The function epsilon for complex Tori and Riemann surfaces

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Abstract

In the framework of the quantization of Kähler manifolds carried out in [3], [4], [5] and [6], one can define a smooth function, called the function *epsilon*, which is the central object of the theory. The first explicit calculation of this function can be found in [10].

In this paper we calculate the function *epsilon* in the case of the complex tori and the Riemann surfaces.

1 Introduction

A quantization of a Kähler manifold (M, ω) is a pair (L, h), where L is a holomorphic line bundle over M and h is a hermitian structure on L such that $\mathrm{curv}(L, h) = -2\pi i\omega$. The curvature $\mathrm{curv}(L, h)$ is calculated with respect to the *Chern connection*, i.e. the unique connection compatible with both the holomorphic and the hermitian structure. Not all manifolds admit such a pair. In terms of cohomology classes, a Kähler manifold admits a quantization if and only if the form ω is integral [7], i.e. its cohomology class $[\omega]_{dR}$ in the de Rham group, is in the image of the natural map $H^2(M, \mathbb{Z}) \hookrightarrow H^2(M, \mathbb{C})$. In particular, when M is compact, the integrality of ω implies, by a well-known theorem of Kodaira, that M is a projective algebraic manifold.

In the framework of the quantization of a Kähler manifold (M, ω) one can define a smooth function $\epsilon_{(L,h)}$ on M, depending on the pair (L, h), which is the central object of the theory and which is one of the main ingredients needed to apply a procedure

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called *quantization by deformation* introduced by Berezin in his foundational paper [1]. The work of Berezin was later developed and generalized in a series of papers [3], [4], [5] and [6] which are the starting point of the present article.

In this paper, we give an explicit calculation of the function *epsilon* in terms of theta functions for the 1-dimensional complex torus (see section 3). We also calculate the function *epsilon* for a Riemann surface of genus $g > 1$ endowed with the hyperbolic metric (see section 4).

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2 Preliminaries

Let (L, h) be a quantization of a Kähler manifold (M, ω). Consider the separable complex Hilbert space $\mathcal{H}_h$ consisting of global holomorphic sections s of L, which are bounded with respect to

$$\langle s, s\rangle_h = \|s\|_h^2 = \int_M h(s(x), s(x))\frac{\omega^n(x)}{n!}$$

(see [3]). Let $x \in M$ and $q \in L^0$ a point of the fibre over x. If one evaluates $s \in \mathcal{H}_h$ at x, one gets a multiple $\delta_q(s)$ of q, i.e. $s(x) = \delta_q(s)q$. The map $\delta_q : \mathcal{H}_h \to \mathbb{C}$ is a continuous linear functional [3] hence by Riesz's theorem, there exists a unique $e_q \in \mathcal{H}_h$ such that $\delta_q(s) = \langle s, e_q\rangle_h$, i.e.

$$s(x) = \langle s, e_q\rangle_h q. \tag{1}$$

It follows, by (1), that

$$e_{cq} = \overline{c}^{-1}e_q, \ \forall c \in \mathbb{C}^*.$$

Definition 2.1. *The holomorphic section e_q is called the coherent states relative to the point q.*

Then, one can define a real valued function on M by the formula

$$\epsilon_{(L,h)}(x) := h(q, q)\|e_q\|_h^2, \tag{2}$$

where $q \in L^0$ is any point on the fibre of x. Let $(s_0, \dots, s_N)$ $(N \leq \infty)$ be a unitary basis for $(\mathcal{H}_h, \langle \cdot\, , \,\cdot \rangle_h)$. Take $\lambda_j \in \mathbb{C}$ such that $s_j(x) = \lambda_j q, j = 0, \dots, N$. Then

$$s(x) = \sum_{j=0}^{N}\langle s, s_j\rangle_h s_j(x) = \sum_{j=0}^{N}\langle s, s_j\rangle_h \lambda_j q = \langle s, \sum_{j=0}^{N}\bar{\lambda}_j s_j\rangle_h q.$$

By (1) it follows that

$$e_q = \sum_{j=0}^{N}\bar{\lambda}_j s_j, \tag{3}$$

and

$$\epsilon_{(L,h)}(x) = h(q,q)\|e_q\|_h^2 = \sum_{j=0}^{N} h(s_j(x), s_j(x)). \tag{4}$$

One can calculate the function $\epsilon_{(L^k,h^k)}$ for every natural number k. Namely, one considers the Kähler form $k\omega$ on M and (L^k, h^k) the quantum line bundle for $(M, k\omega)$, where L^k is the k-tensor power of L and $h^k := h \otimes \cdots \otimes h$, k-times.

