A\omega(t))dt + \int_\eta^1 \exp(A\omega(t))dt]^{-1} \int_0^\eta \exp(A\omega(u))du \\ &= \lim_{A\to\infty}[1 + (\textstyle\int_0^\eta \exp(A\omega(t))dt)^{-1} \int_\eta^1 \exp(A\omega(t))dt]^{-1} \\ &\le \lim_{A\to\infty}[1 + (\eta \exp(A\omega(\eta)))^{-1} \textstyle\int_\eta^1 \exp(A\omega(t))dt]^{-1} \\ &\le \lim_{A\to\infty}[1 + \tfrac{1}{\eta} \textstyle\int_\eta^1 \exp(A \int_\eta^t q(s)ds)dt]^{-1} \\ &\le \lim_{A\to\infty}[1 + \tfrac{A}{\eta} \textstyle\int_\eta^1 (1-s)q(s)ds]^{-1} = 0,\end{aligned}$$

since, $q(s)$ is not identically equal to zero for $s \in [\eta, 1]$. Accordingly, there exists an $A_1 \in R$ such that $\alpha\phi_A(\eta) = \alpha\zeta < 1$ for $A_1 < A < \infty$.

Finally, if $\alpha\eta < 1$ we see from lemma 4 that $\alpha\phi_A(\eta) \le \alpha\eta < 1$ for $A \ge 0$. It follows that there exists $A_1 \le 0$ such that $\alpha\phi_A(\eta) = \alpha\zeta < 1$ for $A_1 < A < \infty$. This completes the proof of the lemma. ∎

We summarize our results for the three-point boundary value problem (4)-(5) in the following.

Theorem 6. *Let $q(t)$ in the three-point boundary value problem (4)-(5) be not identically zero on $[\eta, 1]$. Then there exists an $A_1 \in R \cup \{-\infty\}$ such that the three-point boundary value problem (4)-(5) has a unique solution for $A_1 < A < \infty$. Moreover, $A_1 \le 0$ if $\alpha\eta < 1$ and $A_1 = -\infty$ if $\alpha \le 1$.*

Proof:- We see from lemma 5 that there exists an $A_1 \in R$ such that $\alpha\phi_A(\eta) < 1$ for $A_1 < A < \infty$. It then follows from Theorem 3 that the three-point boundary value problem (13)-(14) has a unique solution for $A_1 < A < \infty$. Since now the three-point boundary value problem (4)-(5) is equivalent to the three-point boundary value problem (13)-(14) the theorem follows. Further, if $\alpha\eta < 1$, we can take $A_1 \le 0$ in view of lemmas 4, 5. When $\alpha \le 1$ we have $\alpha\phi_A(\eta) < 1$ for all $A \in R$ since $\phi_A(\eta) < 1$ for $\eta \in (0,1)$ and thus $A_1 = -\infty$. This completes the proof of the theorem. ∎

Remark 1 We see from lemma 5 that if $\alpha\zeta = \alpha\phi_A(\eta) > 1$ then we must have $A \le A_1$. Now, in this case to obtain the existence of a solution of the three-point boundary value problem (4)-(5) we shall need to apply Theorem 2 and we need to find values of $A \le A_1$ for which both $\| \sqrt{2s}P(s) \|_2$, $H(\alpha, \zeta, \mathbf{p})$ are less than one. We do this for particular examples that we study in the following.

We, next, apply the results for the three-point boundary value problem (4)-(5) to the following example:

$$x''(t) = t^{-\frac{1}{4}}x(t) + At^{-\frac{1}{4}}x'(t) + e(t),\ 0 < t < 1,$$
$$x(0) = 0,\ x(1) = \alpha x(\eta),$$

where $\alpha \in R$, and $\eta \in (0,1)$ are given. We shall compute an A_0 such that for $A_0 < A < \infty$ this boundary value problem has a unique solution.

