Another curvature in synthetic differential geometry

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Abstract

Although the second Bianchi identity has been discussed in somewhat nonstandard literature of synthetic differential geometry (cf. Lavendhomme [1991] and Kock [1996]), it still remains to be couched and established within the standard realm of synthetic discourses. The principal objective of this paper is to show that a slightly modified version of curvature form enjoys the identity. Our discussion will be carried out within the appropriate framework of vector bundles.

0 Introduction

Although Kock [1996] and Lavendhomme [1991] have established the second Bianchi identity in their own synthetic discourses, they have approached the identity somewhat nonstandardly. The identity still remains to be established on the main street of synthetic differential geometry. By our locution "the main street of synthetic differential geometry" we have in mind Lavendhomme's [1996] celebrated textbook on synthetic differential geometry up to Chapter 5 (but not later chapters) as its quintessence. This locution is not intended at all to lessen their somewhat nonstandard approaches to synthetic differential geometry, let alone to insist that their approaches are of little geometric interest. We would like to contend exactly that any story of curvature form could not be finished without the second Bianchi identity even touched.

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While we do not commit ourselves to founding synthetic differential geometry solely upon neighborhood relations and we do continue to account tangent vectors its basic ingredients, we gladly acknowledge our great indebtedness to Kock's [1996] inspiring paper, in which he deduced the classical second Bianchi identity from a combinatorial one to be traced back to the so-called homotopy addition theorem (cf. Whitehead [1978, Chap. IV, §6]). In Section 5 we will also elicit the classical second Bianchi identity from a combinatorial one, yet our combinatorial variant is not simplicial but cubical. The identity will be established within the framework of linear connections on vector bundles (cf. Moerdijk and Reyes [1991, Chap. 5, Definitions 3.1 and 3.4.10]). Since we suspect that the curvature form of standard synthetic differential geometry (cf. Lavendhomme [1996, §5.3, Definition 5]) is not expected to satisfy any meaningful version of the second Bianchi identity while the torsion form of standard synthetic differential geometry (cf. Lavendhomme [1996, §5.3, Definition 3]) was shown to satisfy the first Bianchi identity (cf. Nishimura [1998b]), we have to introduce another curvature form in Section 3. Section 4 is devoted to induced connections. The first two sections are a laconic review on vector bundles and linear connections in synthetic context.

1 Vector Bundles

A mapping $\xi : E \to M$ of microlinear spaces is called a *vector bundle* providing that:

(1.1) $E_m = \xi^{-1}(m)$ is an $\mathbb{R}$-module for any $m \in M$, where $\mathbb{R}$ is the set of real numbers pervious to the so-called general Kock axiom (cf. Lavendhomme [1996, §§2.1.3]).

(1.2) The $\mathbb{R}$-module E_m is Euclidean for any $m \in M$ (cf. Lavendhomme [1996, §1.1, Definition 1]).

We call M the *base space* of ξ and E_m the *fiber over* m. The totality of mappings $\zeta : M \to E$ with $\xi \circ \zeta = \mathrm{id}_M$ (id_M denotes the identity transformation of M) is denoted by Sec ξ.

If $\xi : E \to M$ and $\eta : F \to N$ are vector bundles, then a pair $(\bar{\varphi}, \varphi)$ of maps $\bar{\varphi} : E \to F$ and $\varphi : M \to N$ is called a *bundle map* from ξ to η providing that $\eta \circ \bar{\varphi} = \varphi \circ \xi$ and $\bar{\varphi}$ induces a linear map $E_m \to F_{\varphi(m)}$ for each $m \in M$. In particular, if $M = N$ and φ is id_M, then the bundle map $(\bar{\varphi}, \varphi)$ is called a *strong bundle map* from ξ to η.

