Reduction of Hopf bifurcation problems with symmetries

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Abstract

We consider a family of Γ -equivariant differential equations and look for Hopf bifurcation. We reformulate this problem in the usual way as an operator equation and perform a Liapunov-Schmidt reduction determined by the "semisimple" part of its linearization. In a second reduction step we construct a bifurcation equation which furtunately can be formulated directly by means of the original problem.

1 Introduction

We consider a family of autonomous differential equations

$$\dot{x} = f(x, \lambda) \tag{1}$$

where $x \in \mathbf{R}^n, \lambda \in \mathbf{R}^m, f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ is of class C^∞ and $f(0, \lambda) = 0$, for all λ. Furthermore let the system (1) be Γ-equivariant, that is: there exists a compact group $\Gamma \subset O(n)$, such that

$$f(\gamma x, \lambda) = \gamma f(x, \lambda)\,,\ \forall \gamma \in \Gamma \text{ and } \forall (x, \lambda) \in \mathbf{R}^n \times \mathbf{R}^m.$$

We want to study Hopf bifurcation in the given family under the following condition: $A := D_1 f(0, 0)$ has eigenvalues μi for some integers $\mu \in \mathrm{Z}$ while 0 is not an eigenvalue

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of A . Without loss of generality we may assume that f is in normal form of order k with respect to the semisimple part S of A, i.e.,

$$f(x,\lambda) = f_{NF}(x,\lambda) + o(\| x \|^k)$$

where

$$e^{tS} f_{NF}(x,\lambda) = f_{NF}(e^{tS}x,\lambda)\,,\, t \in \mathbf{R}$$

f_{NF} is Γ -equivariant, too. This will become clear by theorem 3.1.

We formulate the bifurcation problem as an operator equation in spaces of 2π-periodic functions, and perform a reduction, via some Liapunov-Schmidt procedure, to the kernel of the "semisimple" part of the linearized operator equation. Though we do not, in this way, aspire to the strongest Liapunov-Schmidt reduction, we shall gain some advantage in setting up the bifurcation equation. Namely, we can construct a bifurcation equation in a way avoiding any regard to the multiplicity or resonance cases of the eigenvalues.

These considerations are based on a joint work by A. Vanderbauwhede and J.-C. van der Meer [11] in the context of Hamiltonian systems. In contrast, our investigations do not exploit any symplectic structure. But we show that the Γ-equivariance (like the symplectic structure in [11]) is preserved by all reduction steps generating the bifurcation equation.

Meanwhile the results of this paper have been extended to other classes of autonomous systems: on the one side conservative systems and on the other side reversible systems - cf. [4]. The general result obtained in [4] has been used to study Hopf bifurcation at k-fold resonances in conservative and reversible systems - cf. [5] and [6]. A further extension to equivariant reversible systems is described in [7].

2 Some linear algebra

In this section we put together some facts about the splitting of a linear operator on $\mathbf{R}^n$ in its semisimple and nilpotent part, cf. [2] and [3].

Lemma 2.1. *For any linear operator A on $\mathbf{R}^n$ there are unique linear operators S and N such that $A = S + N$ and $SN = NS$, where S and N are semisimple and nilpotent, respectively.*

S is called the semisimple part of A and N the nilpotent one.
In the case of equivariant operators we can add the following:

Lemma 2.2. *Let $A \in \mathrm{L}(\mathbf{R}^n)$ be equivariant with respect to a compact group $\Gamma \subset O(n)$, $A = S + N$ as above, then S and N are equivariant, too.*

PROOF: For every $\gamma \in \Gamma$ $\gamma S \gamma^{-1}$ is semisimple and $\gamma N \gamma^{-1}$ is nilpotent. Obviously $A = \gamma A \gamma^{-1} = \gamma S \gamma^{-1} + \gamma N \gamma^{-1}$ and this splitting is unique.

∎

Lemma 2.3. *For semisimple operators S in $\mathbf{R}^n$ there holds:*
(i) $\mathbf{R}^n = kerS \oplus imS$, and
(ii) $S : imS \to imS$ is bijective.

Lemma 2.4. *Let $A = S + N$ as above, then*
(i) $kerA = kerS \cap kerN$,
(ii) $imA = imS \oplus (kerS \cap kerN)$,
(iii) A, restricted to imS, is injective.