We need to make the following change of independent variable in the equations (6)-(7):

$$s = \phi_A(t) = [\int_0^1 \exp(\frac{4A}{3}t^{\frac{3}{4}})dt]^{-1}\int_0^t \exp(\frac{4A}{3}u^{\frac{3}{4}})du, \tag{15}$$

and define $y(s) = x(\phi_A^{-1}(s))$, $f(s) = e(\phi_A^{-1}(s))(\phi_A'(t))^{-2}$ for $s \in [0,1]$ or equivalently $y(s) = x(t)$, $f(s) = e(t)(\phi_A'(t))^{-2}$ where $s = \phi_A(t)$. With this change of variable, equations (6)-(7) become

$$y''(s) = (\phi_A'(t))^{-2}t^{-\frac{1}{4}}y(s) + f(s),\ 0 < s < 1, \tag{16}$$
$$y(0) = 0,\ y(1) = \alpha y(\zeta),\ \text{where } \zeta = \phi_A(\eta) \tag{17}$$

Now, in this case, we have $\mathbf{p}(s) = (\phi_A'(t))^{-2}t^{-\frac{1}{4}}$, where $s = \phi_A(t)$ and $P(s) = \int_s^1 \mathbf{p}(u)du$ for $0 < s < 1$. We, next, calculate $P(s)$ below:

$P(s) = \int_s^1 \mathbf{p}(u)du = \int_{u=s}^{u=1}(\phi_A'(v))^{-2}v^{-\frac{1}{4}}\frac{du}{dv}dv$, where $u = \phi_A(v)$.

Now $\phi_A'(v) = [\int_0^1 \exp(\frac{4A}{3}t^{\frac{3}{4}})dt]^{-1}\exp(\frac{4A}{3}v^{\frac{3}{4}})$ and $\frac{du}{dv} = \phi_A'(v)$.

So $P(s) = \int_{u=s}^{u=1}(\phi_A'(v))^{-1}v^{-\frac{1}{4}}dv = [\int_0^1 \exp(\frac{4A}{3}t^{\frac{3}{4}})dt]\int_{u=s}^{u=1}\exp(-\frac{4A}{3}v^{\frac{3}{4}})v^{-\frac{1}{4}}dv$

$= -\frac{1}{A}[\int_0^1 \exp(\frac{4A}{3}t^{\frac{3}{4}})dt]\int_{u=s}^{u=1}\exp(w)dw$, where $w = -\frac{4A}{3}v^{\frac{3}{4}}$

$= -\frac{1}{A}[\int_0^1 \exp(\frac{4A}{3}t^{\frac{3}{4}})dt]\exp(w)\mid_{u=s}^{u=1} = \frac{1}{A}[\int_0^1 \exp(\frac{4A}{3}t^{\frac{3}{4}})dt]\{\exp(-\frac{4A}{3}t^{\frac{3}{4}}) - \exp(-\frac{4A}{3})\}$,

$s = \phi_A(t)$.

Now, the three-point boundary value problem (6)-(7) is equivalent to the three-point boundary value problem (16)-(17). We need to calculate $\| \sqrt{2s}P(s) \|_2$, $H(\alpha,\zeta,\mathbf{p})$, where $\zeta = \phi_A(\eta)$, to apply Theorem 2 and need to determine for what values of A each one of them is less than 1. The following lemmas determine range of values of A for which each one of them is less than 1.

Lemma 7. *There exists an $A_0 > 0$ such that $\| \sqrt{2s}P(s) \|_2 < 1$ for $-A_0 < A < \infty$. ($1.44375 < A_0 < 1.444$)*