If M is a microlinear space, then its tangent bundle $\tau_M : M^D \to M$ is a vector bundle, where τ_M assigns, to each $t \in M^D$, $t(0) \in M$ (cf. Lavendhomme [1996, §3.1, Proposition 4]). If M is a microlinear space and $\mathcal{A}$ is an Euclidean $\mathbb{R}$-module which is microlinear, then the trivial bundle $M \times \mathcal{A} \to M$ is a vector bundle. Various algebraic constructions on vector bundles in standard differential geometry (cf. Greub, Halperin and Vanstone [1972, Chap II, §2]) can be carried over to our synthetic context. If $\xi : E \to M$ and $\eta : F \to M$ are vector bundles over the same base space M, then their Whitney sum $\xi \oplus \eta$ and the natural projection $\pi_{\mathcal{L}(\xi,\eta)} : \mathcal{L}(\xi, \eta) \to M$ are vector bundles, where $\mathcal{L}(\xi, \eta)$ denotes the totality $\mathcal{L}(\xi, \eta)$ of linear maps from E_m to F_m for $m \in M$ (cf. Lavendhomme [1996, §1.1, Propositions 4 and 5; §2.3, Proposition 1]). In particular, the dual bundle ξ^* of ξ (in case that

η is the trivial bundle $M \times \mathbb{R} \to M$) and the mapping $\pi_{\mathcal{L}(\xi)} : \mathcal{L}(\xi) \to M$ with $\mathcal{L}(\xi) = \mathcal{L}(\xi, \xi)$ are vector bundles.

If $\psi : N \to M$ is a map of microlinear spaces and $\xi : E \to M$ is a vector bundle, then a *differential n-form on N with values in ξ relative to ψ* is a map Ξ from N^{D^n} to E satisfying the following conditions:

(1.3) For any $\gamma \in N^{D^n}$ $\Xi(\gamma)$ lies in $E_{(\psi(0,...,0))}$.

(1.4) Ξ is n-homogeneous in the sense that $\Xi(\alpha_i \gamma) = \alpha \Xi(\gamma)$ $(1 \leq i \leq n)$.

(1.5) Ξ is alternating in the sense that $\Xi(\Sigma_\sigma(\gamma)) = \varepsilon_\sigma \Xi(\gamma)$ for any permutation σ of $\{1, ..., n\}$, where $\Sigma_\sigma(\gamma)(d_1, ..., d_n) = \gamma(d_{\sigma(1)}, ..., d_{\sigma(n)})$ for any $(d_1, ..., d_n) \in D^n$, and ε_σ is the sign of the permutation σ.

We denote by $A^n(N \xrightarrow{\psi} M; \xi)$ the totality of differential n-forms on N with values in ξ relative ψ. If $N = M$ and ψ is the identity map id_M of M, then $A^n(N \xrightarrow{\psi} M; \xi)$ is denoted also by $A^n(M; \xi)$. If ξ is furthermore a trivial bundle $M \times \mathbb{R} \to M$, then $A^n(M; \xi)$ is denoted simply by $A^n(M)$.

2 Linear Connection

Let $\xi : E \to M$ be a vector bundle. We denote by K_ξ the mapping which assigns, to each $\bar{t} \in E^D$, $(\xi \circ \bar{t}, \bar{t}(0)) \in M^D \times_M E$. Both E^D and $M^D \times_M E$ can be regarded naturally as vector bundles over E and over M^D, and K_ξ is linear with respect to both vector bundle structures (cf. Moerdijk and Reyes [1991, Chap. V, Proposition 3.4.8]). A (*linear*) *connection* on ξ is a mapping $\nabla : M^D \times_M E \to E^D$ pursuant to the following conditions:

(2.1) It is a section of K_ξ. I.e., $K_\xi \circ \nabla$ is the identity transformation of $M^D \times_M E$.

(2.2) It is homogeneous with respect to both vector bundle structures $\odot$ over E and $\cdot$ over M^D.

(2.3) For any $x \in M$ and any $(t, d) \in M^D \times D$, the mapping $u \in E_x| \to \nabla(t, u)(d) \in E_{t(d)}$, denoted by $p^\nabla_{(t,d)}$ or $p_{(t,d)}$, is bijective. Its inverse is denoted by $q^\nabla_{(t,d)} = q_{(t,d)} : E_{t(d)} \to E_x$. We call $p_{(t,d)}$ the *parallel transport* from $t(0)$ to $t(d)$ along t while $q_{(t,d)}$ is called the *parallel transport* from $t(d)$ to $t(0)$ along t.