3 An equivariant normal form theorem

The basic idea of normal form theory is to use coordinate transformations to get the analytic expression for the vectorfields $f(\cdot, \lambda)$ as simple as possible. It is usual to define classes of normal forms depending on $A = D_1 f(0, 0)$.
The main object of this section is to generalize results obtained in [9] to normal forms with respect to the semisimple part of A and to the equivariant setting.
We adopt the notation of [9].

Theorem 3.1. *Let f be the family of Γ-equivariant vector fields defined in the introduction. Then, for each $k \geq 2$, there is a neighborhood Ω_k of the origin in $\mathbf{R}^m$ and a mapping $\Phi \in C^\infty(\mathbf{R}^n \times \Omega_k, \mathbf{R}^n)$ such that for each $\lambda \in \Omega_k$ the following holds:*

(i) $\Phi_\lambda := \Phi(\cdot, \lambda)$ *is a Γ-equivariant diffeomorphism on $\mathbf{R}^n$, with $\Phi_\lambda(0) = 0$,*

(ii) $g_\lambda(\cdot) := \Phi_\lambda^* f(\cdot, \lambda)$ *is Γ-equivariant,*

(iii) $Dg_0(0) = A := D_1 f(0, 0)$,

(iv) $T_k g_\lambda(e^{tS}x) = e^{tS} T_k g_\lambda(x)$ *where $T_k g_\lambda$ is the k-th Taylor polynomial of g_λ at $x = 0$.*

Again, S is the semisimple part of A, Φ_λ^* is the pull back under Φ_λ.

In fact the whole statement is an equivariant version of theorem 2.4 in [9], we can follow the general idea of the proof given there.

Outline of the proof:
(ii) is a straightforward consequence of (i).

$$\begin{array}{rcl} \text{Let } adA : C^\infty(\mathbf{R}^n, \mathbf{R}^n) & \to & C^\infty(\mathbf{R}^n, \mathbf{R}^n) \\ \Phi & \mapsto & A\Phi(\cdot) - D\Phi(\cdot)A(\cdot) \end{array}$$

and $ad_l A := adA\,|_{H_l(\mathbf{R}^n)}$, where $H_l(\mathbf{R}^n)$ is the linear space of l-homogeneous polynomials from $\mathbf{R}^n$ to $\mathbf{R}^n$. Let π_l be a projector on $H_l(\mathbf{R}^n)$ with $ker\,\pi_l = im\,ad_l A$.

If $A = S + N$ is the unique splitting of A as above, then $ad_l A = ad_l S + ad_l N$ is such a splitting as well. As a consequence of lemmas 2.3 and 2.4 we get $H_l(\mathbf{R}^n) = im\,ad_l S \oplus ker\,ad_l S$ and $im\,ad_l A = im\,ad_l S \oplus (ker\,ad_l S \cap im\,ad_l N)$. Hence $im\,\pi_l \subset ker\,ad_l S$ by choosing an appropriate π_l. Moreover, since $\gamma\,ad_l S = ad_l S\,\gamma$, $\gamma\,ad_l N = ad_l N\,\gamma$ and Γ is compact, we can choose π_l such that $\gamma\,\pi_l = \pi_l\,\gamma$. Now we can explain the procedure in three steps.
First step:
We construct linear transformations Ψ_λ, (λ close to 0) such that
(a) $\tilde{f}_\lambda = \Psi_\lambda^* f_\lambda$ is Γ-equivariant,

(b) $D\tilde{f}_0 = A$ and $D\tilde{f}_\lambda - A \in im\,\pi_1$.
In order to do this we define $\mathrm{L}(\mathbf{R}^n)_\Gamma := \{B \in \mathrm{L}(\mathbf{R}^n) : \gamma\, B = B\,\gamma\,,$
$\forall\,\gamma \in \Gamma\}$ and $(im\,ad_1 A)_\Gamma := \{B \in im\,ad_1 A : \gamma\, B = B\,\gamma\,, \forall\,\gamma \in \Gamma\}$.
Then we consider

$$\begin{array}{rcl} F : \mathrm{L}(\mathbf{R}^n)_\Gamma \times \mathbf{R}^m & \to & (im\,ad_1 A)_\Gamma \\ (B, \lambda) & \mapsto & (id - \pi_1)(B^{-1} Df_\lambda(0) B - A). \end{array}$$