Proof:- Let us define $F(A)$ by

$$F(A) = \| \sqrt{2s}P(s) \|_2^2 = \int_0^1 2s \mid P(s) \mid^2 ds = \int_{s=0}^{s=1} 2s \mid P(s) \mid^2 \frac{ds}{dt}dt$$
$$= \frac{2}{A^2}[\int_0^1 \exp(\frac{4A}{3}t^{\frac{3}{4}})dt]^2 \int_{s=0}^{s=1}\{\exp(-\frac{4A}{3}t^{\frac{3}{4}}) - \exp(-\frac{4A}{3})\}^2\phi_A(t)\phi_A'(t)dt,$$

where $s = \phi_A(t)$ and $\phi_A(t)$ is defined in equation (15). We also note from equation (15) that if $s = \phi_A(t)$ then $s = 0$ implies $t = 0$ and $s = 1$ implies $t = 1$. It then follows, using equation (15), again that

$$F(A) = \frac{2}{A^2}\int_0^1\{\exp(-\frac{4A}{3}t^{\frac{3}{4}}) - \exp(-\frac{4A}{3})\}^2(\int_0^t \exp(\frac{4A}{3}u^{\frac{3}{4}})du)\exp(\frac{4A}{3}t^{\frac{3}{4}})dt.$$

Claim 1:- $F(A) < 1$ for $0 \leq A < \infty$.

Proof of Claim 1:- We first note that $e^{-x} \geq 1 - x$ for $x \geq 0$, so that $1 - e^{-x} \leq x$ for $x \geq 0$. Then,

$$\begin{aligned} F(A) &= \tfrac{2}{A^2}\int_0^1\{\exp(-\tfrac{4A}{3}t^{\frac{3}{4}}) - \exp(-\tfrac{4A}{3})\}^2(\int_0^t \exp(\tfrac{4A}{3}u^{\frac{3}{4}})du)\exp(\tfrac{4A}{3}t^{\frac{3}{4}})dt \\ &\leq \tfrac{2}{A^2}\int_0^1\{\exp(-\tfrac{4A}{3}t^{\frac{3}{4}}) - \exp(-\tfrac{4A}{3})\}^2\exp(\tfrac{8A}{3}t^{\frac{3}{4}})tdt \\ &= \tfrac{2}{A^2}\int_0^1\{1 - \exp(-\tfrac{4A}{3} + \tfrac{4A}{3}t^{\frac{3}{4}})\}^2 tdt \\ &\leq \tfrac{2}{A^2}\int_0^1 \tfrac{16A^2}{9}(1 - t^{\frac{3}{4}})^2 tdt = \tfrac{32}{9}\int_0^1(1 - t^{\frac{3}{4}})^2 tdt \\ &= \tfrac{32}{9}\int_0^1(t - 2t^{\frac{7}{4}} + t^{\frac{5}{2}})dt = \tfrac{32}{9}(\tfrac{1}{2} - 2\tfrac{4}{11} + \tfrac{2}{7}) = \tfrac{16}{77} < 1. \end{aligned}$$

So $F(A) < 1$ for all $A > 0$. This proves claim 1.

Claim 2:- There exists an $A_0 > 0$ such that $\| \sqrt{2s}P(s) \|_2 < 1$ for $-A_0 < A \leq 0$.

Proof of Claim 2:- We first show analytically the existence of such an A_0 and then improve the value of A_0 using numerical calculations and some estimates and graphs that we drew using Maple and MathCad. We are considering the case $A \leq 0$, so let us set $A = -B$ so that $B \geq 0$. We then proceed as in claim 1 to get:

$$\begin{aligned} F(A) &= \tfrac{2}{A^2}\int_0^1\{\exp(-\tfrac{4A}{3}t^{\frac{3}{4}}) - \exp(-\tfrac{4A}{3})\}^2(\int_0^t \exp(\tfrac{4A}{3}u^{\frac{3}{4}})du)\exp(\tfrac{4A}{3}t^{\frac{3}{4}})dt \\ &= \tfrac{2}{B^2}\int_0^1\{\exp(\tfrac{4B}{3}t^{\frac{3}{4}}) - \exp(\tfrac{4B}{3})\}^2(\int_0^t \exp(-\tfrac{4B}{3}u^{\frac{3}{4}})du)\exp(-\tfrac{4B}{3}t^{\frac{3}{4}})dt \\ &\leq \tfrac{2}{B^2}\int_0^1\{\exp(\tfrac{4B}{3}t^{\frac{3}{4}}) - \exp(\tfrac{4B}{3})\}^2 tdt \\ &= \tfrac{2}{B^2}\int_0^1 \exp(\tfrac{8B}{3})\{\exp(\tfrac{4B}{3}t^{\frac{3}{4}} - \tfrac{4B}{3}) - 1\}^2 tdt \\ &= \tfrac{2}{B^2}\int_0^1 \exp(\tfrac{8B}{3})\{1 - \exp(-\tfrac{4B}{3}(1 - t^{\frac{3}{4}})\}^2 tdt \\ &\leq \tfrac{2}{B^2}\exp(\tfrac{8B}{3})\int_0^1 \tfrac{16B^2}{9}(1 - t^{\frac{3}{4}})^2 tdt = \tfrac{32}{9}\exp(\tfrac{8B}{3})\int_0^1(1 - t^{\frac{3}{4}})^2 tdt \\ &= \tfrac{16}{77}\exp(\tfrac{8B}{3}). \end{aligned}$$

Now, $\frac{16}{77}\exp(\frac{8B}{3}) < 1$ if $B < \frac{3}{8}\ln\frac{77}{16} \approx .589206$. So when $A \leq 0$, and $A > -.589206$ we have $F(A) < 1$. So we have shown analytically that $A_0 = -.589206$. Now, when we graphed $F(A)$ using MathCad we found that $F(A) < 1$ when $-1.44374854 < A < \infty$ and that the graph of $F(A)$ is a decreasing graph. Using Scientific Workplace's Maple we found that $F(-1.44375) = .99993$ and $F(-1.444) = 1.0002$. So we conclude that $F(A) < 1$ when $-1.44375 \leq A < \infty$.

We found that Maple could not graph $F(A)$, so we estimated $F(A)$ by a simpler function and using the graph of this simpler function we found that $F(A) < 1$ when $-1.4025 \leq A < \infty$. We present some of these details below. We note that for $x \geq 0$, $1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} \leq \exp(-x) \leq 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}$. Using this we see for $A \leq 0$, $A = -B$ with $B \geq 0$ that

$$\begin{aligned} F(A) = F(-B) &= \tfrac{2}{B^2}\int_0^1\{\exp(\tfrac{4B}{3}t^{\frac{3}{4}}) - \exp(\tfrac{4B}{3})\}^2(\int_0^t \exp(-\tfrac{4B}{3}u^{\frac{3}{4}})du)\exp(-\tfrac{4B}{3}t^{\frac{3}{4}})dt \\ &\leq \tfrac{32}{9}\exp(\tfrac{8B}{3})\int_0^1(1 - t^{\frac{3}{4}})^2(1 - \tfrac{2B}{3}(1 - t^{\frac{3}{4}}) + \tfrac{8B^2}{27}(1 - t^{\frac{3}{4}})^2 - \tfrac{8B^3}{81}(1 - t^{\frac{3}{4}})^3 \\ &\quad + \tfrac{32B^4}{1215}(1 - t^{\frac{3}{4}})^4)^2(t - \tfrac{16}{21}Bt^{\frac{7}{4}} + \tfrac{16}{45}B^2t^{\frac{5}{2}} - \tfrac{128}{1053}B^3t^{\frac{13}{4}} + \tfrac{8}{243}B^4t^4)(1 - \tfrac{4}{3}Bt^{\frac{3}{4}} \\ &\quad + \tfrac{8}{9}B^2t^{\frac{3}{2}} - \tfrac{32}{81}B^3t^{\frac{9}{4}} + \tfrac{32}{243}B^4t^3)dt = G(B). \end{aligned}$$