If the vector bundle $\xi : E \to M$ is a trivial bundle $M \times \mathcal{A} \to M$, and if $\nabla(t, (t(0), a))(d) = (t(d), a)$ for any $t \in M^D$, any $a \in \mathcal{A}$ and any $d \in D$, then the connection ∇ is called *trivial*.

As Lavendhomme [1996, §§5.3.1] pointed out, his theory of covariant exterior differentiation can be generalized easily so as to yield mappings

$$d_\nabla : A^k(N \xrightarrow{\varphi} M; \xi) \to A^{k+1}(N \xrightarrow{\varphi} M; \xi),$$

where $\varphi : N \to M$ is a mapping from another microlinear space N to M. The covariant exterior differentiation d_∇ is a natural generalization of the exterior differentiation d (cf. Lavendhomme [1996, §4.2]), in which the vector bundle ξ is trivial and the connection ∇ is also trivial.

3 Two Curvatures

The principal objective of this section is to introduce another curvature by somewhat modifying the well-known curvature in synthetic differential geometry (cf. Lavendhomme [1996, §5.3, Definition 5]). It is this modified curvature that is to be shown in Section 5 to satisfy the second Bianchi identity. Now let us review the familiar curvature within the slightly more general context of vector bundles. A vector bundle $\xi : E \to M$ and a connection ∇ on ξ are chosen once and for all in this section.

The vector bundle $\tau_E : E^D \to E$ can be decomposed as the Whitney sum $V(E^D) \oplus H(E^D)$ with $V(E^D) = \{\bar{t} \in E^D | \bar{t}$ is tangent to $E_{\xi\circ\bar{t}(0)}\}$ and $H(E^D) = \{\nabla(t,u) | (t,u) \in M^D \times_M E\}$. Therefore any tangent vector $\bar{t}$ on E can be decomposed into a vertical tangent vector $\omega_1(\bar{t})$ on E (i.e., an element of $V(E^D)$) and a horizontal tangent vector $\bar{t} - \omega_1(\bar{t})$ on E (i.e., an element of $H(E^D)$). Since $V(E^D)$ can naturally be identified with $E \times_M E$ in such a way that $(v,w) \in E \times_M E$ gives rise to a tangent vector $d \in D | \to v + dw$ to E, the second compoment of $\omega_1(\bar{t})$, regarded as an element of $E \times_M E$, is denoted by $\omega(\bar{t})$, whereby we have the *connection form* $\omega : E^D \to E$.

The following proposition is merely a variant of Lavendhomme [1996, §5.2, Proposition 7].

Proposition 3.1. *For any $\bar{t} \in E^D$ and any $d \in D$ with $t = \xi \circ \bar{t}$, we have*

(3.1) $q_{(t,d)}(\bar{t}(d)) = \bar{t}(0) + d\omega(\bar{t})$.

Proof. Consider the mapping

$$(d, d') \in D(2) | \to p_{(t,d)}(\bar{t}(0) + d'\omega(\bar{t})) \in E,$$

which coincides with $\nabla(t, \bar{t}(0))$ on the first axis and which coincides with $\omega_1(\bar{t})$ on the second axis. Therefore the mapping

$$d \in D | \to p_{(t,d)}(\bar{t}(0) + d\omega(\bar{t})) \in E$$

coincides with $\bar{t}$, which means the desired proposition. ■

The connection form ω is surely an element of $A^1(E \xrightarrow{\xi} M; \xi)$, and its covariant exterior derivative $d_\nabla \omega \in A^2(E \xrightarrow{\xi} M; \xi)$ is called the *curvature form of the first kind* and denoted by Ω, for which we have

Proposition 3.2. *For any $\bar{\gamma} \in E^{D^2}$ and any $(d_1, d_2) \in D^2$ with $\gamma = \xi \circ \bar{\gamma}$, $t_1 = \gamma(\cdot, 0)$, $t_2 = \gamma(d_1, \cdot)$, $t_3 = \gamma(0, \cdot)$ and $t_4 = \gamma(\cdot, d_2)$, we have*

(3.2) $d_1 d_2 \Omega(\bar{\gamma}) = q_{(t_1,d_1)} \circ q_{(t_2,d_2)}(\bar{\gamma}(d_1, d_2)) - q_{(t_3,d_2)} \circ q_{(t_4,d_1)}(\bar{\gamma}(d_1, d_2))$.