By the implicit function theorem the equation $F(B, \lambda) = 0$ can be solved for $B = \tilde{B}(\lambda)$, near the point $(B, \lambda) = (id, 0)$. Then $\Psi_\lambda := \tilde{B}(\lambda)$ are possible transformations.
Second step:
We can construct Φ_λ such that $Dg_\lambda - A \in im\,\pi_1$ and $D^l g_\lambda \in im\,\pi_l$ (recall $g_\lambda(\cdot) = \Phi_\lambda^* f(\cdot, \lambda))$, whence $T_l g_\lambda(e^{tS}x) = e^{tS} T_l g_\lambda(x)$, for $1 \leq l \leq k$. For more details see in [9].
Third step:
We have to ensure, by choise of Φ_λ, the Γ-equivariance of g_λ. This can be realized by means of Γ- equivariant Φ_λ among those constructed in the second step.

In [9] there is shown that the l-homogeneous part $\tilde{T}_l \Phi_\lambda$ of $T_l \Phi_\lambda$ can be obtained as a solution of the equation (in X):

$$\begin{aligned} ad_l A.X = -(id - \pi_l)\tilde{T}_l[(T_l \tilde{f}_\lambda - A_\lambda) \circ T_{l-1}\Phi_\lambda - \\ -D(T_{l-1}\Phi_\lambda - id)(\cdot)(T_{l-1} g_\lambda - A_\lambda)(\cdot)], \end{aligned} \tag{2}$$

$A_\lambda := D\tilde{f}_\lambda(0)$. Let us denote the right hand side of (2) by Y. If $T_{l-1}\Phi_\lambda$ is Γ-equivariant, then $T_{l-1} g_\lambda$ is Γ-equivariant, too. In this case we have $\gamma Y = Y\gamma$ for all $\gamma \in \Gamma$. Hence for any solution X of (2) $\int_\Gamma \gamma X \gamma^{-1} d\mu$, where μ is a normalized Haar measure on Γ, is a Γ-equivariant solution of (2). Using $T_1\Phi_\lambda = D\Phi_\lambda(0) = id$ we see by induction that $T_k \Phi_\lambda$ is Γ-equivariant. Then, see [9], there is a diffeomorphism Φ_λ, which has the constructed $T_k\Phi_\lambda$ as its k-th Taylor polynomial. Using a Γ-invariant cut function (in the construction of Φ_λ given in [9]) Φ_λ will be Γ- equivariant.

$\blacksquare$

Remark For single vector fields it is shown in [8] that there exists a normal form with respect to the semisimple part of the linearization at the equilibrium point. In [1] an equivariant version of a normal form theorem is presented for single vector fields, too.

4 The Liapunov-Schmidt reduction

Let us formulate the bifurcation problem as an operator equation in standard way: The original problem of finding all small nearly 2π-periodic solutions of (1) means to find all small 2π-periodic solutions of $\sigma\dot{x} = f(x,\lambda)$ for (σ,λ) close to (1,0). This is equivalent to solve the equation

$$F(x,\sigma,\lambda) = 0$$

for (x,σ,λ) close to (0,1,0) with

$$\begin{array}{rcl} F : C^1_{2\pi} \times \mathbf{R}^1 \times \mathbf{R}^m & \to & C^0_{2\pi} \\ (x,\sigma,\lambda) & \mapsto & -\sigma\dot{x} + f(x,\lambda). \end{array}$$