We say that a quantization (L,h) of a Kähler manifold (M,ω) is *regular* if, for any natural number k, $\epsilon_{(L^k,h^k)}$ is constant. If a manifold (M,ω) admits a regular quantization then on can define a $*$-product on $C^\infty(M)$ the algebra of smooth functions on the manifold M (see [3], [4], [5] and [6]). One of the main tool in constructing this $*$-product is the following Rawnsley's result [10], saying that, if the above regulariy condition is satisifed, then the Kähler forms $k\omega$ are projectively induced i.e. for every natural number k there exists a natural number $N(k)$ and a holomorphic map into the complex $N(k)$-dimensional projective space

$$\phi_k : M \to \mathbb{P}^{N(k)}(\mathbb{C})$$

such that $\phi_k^*\Omega_k = k\omega$, for Ω_k the Fubini-Study form on $\mathbb{P}^{N(k)}(\mathbb{C})$.

3 Quantization of complex tori

Let $M = V/\Lambda$ be an n-dimensional complex torus, where V is an n-dimensional complex vector space and Λ is a $2n$-lattice on V. Let H be a hermitian form on V and

$$\omega := \frac{i}{2}\partial\bar{\partial}H.$$

Since ω is invariant by translations it descends to a globally defined Kähler form ω on M which makes (M,ω) into a homogeneous Kähler manifold. It is well-known [9] that ω is integral iff the imaginary part of H takes integral values on Λ, i.e. $\Im H(\Lambda,\Lambda) \subset \mathbb{Z}$. Under this hypothesis it follows by [7] that (M,ω) admits a quantization (L,h). On the other hand a theorem in [11] asserts that ω can not projectively induced and so by the discussion at the end of the previous section the quantization (L,h) can not be regular.

An explicit description of the line bundle L and of the hermitian structure h can be found in [9, Chapter 1] to whom we refer for the proof of the following assertions. First of all the global holomorphic sections of L can be seen as holomorphic functions θ on V satisfying

$$\theta(v+\lambda) = A(\lambda, v)\theta(v), \tag{5}$$

where

$$A(\lambda, v) = \chi(\lambda)e^{\pi H(v,\lambda) + \frac{\pi}{2}H(\lambda,\lambda)}$$

and $\chi : \Lambda \to S^1$ belongs to the group of semicharacter of H, i.e.

$$\chi(\lambda+\mu) = \chi(\lambda)\chi(\mu)e^{\pi i \Im H(\lambda,\mu)},\ \forall \lambda, \mu \in \Lambda. \tag{6}$$

Given θ a holomorphic section of L define

$$h(\theta(v), \theta(v)) = e^{-\pi H(v,v)}|\theta(v)|^2.$$

It follows easily by (5) that the function h is invariant under the action of the lattice, i.e.

$$h(\theta(v+\lambda),\theta(v+\lambda)) = h(\theta(v),\theta(v))\ \forall\lambda\in\Lambda,$$

and so it defines a hermitian structure on L. Furthermore,

$$\operatorname{curv}(L,h) = -\partial\bar\partial\log h = \pi\partial\bar\partial H = -2\pi i\omega,$$

which shows that (L,h) is a quantization for $(V/\Lambda,\omega)$.

3.1 The function *epsilon* for the 1-dimensional complex torus

Let

$$\Lambda = \{p+iq \mid \ p,q\in\mathbb{Z}\}$$

be the lattice in $\mathbb{C}$ generated by $(1,0)$ and $(0,1)$ and $\mathbb{C}/\Lambda$ be the 1-dimensional complex torus. Let $H(z,w) = z\bar w$ be the standard hermitian form on $\mathbb{C}$ and

$$\omega = \frac{i}{2}\partial\bar\partial|z|^2 = \frac{i}{2}dz\wedge d\bar z$$

the flat Kähler form on $\mathbb{C}/\Lambda$. A simple calculation shows that

$$\Im H(\lambda,\mu) = mq - pn,\ \forall\lambda = p+iq, \mu = m+in,$$

i.e. H is integral on the lattice. By the previous section there exists a holomorphic line bundle L whose global holomorphic sections can be identified with the holomorphic functions θ on $\mathbb{C}$ such that

$$\theta(z+\lambda) = A(\lambda,z)\theta(z) = e^{i\pi pq}e^{\pi z\bar\lambda+\frac{\pi}{2}|\lambda|^2}\theta(z),\ \forall\lambda = p+iq\in\Lambda,$$

where we choose

$$\chi(\lambda) = e^{i\pi pq},\ \forall\lambda = p+iq\in\Lambda$$

as a semicharacter of H.