Proof. By the definition of covariant exterior differentiation, we have

(3.3) $d_1 d_2 \Omega(\bar{\gamma}) = d_1 \omega(\bar{\gamma}(\cdot, 0)) + d_2 q_{(t_1,d_1)}(\omega(\bar{\gamma}(d_1, \cdot)))$
$- d_1 q_{(t_3,d_2)}(\omega(\bar{\gamma}(\cdot, d_2))) - d_2 \omega(\bar{\gamma}(0, \cdot))$.

By Proposition 3.1 we have

(3.4) $d_1\omega(\bar{\gamma}(\cdot,0)) = q_{(t_1,d_1)}(\bar{\gamma}(d_1,0)) - \bar{\gamma}(0,0)$

(3.5) $$d_2 q_{(t_1,d_1)}(\omega(\bar{\gamma}(d_1,\cdot))) = q_{(t_1,d_1)}\{q_{(t_2,d_2)}(\bar{\gamma}(d_1,d_2)) - \bar{\gamma}(d_1,0)\} \\ = q_{(t_1,d_1)} \circ q_{(t_2,d_2)}(\bar{\gamma}(d_1,d_2)) - q_{(t_1,d_1)}(\bar{\gamma}(d_1,0))$$

(3.6) $$d_1 q_{(t_3,d_2)}(\omega(\bar{\gamma}(\cdot,d_2))) = q_{(t_3,d_2)}\{q_{(t_4,d_1)}(\bar{\gamma}(d_1,d_2)) - \bar{\gamma}(0,d_2)\} \\ = q_{(t_3,d_2)} \circ q_{(t_4,d_1)}(\bar{\gamma}(d_1,d_2)) - q_{(t_3,d_2)}(\bar{\gamma}(0,d_2))$$

(3.7) $d_2\omega(\bar{\gamma}(0,\cdot)) = q_{(t_3,d_2)}(\bar{\gamma}(0,d_2)) - \bar{\gamma}(0,0)$.

Therefore the desired conclusion follows. ■

If ξ is the tangent bundle of M, then the curvature form of the first kind is no other than that of Lavendhomme (1996, §5.3, Definition 5). Now we introduce another curvature form, to be called the *curvature form of the second kind* and to be denoted by $\tilde{\Omega}$, as follows:

(3.8) $\tilde{\Omega}(\bar{\gamma}) = \Omega(h(\bar{\gamma}))$ for any microsquare $\bar{\gamma}$ on E,

where $h(\bar{\gamma})$ denotes the *horizontal component* of $\bar{\gamma}$ [cf. Moerdijk and Reyes (1991, Chap. V, §6)] in the sense that

(3.9) $h(\bar{\gamma})(d_1,d_2) = p_{(\gamma(d_1,\cdot),d_2)} \circ p_{(\gamma(\cdot,0),d_1)}(\bar{\gamma}(0,0))$

with $\gamma = \xi \circ \bar{\gamma}$. For the curvature form of the second kind, we have

Proposition 3.3. *Using the same notation as in Proposition 3.2, we have*

(3.10) $d_1 d_2 \tilde{\Omega}(\bar{\gamma}) = \bar{\gamma}(0,0) - q_{(t_3,d_2)} \circ q_{(t_4,d_1)} \circ p_{(t_2,d_2)} \circ p_{(t_1,d_1)}(\bar{\gamma}(0,0))$,

so that $\tilde{\Omega}(\bar{\gamma})$ *depends only on* $\gamma = \xi \circ \bar{\gamma}$ *and* $v = \bar{\gamma}(0,0)$, *which enables us to regard* $\tilde{\Omega}$ *as a function from* M^{D^2} *to* $\mathcal{L}(\xi)$ *in the sense that* $\tilde{\Omega}(\gamma)(v) = \tilde{\Omega}(\bar{\gamma})$.