Pointwise, $F(x,\sigma,\lambda)$ is defined by

$$F(x,\sigma,\lambda)(t) := -\sigma\dot{x}(t) + f(x(t),\lambda).$$

The spaces $C^0_{2\pi}$ and $C^1_{2\pi}$ are endowed with the usual norm. F is smooth and there holds $F(0,\sigma,\lambda) \equiv 0$.
It is known (see in [10]) that F is $\Gamma \times S^1$-equivariant, that is

$$\tilde{T}_\gamma T_\phi F(x,\sigma,\lambda) = F(\tilde{T}_\gamma T_\phi x,\sigma,\lambda),$$

where $\tilde{T}_\gamma$ and T_ϕ for all $\gamma \in \Gamma$ and $\phi \in S^1$ are defined by

$$\begin{array}{rcl} \tilde{T}_\gamma : C^0_{2\pi} & \to & C^0_{2\pi} \\ x & \mapsto & x_\gamma, \qquad x_\gamma(t) := \gamma x(t) \end{array}$$

and

$$\begin{array}{rcl} T_\phi : C^0_{2\pi} & \to & C^0_{2\pi} \\ x & \mapsto & x^\phi, \qquad x^\phi(t) := x(t+\phi). \end{array}$$

Let again $D_1 f(0,0) =: A = S + N$, and $L_A := D_1 F(0,1,0)$. We split $L_A = L_S + L_N$, where $(L_S x)(t) := -\dot{x}(t) + Sx(t)$ and $(L_N x)(t) := Nx(t)$. We call L_S the "semisimple" part of L_A.

Lemma 4.1. *L_S is a Fredholm operator with index zero, and*

$$C^0_{2\pi} = kerL_S \oplus imL_S, \quad C^1_{2\pi} = kerL_S \oplus (imL_S \cap C^1_{2\pi}). \tag{3}$$

Proof: We know from classical Floquet theory that L_S is a Fredholm operator with index zero. Using any inner product $\langle \cdot, \cdot \rangle$ on $\mathbf{R}^n$ and denoting by S^T the adjoint of S the same theory shows that

$$kerL_S = \{z \in C^1_{2\pi} : z(t) = e^{tS} z_0,\ z_0 \in ker(e^{2\pi S} - id)\}$$

while

$$imL_S = \{z \in C^0_{2\pi} : \int_0^{2\pi} \langle z(t), \tilde{z}(t) \rangle dt = 0,\ \forall \tilde{z} \in kerL_S{}^*\},$$

where ${L_S}^*$ is the formal adjoint of L_S, defined by $({L_S}^* x)(t) := \dot{x}(t) + S^T x(t)$. So

$$ker {L_S}^* = \{\tilde{z} \in C^1_{2\pi} : \tilde{z}(t) = e^{-tS^T} \tilde{z}_0,\ \tilde{z}_0 \in ker(e^{2\pi S^T} - id)\}$$

Now suppose that $z \in ker\, L_S \cap im\, L_S$; then $z(t) = e^{tS} z_o$, with $z_o \in ker\,(e^{2\pi S} - id)$ such that

$$\frac{1}{2\pi} \int_0^{2\pi} \langle e^{tS} z_o, e^{-tS^T} \tilde{z}_o \rangle dt = \langle z_o, \tilde{z}_o \rangle = o$$

for all $\tilde{z}_o \in ker\,(e^{2\pi S^T} - id)$. This proves that $z_o \in ker\,(e^{2\pi S} - id) \cap im\,(e^{2\pi S} - id)$, and since S is semisimple we conclude that $z_o = 0$ and hence also $z(t) \equiv 0$. This proves (3).

■

Moreover, we can prove

Lemma 4.2. *L_A is an isomorphism from $imL_S \cap C^1_{2\pi}$ onto imL_S .*

PROOF: L_S is an isomorphism from $imL_S \cap C^1_{2\pi}$ onto imL_S, and therefore the result follows from the fact that $L_A = L_S + L_N$, with L_N nilpotent and commuting with L_S.

■

A straightforward calculation confirms

$$\begin{array}{lcl} \tilde{T}_\gamma L_S = L_S \tilde{T}_\gamma & , & \tilde{T}_\gamma L_N = L_N \tilde{T}_\gamma, \\ T_\phi L_S = L_S T_\phi & , & T_\phi L_N = L_N T_\phi. \end{array} \tag{4}$$

Let P_0 and P_1 be projectors on $C^0_{2\pi}$ and $C^1_{2\pi}$, respectively, defining the splittings (3), such that $imP_0 = imP_1 = kerL_S$. From (4) we get

$$P_i \tilde{T}_\gamma = \tilde{T}_\gamma P_i \text{ and } P_i T_\phi = T_\phi P_i\ ,\ i \in \{0, 1\} \tag{5}$$