More generally, given any natural number k let L^k be the k-th tensor power of L.

The global holomorphic sections of L^k, can be seen as the holomorphic functions θ on $\mathbb{C}$ satisfying

$$\theta(z+\lambda) = e^{ki\pi pq}e^{k\pi z\bar\lambda+\frac{k\pi}{2}|\lambda|^2}\theta(z),\ \forall\lambda = p+iq\in\Lambda, \tag{7}$$

and the hermitian structure h^k such that $\operatorname{curv}(L^k,h^k) = -2\pi ki\omega$ is given by

$$h^k(\theta(z),\theta(z)) = e^{-k\pi|z|^2}|\theta(z)|^2, \forall\theta\in H^0(L^k).$$

By the Riemann-Roch theorem $\mathcal{H}_{h^k}$ is k-dimensional. Given $j = 0,\dots,k-1$ define

$$\theta_j(z) = e^{k\frac{\pi}{2}z^2}\sum_{m\in\mathbb{Z}} e^{\frac{-\pi}{k}(km+j)^2+2\pi i(km+j)z}$$

It is not hard to see that the functions θ_j's satisfy the functional equation (7). Furthermore

Proposition 3.1. $\left\{(\frac{2}{k})^{\frac{1}{4}}\theta_0, \ldots, (\frac{2}{k})^{\frac{1}{4}}\theta_{k-1}\right\}$ *form a unitary basis for* $(\mathcal{H}_{h^k}, \langle \cdot\, , \,\cdot \rangle_{h^k})$.

Proof : For $a, b = 0, 1, \ldots, k-1$

$$\begin{aligned}&\langle \theta_a, \theta_b \rangle_{h^k} \\ &\quad = \sum_{m,p\in\mathbb{Z}} e^{\frac{-\pi}{k}((km+a)^2+(kp+b)^2)} \int_{\mathbb{C}/\Lambda} e^{-k\pi|z|^2} e^{\frac{k\pi}{2}(z^2+\bar{z}^2)} e^{2\pi i(km+a)z} e^{-2\pi i(km+a)\bar{z}} k\omega.\end{aligned}$$

If $z = x + iy$, the previous integral can be written as

$$\sum_{m,p\in\mathbb{Z}} e^{\frac{-\pi}{k}((km+a)^2+(kp+b)^2)} \int_0^1 \int_0^1 e^{-2k\pi y^2} e^{2\pi i(k(m-p)+(a-b))x} e^{-2\pi(k(m+p)+(a+b))y} k dx \wedge dy.$$

Integrating with respect to x we obtain

$$\int_0^1 e^{2\pi i(k(m-p)+(a-b))x} dx = \delta_{0k(m-p)+b-a} = \delta_{mp}\delta_{ab},$$

where the last equality follows from the fact that $b - a$ is divisible by k if and only if $b = a$. Thus,

$$\langle \theta_a, \theta_b \rangle_{h^k} = k\delta_{ab} \sum_{m\in\mathbb{Z}} e^{\frac{-\pi}{k}((km+a)^2+(km+b)^2)} \int_0^1 e^{-2k\pi y^2} e^{-4\pi(km+\frac{a+b}{2})y} dy.$$

Therefore the θ_j's form an orthogonal basis for $(\mathcal{H}_{h^k}, \langle \cdot\, , \,\cdot \rangle_{h^k})$. For $a = b = j$ one gets:

$$\begin{aligned}\|\theta_j\|^2_{h^k} &= k \int_0^1 e^{-2k\pi y^2} \sum_{m\in\mathbb{Z}} e^{\frac{-2\pi}{k}(km+j)^2} e^{-4\pi(km+j)y} dy \\ &= k \sum_{m\in\mathbb{Z}} \int_0^1 e^{-2k\pi(y+m+\frac{j}{k})^2} dy.\end{aligned}$$

By the change of variable $t = y + m + \frac{j}{k}$ one obtains:

$$\|\theta_j\|^2_{h^k} = k \int_{-\infty}^{+\infty} e^{-2k\pi t^2} dt = \sqrt{\frac{k}{2}}.$$