Proof. Simply put $h(\bar{\gamma})$ in place of $\bar{\gamma}$ in Proposition 3.2. ■

Surely, if $\tilde{\Omega}$ claims to deserve its name, it has to be shown to satisfy the following:

Proposition 3.4. *The function* $\tilde{\Omega} : M^{D^2} \to L(\xi)$ *is a differential 2-form with values in* $\pi_{\mathcal{L}(\xi)}$. *I.e.,* $\tilde{\Omega} \in A^2(M;\pi_{\mathcal{L}(\xi)})$.

Proof. We define a function $\mathfrak{h} : M^{D^2} \underset{M}{\times} E \to E^{D^2}$ as follows:

(3.11) $\mathfrak{h}(\gamma,v)(d_1,d_2) = p_{(\gamma(d_1,\cdot),d_2)} \circ p_{(\gamma(\cdot,0),d_1)}(v)$

for any $(\gamma,v) \in M^{D^2} \underset{M}{\times} E$ and any $(d_1,d_2) \in D^2$.

Then it is easy to see that

(3.12) $\mathfrak{h}(\alpha_i\gamma, v) = \alpha_i\mathfrak{h}(\gamma, v)$ for any $\alpha \in \mathbb{R}$ $(i = 1, 2)$.

Since $\tilde{\Omega}(\gamma)(v) = \Omega(\mathfrak{h}(\gamma, v))$ and Ω is 2-homogeneous, $\tilde{\Omega}$ is also 2-homogeneous. To show that $\tilde{\Omega}$ is alternating, we let $v_0 = v$ and define v_1 and v_2 in order as follows:

(3.13) $v_1 = q_{(t_3,d_2)} \circ q_{(t_4,d_1)} \circ p_{(t_2,d_2)} \circ p_{(t_1,d_1)}(v_0)$

(3.14) $v_2 = q_{(t_1,d_1)} \circ q_{(t_2,d_2)} \circ p_{(t_4,d_1)} \circ p_{(t_3,d_2)}(v_1)$.

On the one hand it follows directly from (3.13) and (3.14) that

(3.15) $v_2 = v_0$.

On the other hand we can calculate v_1 and v_2 in order by making use of Proposition 3.3:

(3.16) $v_1 = v_0 - d_1d_2\tilde{\Omega}(\gamma)(v_0)$

(3.17)
$$\begin{aligned} v_2 &= v_1 - d_1d_2\tilde{\Omega}(\Sigma(\gamma))(v_1) \\ &= v_0 - d_1d_2\tilde{\Omega}(\gamma)(v_0) - d_1d_2\tilde{\Omega}(\Sigma(\gamma))(v_0 - d_1d_2\tilde{\Omega}(\gamma)(v_0)) \quad [(3.16)] \\ &= v_0 - d_1d_2\tilde{\Omega}(\gamma)(v_0) - d_1d_2\tilde{\Omega}(\Sigma(\gamma))(v_0). \end{aligned}$$

It follows from (3.15) and (3.17) that

(3.18) $\tilde{\Omega}(\gamma)(v_0) + \tilde{\Omega}(\Sigma(\gamma))(v_0) = 0$,

which means that $\tilde{\Omega}$ is alternating. ∎

4 Induced Connections

Now we define some induced connections. Let $\xi : E \to M$ and $\eta : F \to M$ be vector bundles over the same base space M with linear connections ∇ and ∇' bestowed upon them. First we define an induced connection $\nabla \oplus \nabla'$ on the Whitney sum $\xi \oplus \eta$ as follows:

(4.1) $(\nabla \oplus \nabla')(t, v_\xi \oplus v_\eta)(d) = \nabla(t, v_\xi)(d) \oplus \nabla'(t, v_\eta)(d)$

for any $t \in M^D$, any $v_\xi \in E_{t(0)}$, any $v_\eta \in F_{t(0)}$ and any $d \in D$.