Now we have got all the ingredients nessecary for the application of a kind of Liapunov-Schmidt reduction:

$$F(x, \sigma, \lambda) = 0$$

is equivalent to the system

$$\begin{aligned} (id - P_0) F(w + v, \sigma, \lambda) &= 0 \\ P_0 F(w + v, \sigma, \lambda) &= 0, \end{aligned}$$

where $w := P_1 x$ and $v := (id - P_1)x$. Because of lemma 4.2 and the implicit function theorem the first equation can be solved for $v = \tilde{v}(w, \sigma, \lambda)$. Then the second equation yields the bifurcation equation

$$\hat{G}(w, \sigma, \lambda) := P_0 F(w + \tilde{v}(w, \sigma, \lambda), \sigma, \lambda) = 0 \tag{6}$$

with (w, σ, λ) close to (0,1,0). Because of (5) $\hat{G}$ is $\Gamma \times S^1$-equivariant (cf. [10]).

Let $U := ker(e^{2\pi S} - id)$. Then $\zeta : U \to kerL_S \subset C^0_{2\pi}$ defined by $(\zeta u)(t) := e^{tS} u$ is an isomorphism. Therefore the bifurcation equation (6) is equivalent to:

$$G(u, \sigma, \lambda) := \zeta^{-1} P_0 F(\zeta(u) + \tilde{v}(\zeta(u), \sigma, \lambda), \sigma, \lambda) = 0. \tag{7}$$

Using the definition of U and the Γ-equivariance of S one can easily verify that

$$(\gamma,\phi,u)\in\Gamma\times S^1\times U\mapsto(\gamma,\phi)\cdot u:=\gamma e^{\phi S}u$$

defines a linear $\Gamma\times S^1$- action on U; from $\tilde{T}_\gamma T_\phi\zeta(u)=\zeta((\gamma,\phi)\cdot u)$ and the $\Gamma\times S^1$-equivariance of $\tilde{G}$ we get then the $\Gamma\times S^1$- equivariance of G:

$$(\gamma,\phi)\cdot G(u,\sigma,\lambda)=G((\gamma,\phi)\cdot u,\sigma,\lambda)\,,\ \forall(\gamma,\phi)\in\Gamma\times S^1.$$

Lemma 4.3. *If $z\in imL_S\cap C^1_{2\pi}$, then $\dot{z}\in imL_S$.*

PROOF: Let $x\in C^1_{2\pi}$ be such that $L_Sx=z$, i.e. $-\dot{x}(t)+Sx(t)=z(t)$; since z is of class C^1 it follows that x is of class C^2, and differentiation gives $L_S\dot{x}=\dot{z}$. We can conclude that $\dot{z}\in imL_S$.

■

Using lemma 4.3 we can simplify the expression of G:

$$\begin{aligned}G(u,\sigma,\lambda)&=\zeta^{-1}P_0F(\zeta(u)+\tilde{v}(\zeta(u),\sigma,\lambda),\sigma,\lambda)\\&=\zeta^{-1}P_0[-\sigma\frac{d}{dt}(\zeta(u))-\sigma\frac{d}{dt}(\tilde{v}(\ldots))+f(\zeta(u)+\tilde{v}(\ldots),\lambda)\\&=-\sigma Su+\zeta^{-1}P_0f(\zeta(u)+\tilde{v}(\zeta(u),\sigma,\lambda),\lambda)\end{aligned}\tag{8}$$

Lemma 4.4. *We have $v(u,\sigma,\lambda):=\tilde{v}(\zeta(u),\sigma,\lambda)=o(\|\,u\,\|^k)$, as $u\to 0$.*

PROOF: $u\in U$ implies $f_{NF}(u,\lambda)\in U$ and hence

$$f_{NF}(\zeta(u),\lambda)=f_{NF}(e^{tS}u,\lambda)=e^{tS}f_{NF}(u,\lambda)\in kerL_S.\tag{9}$$

Let $H(u,v,\sigma,\lambda):=(id-P_0)F(\zeta(u)+v,\sigma,\lambda)$. Then (9) yields $H(u,0,\sigma,\lambda)=(id-P_0)f_R(\zeta(u),\lambda)=o(\|\,u\,\|^k)$, where $f_R:=f-f_{NF}$. The statement then follows by calculating the derivatives $D_1^jv(0,1,0)$ $(1\leq j\leq k)$, using the identity $H(u,v(u,\sigma,\lambda),\sigma,\lambda)=0$.