■

By (4) and 3.1, the function epsilon can be calculated as

$$\epsilon_{(L^k,h^k)}(z) = e^{-k\pi|z|^2} \sqrt{\frac{2}{k}} \sum_{j=0}^{k-1} |\theta_j(z)|^2.$$

Remark 3.2. *The previous calculation can be generalized to the case where* Λ *is a general lattice in* $\mathbb{C}$. *Similar calculations can be found in [2].*

4 Quantization of Riemann surfaces

Let Σ_g be a compact Riemann surface of genus $g \geq 2$. One can realize Σ_g as the quotient $\mathbb{D}/\Gamma$ of the unit disk $\mathbb{D} \subset \mathbb{C}$ under the fractional linear transformations of a Fuchsian subgroup Γ of

$$\mathrm{SU}(1,1) = \{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \mid \ |a|^2 - |b|^2 = 1\}.$$

Here the action of $\gamma = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \Gamma$ is given by $z \mapsto \gamma(z) = \frac{az+b}{\bar{b}z+\bar{a}}$. It is immediate to check that the Kähler form

$$\omega_{hyp} = \frac{i}{\pi} \frac{dz \wedge d\bar{z}}{(1 - z\bar{z})^2}$$

is invariant under fractional linear transformations, so it defines a Kähler form on Σ_g, denoted by the same symbol ω_{hyp}. Let L be the canonical bundle over Σ_g, i.e. the holomorphic line bundle whose global holomorphic sections are the holomorphic forms of type $(1,0)$ on Σ_g. Let $p : \mathbb{D} \to \mathbb{D}/\Gamma$ be the natural projection map. The line bundle $p^*(L)$ is holomorphically trivial and its global holomorphic sections are the form of type $(1,0)$ on $\mathbb{D}$, i.e. $f(z)dz$ where $f(z)$ is a holomorphic function on $\mathbb{D}$. Hence, the global holomorphic sections of L can be seen as the forms $s = fdz$ invariant by the action of Γ, i.e.

$$f(\gamma(z))d(\gamma(z)) = f(\gamma(z))\gamma'(z)dz = f(z)dz, \forall \gamma \in \Gamma, \tag{8}$$

where $\gamma'(z)$ denotes the derivative of $\gamma(z)$ with respect to z (if $\gamma(z) = \frac{az+b}{\bar{b}z+\bar{a}}$ then $\gamma'(z) = (\bar{b}z + \bar{a})^{-2}$). In other words if

$$\sigma : \mathbb{D} \to \mathbb{D} \times \mathbb{C} : z \to (z, 1)$$

is the section of the trivial bundle over $\mathbb{D}$, then the space of holomorphic sections of L can be identify with the space of all $s = f\sigma$, where f is a holomorphic function on $\mathbb{D}$ such that

$$f(\gamma(z)) = (\gamma'(z))^{-1} f(z).$$

More generally, given k a natural number, one can show that the global holomorphic sections of L^k can be seen as $s = f\sigma$, where f is holomorphic function on $\mathbb{D}$, such that

$$f(\gamma(z)) = (\gamma'(z))^{-k} f(z). \tag{9}$$

Given such a section $s = f\sigma$ define

$$h^k(s(z), s(z)) = (1 - |z|^2)^{2k} |f(z)|^2.$$

One can easily check that

$$(1 - |\gamma(z)|^2)^{2k} = |\gamma'(z)|^{2k} (1 - |z|^2)^{2k}, \tag{10}$$

so

$$h^k(s(\gamma(z)), s(\gamma(z))) = h^k(s(z), s(z)), \forall \gamma \in \Gamma.$$

Therefore h^k defines a hermitian structure on L^k. Moreover

$$\mathrm{curv}(L, h) = -2\partial\bar{\partial} \log(1 - |z|^2) = \frac{2dz \wedge d\bar{z}}{(1 - |z|^2)^2} = -2\pi i \omega_{hyp}, \tag{11}$$

which shows that the pair (L^k, h^k) is a quantization for $(\Sigma_g, k\omega_{hyp})$.