Proposition 4.1. *For any $\gamma_\xi \in E^D$ and any $\gamma_\eta \in F^D$ with $\xi^D(\gamma_\xi) = \eta^D(\gamma_\eta)$, we have*

(4.2) $\omega_{\xi\oplus\eta}(\gamma_\xi + \gamma_\eta) = \omega_\xi(\gamma_\xi) \oplus \omega_\eta(\gamma_\eta)$,

where $\omega_{\xi\oplus\eta}$, ω_ξ and ω_η denote the connection forms of $\nabla \oplus \nabla'$, ∇ and ∇' respectively.

Proof. Let $t = \xi^D(\gamma_\xi) = \eta^D(\gamma_\eta)$. For any $d \in D$, we have

(4.3)
$$\begin{aligned} q^{\nabla\oplus\nabla'}_{(t,d)}(\gamma_\xi(d) \oplus \gamma_\eta(d)) &= (\gamma_\xi(0) + d\omega_\xi(\gamma_\xi)) \oplus (\gamma_\eta(0) + d\omega_\eta(\gamma_\eta)) \\ &= (\gamma_\xi(0) \oplus \gamma_\eta(0)) + d(\omega_\xi(\gamma_\xi) \oplus \omega_\eta(\gamma_\eta)). \end{aligned}$$

Therefore the desired proposition follows from Proposition 3.1. ∎

Corollary 4.2. *For any $\mu \in \mathrm{Sec}\,\xi$ and any $\nu \in \mathrm{Sec}\,\eta$, we have*

(4.4) $d_{\nabla \oplus \nabla'}(\mu + \nu) = d_\nabla \mu + d_{\nabla'} \nu.$

We now define an induced connection $\hat{\nabla}$ on $\pi_{\mathcal{L}(\xi,\eta)}$ as follows:

(4.5) $\hat{\nabla}(t, \hat{v})(d)(v) = p^{\nabla'}_{(t,d)}(\hat{v}(q^{\nabla}_{(t,d)}(v)))$

for any $t \in M^D$, any $d \in D$, any $\hat{v} \in \mathcal{L}(\xi, \eta)_{t(0)}$ and any $v \in E_{t(0)}$.

Proposition 4.3. *For any $\delta \in \mathcal{L}(\xi,\eta)^D$ and any $\gamma \in E^D$ with $(\pi_{\mathcal{L}(\xi,\eta)})^D(\delta) = \xi^D(\gamma)$, we have*

(4.6) $\omega_\eta(\delta(\gamma)) = \hat{\omega}(\delta)(\gamma(0)) + \delta(0)(\omega_\xi(\gamma)),$

where $\hat{\omega}$ denote the connection form of $\hat{\nabla}$ and $\delta(\gamma)$ denotes the mapping $d \in D| \to \delta(d)(\gamma(d))$.

Proof. Let $t = (\pi_{\mathcal{L}(\xi,\eta)})^D(\delta) = \xi^D(\gamma)$. For any $d \in D$, we have

$$(4.7)\quad \begin{aligned} q^{\nabla'}_{(t,d)}(\delta(d)(\gamma(d))) &= q^{\hat{\nabla}}_{(t,d)}(\delta(d))(q^{\nabla}_{(t,d)}(\gamma(d))) \\ &= (\delta(0) + d\hat{\omega}(\delta))(\gamma(0) + d\omega_\xi(\gamma)) \\ &= \delta(0)(\gamma(0)) + d\{\hat{\omega}(\delta)(\gamma(0)) + \delta(0)(\omega_\xi(\gamma))\}. \end{aligned}$$

Therefore the desired proposition follows from Proposition 3.1. ∎

Corollary 4.4. *For any $\mu \in \mathrm{Sec}\,\xi$ and any $\iota \in \mathrm{Sec}\,\pi_{\mathcal{L}(\xi,\eta)}$, we have*

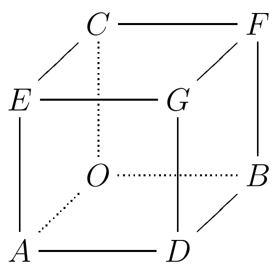
(4.8) $d_{\nabla'}(\iota(\mu)) = (d_{\hat{\nabla}}\iota)(\mu) + \iota(d_\nabla \mu).$

If η is the trivial bundle $M \times \mathbb{R} \to M$ and the connection ∇' is trivial, then the connection $\hat{\nabla}$ is usually denoted by ∇^*. If $\xi = \eta$ and $\nabla = \nabla'$, then the connection $\hat{\nabla}$ is usually denoted by $\tilde{\nabla}$.