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5 A further reduction

Let $\ll\cdot,\cdot\gg$ be a $\Gamma\times S^1$-invariant inner product on U and Σ the corresponding unit sphere. (Such an inner product exists since $\Gamma\times S^1$ is compact - cf.[1].) Clearly Σ is invariant under the action of $\Gamma\times S^1$. Since $G(o,\sigma,\lambda)\equiv 0$ we see that if $(u,\sigma,\lambda)=(r\theta,\sigma,\lambda)$ is a solution of (7) close to $(0,1,0)$ and with $r\neq 0$, then

$$\tilde{g}(r,\theta,\sigma,\lambda)=0\,,\tag{10}$$

where the mapping $\tilde{g}$ is defined by

$$\ll S\theta,G(r\theta,\sigma,\lambda)\gg=r\tilde{g}(r,\theta,\sigma\lambda)\,.\tag{11}$$

Lemma 5.1. *$\tilde{g}$ has following properties:*

(i) $\tilde{g} : \mathbf{R} \times \Sigma \times \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}$ *is well-defined and smooth in a neighborhood of* $\{0\} \times \Sigma \times \{1\} \times \{0\}$;

(ii) $\tilde{g}(0, \theta, \sigma, 0) = -\sigma \ll S\theta, S\theta \gg + \ll S\theta, A\theta \gg$,

(iii) $\tilde{g}$ *is* $\Gamma \times S^1$*-invariant, i.e.,* $\tilde{g}(r, (\gamma\phi) \cdot \theta, \sigma, \lambda) = \tilde{g}(r, \theta, \sigma, \lambda)$.

Proof: (i) follows immetiately from (11), which also gives $\tilde{g}(0, \theta, \sigma, 0) = \ll S\theta, D_1 G(0, \sigma, 0)\theta \gg = \ll S\theta, -\sigma S\theta + A\theta \gg$. This proves (ii). Finally, (iii) follows from the $\Gamma \times S^1$-invariance of $\ll \cdot, \cdot \gg$.

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Let now $\Sigma_o := \{\theta \in \Sigma : \tilde{g}(0, \theta, 1, 0) = 0\} = \{\theta \in \Sigma : \ll S\theta, N\theta \gg = 0\}$ then Σ_o is a compact, $\Gamma \times S^1$-invariant subset of Σ, which is nonempty since it contains $ker\, N \cap \Sigma$. From lemma 5.1 (ii) we see that for each $\theta \in \Sigma_o$ we have $\tilde{g}(0, \theta, 1, 0) = 0$ and $D_3 \tilde{g}(0, \theta, 1, 0) = - \ll S\theta, S\theta \gg \neq 0$. It follows then from the implicit function theorem that for $(r, \theta, \sigma, \lambda)$ near $\{0\} \times \Sigma_o \times \{1\} \times \{0\}$ the equation (10) has a unique solution $\sigma = \sigma^*(r, \theta, \lambda)$; more precisely, there exist $r_o > 0$, a $\Gamma \times S^1$-invariant neighborhood V of Σ_o in Σ, a neighborhood ω of $\lambda = 0$ in $\mathbf{R}^m$, and a smooth $\Gamma \times S^1$-invariant function $\sigma^* : (-r_o, r_o) \times V \times \omega \to \mathbf{R}$ with $\sigma^*(0, \theta, 0) = 1$ for $\theta \in \Sigma_o$ and such that for each $(r, \theta, \lambda) \in (-r_o, r_o) \times V \times \omega$ the equation (10) has the unique solution $\sigma = \sigma^*(r, \theta, \lambda)$ near 1.
We would like $\sigma^*(r, \theta, \lambda)$ to be defined for all $\theta \in \Sigma$, and therefore we make the following construction. Let V_1 and V_2 be two open $\Gamma \times S^1$-invariant neighborhoods of Σ_o in Σ, such that $\Sigma_o \subset V_1 \subset \bar{V_1} \subset V_2 \subset \bar{V_2} \subset V$, and let $\chi : \Sigma \to \mathbf{R}$ be a smooth $\Gamma \times S^1$-invariant cut-off function with the following properties: $0 \leq \chi(\theta) \leq 1$ for all $\theta \in \Sigma$, $\chi(\theta) = 1$ for $\theta \in V_1$, $\chi(\theta) = 0$ for $\theta \in \Sigma \setminus V_2$; the construction of such cut-off function is standard. Then we define $\tilde{\sigma}^* : (-r_o, r_o) \times \Sigma \times \omega \to \mathbf{R}$ by