4.1 The function epsilon for the Riemann surfaces

Given a natural number k define a function on $\mathbb{D} \times \mathbb{D}$ by the formula

$$e^k(z,w) = \frac{2k-1}{2k} \sum_{\gamma \in \Gamma} (1-\gamma(z)\bar{w})^{-2k} (\gamma^{'}(z))^k. \tag{12}$$

Classical theorems going back to Poincare (see [8, pp. 101-104]) assert that the series (12) converges almost uniformly for all $z \in \mathbb{D}$. It is easily seen that for every $w \in \mathbb{D}$

$$e^k(\gamma(z), w) = (\gamma^{'}(z))^{-k} e^k(z,w), \ \forall \gamma \in \Gamma. \tag{13}$$

Hence $e^k_{\sigma(w)}(z) := e^k(z,w)\sigma(z)$ is a holomorphic section of L^k. Let U be a fundamental domain in $\mathbb{D}$ for the action of Γ. Given any $s = f\sigma$ a holomorphic section for L^k it follows by (9) and (13) that

$$\begin{aligned}
\langle s, e^k_{\sigma(w)} \rangle_{h^k} &= \int_{\Sigma_g} f(z) \overline{e^k(z,w)} (1-|z|^2)^{2k} k\omega_{hyp}(z) \\
&= \frac{2k-1}{2k} \sum_{\gamma \in \Gamma} \int_U f(z)(1-\overline{\gamma(z)}w)^{-2k} (\overline{\gamma^{'}(z)})^k (1-|z|^2)^{2k} k\omega_{hyp}(z) \\
&= \frac{2k-1}{2k} \sum_{\gamma \in \Gamma} \int_U f(\gamma(z))(1-\overline{\gamma(z)}w)^{-2k} (1-|\gamma(z)|^2)^{2k} k\omega_{hyp}(z) \\
&= \int_{\mathbb{D}} f(z)(1-\bar{z}w)^{-2k}(1-|z|^2)^{2k} k\omega_{hyp}(z) = f(w),
\end{aligned}$$

where the last equality follows by a direct calculation (cfr. [5, p10]). Hence

$$\langle s, e^k_{\sigma(w)} \rangle_{h^k} \sigma(w) = f(w)\sigma(w),$$

i.e. $e^k_{\sigma(w)}$ is the coherent state relative to $\sigma(w)$. By the very definition of coherent states one has $\|e^k_{\sigma(z)}\|^2_{h^k} \sigma(z) = e^k(z,z)\sigma(z)$ and by (2)

$$\epsilon_{(L^k,h^k)}(z) = \|e^k_{\sigma(z)}\|^2_{h^k} h^k(\sigma(z),\sigma(z)) = \frac{2k-1}{2k}(1-|z|^2)^{2k} \sum_{\gamma \in \Gamma} (1-\gamma(z)\bar{z})^{-2k} (\gamma^{'}(z))^k.$$

References

[1] F.A. Berezin, *Quantization*, Math. USSR Izvestija 8 (1974), pp. 1109-1165.

[2] M. Bordemann, J. Hoppe, P. Schaller and M. Schlichenmaier, *$gl(\infty)$ and Geometric Quantization*, Comm. Math. Phys. 138 (1991), pp. 209-244.

[3] M. Cahen, S. Gutt, J. H. Rawnsley, *Quantization of Kähler manifolds I: Geometric interpretation of Berezin's quantization*, JGP 7 (1990), pp. 45-62.

[4] M. Cahen, S. Gutt, J. H. Rawnsley, *Quantization of Kähler manifolds II*, Trans. Amer. Math. Soc. 337 (1993), pp. 73-98.

[5] M. Cahen, S. Gutt, J. H. Rawnsley, *Quantization of Kähler manifolds III*, Lett. Math. Phys. 30 (1994), pp. 291-305.

[6] M. Cahen, S. Gutt, J. H. Rawnsley, *Quantization of Kähler manifolds IV*, Lett. Math. Phys. 34 (1995), pp. 159-168.

[7] B. Kostant, *Quantization and Unitary Representation*, Lecture notes in Mathematics 170.

[8] I. Kra, *Automorphic Functions and Klenian Groups*, Benjamin, Reading, Mass., (1972).

[9] H. Lange and C. Birkenhake, *Complex Abelian Varieties*, Springer-Verlag (1992).

[10] J. H. Rawnsley, *Coherent states and Kähler manifolds*, The Quarterly Journal of Mathematics (1977), pp. 403-415.

[11] M. Takeuchi, *Homogeneous Kähler Manifolds in Complex Projective Space*, Japan J. Math. vol. 4 (1978), pp. 171-219.

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