Now, $G(B)$ is such that $.20779 = G(0) \leq G(B) \leq G(1.4025) = .99757 < 1$. (It is easy to see this by graphing $G(B)$ using Maple.) It follows that $F(A) < 1$ when $-1.4025 < A < \infty$. The number -1.4025 can be pushed closer to -1.444 by using higher order polynomials to estimate $\exp(-x)$, $x \geq 0$, by higher degree polynomials. This completes the proof of the lemma. ∎

We, next, define $H(A)$ by

$$H(A) = H(\alpha, \zeta, \mathbf{p}),$$

where α, η are as in (6)-(7), $\mathbf{p}(s) = (\phi'_A(t))^{-2} t^{-\frac{1}{4}}$, where $s = \phi_A(t)$ with $\phi_A(t)$ defined in equation (15) and $\zeta = \phi_A(\eta)$.

Lemma 8. $H(A) = \dfrac{\alpha(\eta - \phi_A(\eta))}{A \mid 1 - \alpha\phi_A(\eta) \mid}$.

Proof:- We first observe that

$$\begin{aligned} \textstyle\int_0^\zeta s\mathbf{p}(s)ds &= \textstyle\int_{s=0}^{s=\zeta} \phi_A(t)(\phi'_A(t))^{-2} t^{-\frac{1}{4}} \frac{ds}{dt} dt = \int_{t=0}^{t=\eta} \phi_A(t)(\phi'_A(t))^{-1} t^{-\frac{1}{4}} dt \\ &= \textstyle\int_0^\eta (\int_0^t \exp(\frac{4A}{3} u^{\frac{3}{4}}) du) \exp(-\frac{4A}{3} t^{\frac{3}{4}}) t^{-\frac{1}{4}} dt \\ &= -\textstyle\frac{1}{A} \int_0^\eta (\int_0^t \exp(\frac{4A}{3} u^{\frac{3}{4}}) du) \frac{d}{dt}(\exp(-\frac{4A}{3} t^{\frac{3}{4}})) dt \\ &= \textstyle\frac{1}{A}(\eta - (\int_0^\eta \exp(\frac{4A}{3} t^{\frac{3}{4}}) dt) \exp(-\frac{4A}{3} \eta^{\frac{3}{4}})). \end{aligned} \tag{18}$$

Next,

$$\begin{aligned} \textstyle\int_\zeta^1 (1-s)\mathbf{p}(s)ds &= \textstyle\int_\eta^1 (1-\phi_A(t))(\phi'_A(t))^{-1} t^{-\frac{1}{4}} dt \\ &= \textstyle[\int_0^1 \exp(\frac{4A}{3} t^{\frac{3}{4}}) dt] \int_\eta^1 (1-\phi_A(t)) \exp(-\frac{4A}{3} t^{\frac{3}{4}}) t^{-\frac{1}{4}} dt \\ &= -\textstyle\frac{1}{A}[\int_0^1 \exp(\frac{4A}{3} t^{\frac{3}{4}}) dt] \int_\eta^1 (1-\phi_A(t)) \frac{d}{dt}(\exp(-\frac{4A}{3} t^{\frac{3}{4}})) dt \\ &= \textstyle\frac{1}{A}[\int_0^1 \exp(\frac{4A}{3} t^{\frac{3}{4}}) dt](1-\phi_A(\eta)) \exp(-\frac{4A}{3} \eta^{\frac{3}{4}}) \\ &\quad -\textstyle\frac{1}{A}[\int_0^1 \exp(\frac{4A}{3} t^{\frac{3}{4}}) dt] \int_\eta^1 \phi'_A(t) \exp(-\frac{4A}{3} t^{\frac{3}{4}})) dt \\ &= \textstyle\frac{1}{A}[\int_0^1 \exp(\frac{4A}{3} t^{\frac{3}{4}}) dt](1-\phi_A(\eta)) \exp(-\frac{4A}{3} \eta^{\frac{3}{4}}) - \frac{1-\eta}{A} \end{aligned} \tag{19}$$