5 Bianchi Identity

The principal objective of this section is to establish the second Bianchi identity of our curvature form of the second kind. Let us begin with a cubical version of Kock's [1996, Theorem 2] simplicial and combinatorial Bianchi identity. As in Section 3, a vector bundle $\xi : E \to M$ and a connection ∇ on ξ are chosen once and for all.

Theorem 5.1. *Let γ be a microcube on M. Let $d_1, d_2, d_3 \in D$. We denote points $\gamma(0,0,0)$, $\gamma(d_1,0,0)$, $\gamma(0,d_2,0)$, $\gamma(0,0,d_3)$, $\gamma(d_1,d_2,0)$, $\gamma(d_1,0,d_3)$, $\gamma(0,d_2,d_3)$, and $\gamma(d_1,d_2,d_3)$, by O, A, B, C, D, E, F and G respectively. These eight points are depicted figuratively as the eight vertices of a cube:*



Then we have

(5.1) $P_{AO} \circ P_{DA} \circ P_{GD} \circ R_{GFBD} \circ R_{GECF} \circ R_{GDAE} \circ P_{DG} \circ P_{AD} \circ P_{OA} \circ R_{OCEA} \circ R_{OBFC} \circ R_{OADB} = \mathrm{id}_O$,

where

(5.2) for any adjacent vertices X, Y of the cube, P_{XY} denotes the parallel transport from X to Y along the line connecting X and Y (e.g., P_{OA} and P_{AO} denote $p_{(\gamma(\cdot,0),d_1)}$ and $q_{(\gamma(\cdot,0),d_1)}$ respectively),

(5.3) for any four vertices X, Y, Z, W of the cube rounding one of the six facial squares of the cube, R_{XYZW} denotes $P_{WX} \circ P_{ZW} \circ P_{YZ} \circ P_{XY}$ (e.g., R_{OADB} denotes $q_{(\gamma(0,\cdot,0),d_2)} \circ q_{(\gamma(\cdot,d_2,0),d_1)} \circ p_{(\gamma(d_1,\cdot,0),d_2)} \circ p_{(\gamma(\cdot,0,0),d_1)}$), and

(5.4) id_O is the identity transformation of E_O.

Proof. Write over (5.1) exclusively in terms of P_{XY}'s, and write off all consecutive $P_{XY} \circ P_{YX}$'s. ■

The above theorem gives rise to the following standard form of the second Bianchi identity.

Theorem 5.2. *We have*

(5.5) $d_{\tilde{\nabla}}\tilde{\Omega} = 0$,

where $d_{\tilde{\nabla}}$ is the covariant exterior differentiation with respect to the induced connection $\tilde{\nabla}$ on $\pi_{\mathcal{L}(\xi)}$, and recall that $\tilde{\Omega} \in A^2(M; \pi_{\mathcal{L}(\xi)})$, as was explained in Proposition 3.4.

Proof. Let γ, d_1, d_2, d_3, O, A, B, C, D, E, F and G be as in Theorem 5.1. Given $v_0 \in E_{\gamma(0,0,0)}$, we define $v_i \in E_{\gamma(0,0,0)}$ $(i = 1,2,3,4,5,6)$ in order as follows:

(5.6) $v_1 = R_{OADB}(v_0)$

(5.7) $v_2 = R_{OBFC}(v_1)$

(5.8) $v_3 = R_{OCEA}(v_2)$

(5.9)
$$\begin{aligned} v_4 &= P_{AO} \circ P_{DA} \circ P_{GD} \circ R_{GDAE} \circ P_{DG} \circ P_{AD} \circ P_{OA}(v_3) \\ &= P_{AO} \circ R_{AEGD} \circ P_{OA}(v_3) \end{aligned}$$