$$\tilde{\sigma}^*(r, \theta, \lambda) := 1 - \chi(\theta)(1 - \sigma^*(r, \theta, \lambda));$$

clearly $\tilde{\sigma}^*$ is $\Gamma \times S^1$-invariant, and $\tilde{\sigma}^*(r, \theta, \lambda) = \sigma^*(r, \theta, \lambda)$ for $(r, \theta, \lambda) \in (-r_o, r_o) \times V_1 \times \omega$.
Next we use $\tilde{\sigma}^*$ to define $\tilde{\sigma} : U \setminus \{0\} \times \mathbf{R}^m \to \mathbf{R}$ by

$$\tilde{\sigma}(u, \lambda) := \tilde{\sigma}^*(\|u\|, \frac{u}{\|u\|}, \lambda), \quad \|u\|^2 = \ll u, u \gg;$$

$\tilde{\sigma}$ is smooth, $\Gamma \times S^1$-invariant, and $\tilde{\sigma}(u, \lambda)$ stays bounded (near 1) as $u \to 0$. We also define $\tilde{G} : U \times \mathbf{R}^m \to U$, $\hat{x} : U \times \mathbf{R}^m \to C^1_{2\pi}$ and $\hat{f} : U \times \mathbf{R}^m \to U$ by

$$\tilde{G}(u, \lambda) := \begin{cases} G(u, \tilde{\sigma}(u, \lambda), \lambda) & \text{if} \quad u \neq 0, \\ 0 & \text{if} \quad u = 0, \end{cases} \tag{12}$$

$$\hat{x}(u, \lambda) := \begin{cases} \zeta(u) + v(u, \tilde{\sigma}(u, \lambda), \lambda) & \text{if} \quad u \neq 0, \\ 0 & \text{if} \quad u = 0, \end{cases} \tag{13}$$

$$\hat{f}(u, \lambda) := \zeta^{-1} P_o f(\hat{x}(u, \lambda), \lambda). \tag{14}$$

These mappings are continuous, smooth for $u \neq 0$, and $\Gamma \times S^1$- equivariant. Moreover, one can easely verify that

$$\ll \tilde{G}(u,\lambda), Su \gg = 0 \tag{15}$$

and

$$\tilde{G}(u,\lambda) = -\tilde{\sigma}\,(u,\lambda)Su + \hat{f}(u,\lambda). \tag{16}$$

Finally, we also have the following.

Lemma 5.2. $\hat{f}(u,\lambda) = f_{NF}(u,\lambda) + o(\|u\|^k)$ *as* $u \to 0$.

PROOF: It follows from lemma 4.4 and the fact that $\tilde{\sigma}\,(u,\lambda))$ stays bounded as $u \to 0$ that $\hat{x}(u,\lambda) = \zeta(u) + o(\|u\|^k)$; we find then from (14) that

$$\begin{aligned}
\hat{f}(u,\lambda) &= \zeta^{-1}P_o f_{NF}(\hat{x}(u,\lambda),\lambda) + \zeta^{-1}P_o f_R(\hat{x}(u,\lambda),\lambda) \\
&= \zeta^{-1}P_o f_{NF}(\zeta(u),\lambda) + o(\|u\|^k) \\
&= f_{NF}(u,\lambda) + o(\|u\|^k).
\end{aligned}$$

■

It follows from lemma 5.2 that $\hat{f}$ and $\hat{x}$ are of class C^k.
We can now formulate and prove our main result.