We, now, get from equations (18) and (19) that

$$\begin{aligned} &\textstyle(1-\zeta) \int_0^\zeta s\mathbf{p}(s)ds + \zeta \int_\zeta^1 (1-s)\mathbf{p}(s)ds \\ &= \textstyle\frac{1-\phi_A(\eta)}{A}(\eta - (\int_0^\eta \exp(\frac{4A}{3} t^{\frac{3}{4}}) dt) \exp(-\frac{4A}{3} \eta^{\frac{3}{4}})) \\ &\quad +\textstyle\frac{\phi_A(\eta)}{A}[\int_0^1 \exp(\frac{4A}{3} t^{\frac{3}{4}}) dt](1-\phi_A(\eta)) \exp(-\frac{4A}{3} \eta^{\frac{3}{4}}) - \frac{(1-\eta)\phi_A(\eta)}{A} \\ &= \textstyle\frac{\eta-\phi_A(\eta)}{A} - \frac{1-\phi_A(\eta)}{A} \exp(-\frac{4A}{3} \eta^{\frac{3}{4}})[\int_0^\eta \exp(\frac{4A}{3} t^{\frac{3}{4}}) dt - \phi_A(\eta)(\int_0^1 \exp(\frac{4A}{3} t^{\frac{3}{4}}) dt)] \\ &= \textstyle\frac{\eta-\phi_A(\eta)}{A} \end{aligned} \tag{20}$$

We, then, get from equation (20) that $H(A) = \dfrac{\alpha(\eta - \phi_A(\eta))}{\mid 1 - \alpha\zeta \mid A} = \dfrac{\alpha(\eta - \phi_A(\eta))}{A \mid 1 - \alpha\phi_A(\eta) \mid}$. This completes the proof of lemma. ∎

Remark 2 We see from Theorem 3 that we do not need to consider $H(A)$ when $\alpha\phi_A(\eta) < 1$ to find the A for which the three-point boundary value problem (6)-(7) has a unique solution. $H(A)$ comes into play only when $\alpha\phi_A(\eta) > 1$. Since in this case the existence condition happens to be $\max\{F(A), H(A)\} < 1$, one needs to find those $A > -1.444$ and $A \leq A_1$, where A_1 is such that $\alpha\phi_A(\eta) < 1$ for $A_1 < A < \infty$. It is accordingly useful to have the simple expression for $H(A)$ given in lemma 8.

We summarize our results for the three-point boundary value problem (6)-(7) in the following.

Theorem 9. *The three-point boundary value problem (6)-(7) has a unique solution for all $A \in R$ if $\alpha \leq 1$ and for $A_1 < A < \infty$, where A_1 as given in lemma 5, if $\alpha > 1$.*

Proof:- We deduce the existence of a solution for the three-point boundary value problem (6)-(7) from the existence of a solution for the three-point boundary value problem (16)-(17). Now, we see from Theorem 3 that the three-point boundary value problem (16)-(17) has a solution for all A for which $\alpha\phi_A(\eta) < 1$. Now if $\alpha \leq 1$ we see that $\alpha\phi_A(\eta) < 1$ for all A and if $\alpha > 1$ we see that there exists an A_1 by lemma 5 such that $\alpha\phi_A(\eta) < 1$ for $A_1 < A < \infty$. This completes the proof of the theorem. ∎

When $\alpha > 1$, our work above indicates that the three-point boundary value problem (6)-(7) has exactly one solution for $A \in (A_1, \infty)$, in view of lemma 5 and Theorem 3. But it is possible that there is another interval I near 0 such that a solution exists for $A \in I$ as we illustrate in some examples below. We should like to remark that since the three-point boundary value problem (6)-(7) is a linear problem our methods indicate that its solution is in fact a unique solution. Also, it is obvious that $\phi_A(\eta) < \eta$ for $A > 0$, and we derive the existence of a solution for the boundary value problem (6)-(7) from the existence of a solution of the three-point boundary value problem (16)-(17), there exist a lot of A for which a solution to the boundary value problem (6)-(7) exists even in the resonance case $\alpha\eta = 1$, because in this case the corresponding three-point boundary value problem (16)-(17) is a non-resonance problem for a lot of A.