Theorem 5.1. *Let* $(x,\sigma,\lambda) \in C^1_{2\pi} \times \mathbf{R} \times \mathbf{R}^m$ *be close to* $(0,1,0)$ *and with* $x \neq 0$. *Then we have* $F(x,\sigma,\lambda) = 0$ *if and only if* $x = \hat{x}(u,\lambda)$ *for some sufficiently small* $u \neq 0$ *satisfying*

$$\sigma Su = \hat{f}(u,\lambda). \tag{17}$$

PROOF: If $F(x,\sigma,\lambda) = 0$ then the Liapunov-Schmidt reduction of section 4 shows that $x = \zeta(u) + v(u,\sigma,\lambda)$ for some (u,σ,λ) satisfying (7) implies (10) (with $u = r\theta$), and since (r,σ,λ) is close to $(0,1,0)$ we find from (10) that $\sigma = \sigma^*(r,\theta,\lambda) = \tilde{\sigma}\,(u,\lambda)$. From this we see that $x = \hat{x}(u,\lambda)$ and $\tilde{G}(u,\lambda) = 0$; together with $\sigma = \tilde{\sigma}\,(u,\lambda)$ this last equation also implies (17).
Suppose conversely that (17) holds, with (u,σ,λ) close to $(0,1,0)$ and $u \neq 0$. Taking the inner product with Su, and using (15) and (16) we find

$$\sigma \ll Su, Su \gg = \ll Su, \hat{f}(u,\lambda) \gg = \tilde{\sigma}\,(u,\lambda) \ll Su, Su \gg,$$

from which we conclude that $\sigma = \tilde{\sigma}\,(u,\lambda)$ (since $Su \neq 0$). Together with (17) and (16) this gives $\tilde{G}(u,\lambda) = 0$ and $G(u,\sigma,\lambda) = 0$. Again by the Liapunov-Schmidt reduction it follows that $(x,\sigma,\lambda) := (\zeta(u) + v(u,\sigma,\lambda),\sigma,\lambda) = (\hat{x}(u,\lambda),\sigma,\lambda)$ solves $F(x,\sigma,\lambda) = 0$.

■

6 Discussion

It follows immediately from theorem 5.1 that we can consider the equation (17) as the bifurcation equation of the given Hopf bifurcation problem; indeed, (17) determines in a unique way all the small periodic solutions we were looking for. Moreover, it follows from lemma 5.2 that we can obtain the Taylor expansion of $\hat{f}(u, \lambda)$ up to any order k by calculating the normal form of the original vector field $f(u, \lambda)$ up to that order. This gives a direct link between the bifurcation equation (17) and original equation (1). Actually, it is possible to reformulate in terms of a reduced differential equation as follows. Consider the (Hopf bifurcation) problem of finding, for all (T, λ) near $(2\pi, 0)$, all small T-periodic solutions of the equation

$$\dot{u} = \hat{f}(u, \lambda). \tag{18}$$

This problem is similar to the original one treated in this paper, except that now the the vector field $\hat{f}(u, \lambda)$ on U is $\Gamma \times S^1$-equivariant, that is, $\hat{f}$ satisfies all the requirements which we imposed on the normal form f_{NF} in the foregoing theory. Therefore, if we work out this theory for (18) we will arrive at exactly the same bifurcation equation (17) as for our original problem. We conclude that for (T, λ) near $(2\pi, 0)$ there exists a 1-1 relation between the small T-periodic solutions of (1) and the small T-periodic solutions of (18). Moreover, these periodic solutions all have the form $u(t) = \zeta(u)(\sigma t) = e^{\sigma t S}S$, for some $\sigma \in \mathbf{R}$ near 1 and $u \in U$; substituting this form of $u(t)$ into (18) gives directly the bifurcation equation (17). We call (18) the reduced equation; it lives on the subspace U, is $\Gamma \times S^1$-equivariant and can be approximated by restricting the normal form f_{NF} of f to $U \times \mathbf{R}^m$.
Therefore, if one can show that the operator $-\sigma S u + f_{NF}(u, \lambda)$ is k - determinated (within the class of $\Gamma \times S^1$-equivariant operators), then we can replace the bifurcation equation (17) by the approximate bifurcation equation

$$-\sigma S u + f_{NF}(u, \lambda) = 0,$$

which is directly related to the original equation (1).

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