Let us now consider the three-point boundary value problem (6)-(7) when $\alpha = 1.5$ and $\eta = .25$. It was shown by Gupta-Trofimchuk in [6] that this problem has a solution if $| A | < .6417036299$. In this case we have $\alpha\eta = (1.5)(.25) = .375 < 1$ and accordingly we see that $1.5\phi_A(.25) < 1$ for $A \geq 0$ and $1.5\phi_A(.25) = .375$ when $A = 0$. This shows that there exists an $A_1 < 0$ such that $1.5\phi_A(.25) < 1$ for $A_1 < A < \infty$. We see using Maple that the graph of $1.5\phi_A(.25)$ is a decreasing graph as a function of A. Now for $A = -3.027$ we see that $\int_0^{.25} \exp(\frac{4}{3}(-3.027)x^{\frac{3}{4}})dx = .11936$ and $\int_0^1 \exp(\frac{4}{3}(-3.027)x^{\frac{3}{4}})dx = .17903$ so that $1.5\phi_A(.25) = 1.5(\frac{.11936}{.17903}) = 1.0001$. Also for $A = -3.026$ we see that $\int_0^{.25} \exp(\frac{4}{3}(-3.026)x^{\frac{3}{4}})dx = .11939$ and $\int_0^1 \exp(\frac{4}{3}(-3.026)x^{\frac{3}{4}})dx = .1791$ so that $1.5\phi_A(.25) = 1.5(\frac{.11939}{.1791}) = .99992$. It follows from Theorem 3 that the boundary value problem has a unique solution when $-3.026 \leq A < \infty$. Here $F(A)$ and $H(A)$ need not be considered because we know that $F(A) > 1$ for $A < -1.444$.

We, next, consider the three-point boundary value problem (6)-(7) when $\alpha = 1.5$ and $\eta = .5$. It was shown by Gupta-Trofimchuk in [6] that this problem has a solution if $| A | < .2097464385$. Now, Theorem 9 applies and a unique solution to three-point boundary value problem (6)-(7) exists for all those A for which $1.5\phi_A(.5) < 1$. Now, we see, using Maple, that $1.5\phi_A(.5) < 1$ when $-1.09 \leq A < \infty$, since the graph of $1.5\phi_A(.5)$ is decreasing on the interval $(-2, \infty)$ and $1.5\phi_A(.5) = .99787$, when $A = -1.09$, $1.5\phi_A(.5) = 1.0$ when $A = -1.1$. It follows from Theorem 3 and lemma 7 that the boundary value problem has a unique solution when $-1.09 \leq A < \infty$. Now, to see for what $A < -1.1$ the three-point boundary value problem (6)-(7) with $\alpha = 1.5$ and $\eta = .5$, has a solution we need to apply Theorem 2. Now, we see from lemma 7 that $F(A) < 1$ when $-1.44375 \leq A < \infty$ and so in particular for $-1.44375 < A < -1.1$. Now, we see from lemma 8 that $A = -1.1$ is a vertical asympote for $H(A)$ since $1.5\phi_A(.5) = 1.0$ when $A = -1.1$. Now we see using Maple that the graph of $H(A)$ is increasing on $(-\infty, -1.1)$. And, we see using Maple that $\int_0^{.5} \exp(\frac{4}{3}(-1.44375)x^{\frac{3}{4}})dx = .27299$ and $\int_0^1 \exp(\frac{4}{3}(-1.44375)x^{\frac{3}{4}})dx = .38228$ so that $H(-1.44375) = \dfrac{1.5(.5 - \frac{.27299}{.38228})}{(-2)\left|1 - \frac{1.5(.27299)}{.38228}\right|} = 2.2565$. Thus $H(A) > 1$ for $-1.44375 < A < -1.1$. Accordingly, our methods do not decide if a solution exists for $A < -1.1$.

We, next, consider the three-point boundary value problem (6)-(7) when $\alpha = 2$ and $\eta = .75$. It was shown by Gupta-Trofimchuk in [6] that this problem has a solution if $\mid A \mid < .3840152114$. In this case we see, using Maple, that $H(A) < 1$ when $-2 < A \leq 1.13$, $3.24 \leq A < \infty$, since the graph of $H(A)$ is increasing on the interval $(-2, 1.15]$ with $H(-2) = .2024$ and $H(1.13) = .99789$, $H(1.14) = 1.0084$ and the graph of $H(A)$ is decreasing on the interval $[3.24, \infty)$ with $H(3.24) = .99184$, $H(3.23) = 1.0006$. Also, we see, again using Maple, that $2\phi_A(.75) < 1$ when $2.14 \leq A < \infty$, since the graph of $2\phi_A(.75)$ is decreasing on the interval $(-2, \infty)$ and $2\phi_A(.75) = .99964$ when $A = 2.14$, $2\phi_A(.75) = 1.0019$ when $A = 2.13$. It follows from Theorem 3 that the boundary value problem (6)-(7) has a unique solution when $2.14 \leq A < \infty$ and has a unique solution for $-A_0 < A \leq 1.13$ by Theorem 2 and lemma 7, where A_0 is as given in lemma 7.

We, next, study this problem when $\alpha = 2$ and $\eta = .6$. It was remarked by Gupta-Trofimchuk in [6] that they did not know if this problem has a solution. In this case we see, using Maple, that $H(A) < 1$ when $-2 < A \leq -.31$, $1.72 \leq A < \infty$, since the graph of $H(A)$ is increasing on the interval $(-2, -.32]$ with $H(-2) = .35592$ and $H(-.31) = .99574$, $H(-.3) = 1.0061$ and the graph of $H(A)$ is decreasing on the interval $[1.72, \infty)$ with $H(1.72) = .99778$, $H(1.71) = 1.0074$. Also, we see, again using Maple, that $2\phi_A(.6) < 1$ when $.69 \leq A < \infty$, since the graph of $2\phi_A(.6)$ is decreasing on the interval $(-2, \infty)$ and $2\phi_A(.6) = .99891$ when $A = .69$, $2\phi_A(.6) = 1.0018$ when $A = .68$. It follows from Theorem 3 that the boundary value problem (6)-(7) has a unique solution when $.69 \leq A < \infty$ and has a unique solution for $-A_0 < A \leq -.31$ by Theorem 2 and lemma 7, where A_0 is as given in lemma 7.

Finally, we study this problem when $\alpha = 8$ and $\eta = .75$. In this case we see, using Maple, that $H(A) < 1$ when $-2 < A \leq 5.48$, $10.59 \leq A < \infty$, since the graph of $H(A)$ is increasing on the interval $(-2, 5.5]$ with $H(-2) = .10693$ and $H(5.48) = .99716$, $H(5.49) = 1.0022$ and the graph of $H(A)$ is decreasing on the interval $[10.55, \infty)$ with $H(10.59) = .99785$, $H(10.58) = 1.0009$. Also, we see, again using Maple, that $8\phi_A(.75) < 1$ when $7.72 \leq A < \infty$, since the graph of $8\phi_A(.75)$ is decreasing on the interval $(-2, \infty)$ and $8\phi_A(.75) = .99991$ when $A = 7.72$, $8\phi_A(.75) = 1.0025$ when $A = 7.71$. It follows from Theorem 3 that the boundary value problem (6)-(7) has a unique solution when $7.72 \leq A < \infty$ and has a unique solution for $-A_0 < A \leq 5.48$ by Theorem 2 and lemma 7, where A_0 is as given in lemma 7.

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