(5.10)
$$\begin{aligned} v_5 &= P_{AO} \circ P_{DA} \circ P_{GD} \circ R_{GECF} \circ P_{DG} \circ P_{AD} \circ P_{OA}(v_4) \\ &= P_{AO} \circ R_{AEGD} \circ P_{EA} \circ R_{ECFG} \circ P_{AE} \circ R_{ADGE} \circ P_{OA}(v_4) \\ &= P_{AO} \circ P_{EA} \circ R_{EGDA} \circ R_{ECFG} \circ R_{EADG} \circ P_{AE} \circ P_{OA}(v_4) \\ &= R_{OCEA} \circ P_{CO} \circ P_{EC} \circ R_{EGDA} \circ R_{ECFG} \circ R_{EADG} \circ P_{CE} \circ P_{OC} \\ &\quad \circ R_{OAEC}(v_4) \\ &= R_{OCEA} \circ P_{CO} \circ P_{EC} \circ R_{EGDA} \circ P_{CE} \circ R_{CFGE} \circ P_{EC} \circ R_{EADG} \circ \\ &\quad P_{CE} \circ P_{OC} \circ R_{OAEC}(v_4) \end{aligned}$$

(5.11)
$$\begin{aligned} v_6 &= P_{AO} \circ P_{DA} \circ P_{GD} \circ R_{GFBD} \circ P_{DG} \circ P_{AD} \circ P_{OA}(v_5) \\ &= P_{AO} \circ P_{DA} \circ R_{DGFB} \circ P_{AD} \circ P_{OA}(v_5) \\ &= R_{OBDA} \circ P_{BO} \circ R_{BDGF} \circ P_{OB} \circ R_{OADB}(v_5). \end{aligned}$$

Now we calculate v_i $(i = 1, ..., 6)$ in order. It follows directly from Proposition 3.3 that

(5.12) $v_1 = v_0 - d_1 d_2 \tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0).$

The calculations of v_2 and v_3 are similar, so we present details of the former calculation but simply register the result of the latter calculation, safely leaving its details to the reader.

(5.13)
$$\begin{aligned} v_2 &= v_1 - d_2 d_3 \tilde{\Omega}(\gamma(0,\cdot,\cdot))(v_1) \quad \text{[Proposition 3.3]} \\ &= v_0 - d_1 d_2 \tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0) - d_2 d_3 \tilde{\Omega}(\gamma(0,\cdot,\cdot)) \\ &\quad (v_0 - d_1 d_2 \tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0)) \quad \text{[(5.12)]} \\ &= v_0 - d_1 d_2 \tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0) - d_2 d_3 \tilde{\Omega}(\gamma(0,\cdot,\cdot))(v_0) \end{aligned}$$

(5.14)
$$\begin{aligned} v_3 &= v_0 - d_1 d_2 \tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0) - d_2 d_3 \tilde{\Omega}(\gamma(0,\cdot,\cdot))(v_0) \\ &\quad + d_1 d_3 \tilde{\Omega}(\gamma(\cdot,0,\cdot))(v_0)). \end{aligned}$$

The three calculations of v_4, v_5 and v_6 are similar, so we present their details only in case of the first and the last, leaving details of the most tedious calculation of v_5 to the reader.

(5.15)
$$\begin{aligned} v_4 &= P_{AO} \circ R_{AEGD} \circ P_{OA}(v_0 - d_1 d_2 \tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0) - \\ &\quad d_2 d_3 \tilde{\Omega}(\gamma(0,\cdot,\cdot))(v_0)) + d_1 d_3 \tilde{\Omega}(\gamma(\cdot,0,\cdot))(v_0)) \quad \text{[(5.14)]} \\ &= P_{AO} \circ R_{AEGD}(P_{OA}(v_0) - d_1 d_2 P_{OA}(\tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0)) \\ &\quad - d_2 d_3 P_{OA}(\tilde{\Omega}(\gamma(0,\cdot,\cdot))(v_0)) + d_1 d_3 P_{OA}(\tilde{\Omega}(\gamma(\cdot,0,\cdot))(v_0))) \\ &= P_{AO}(P_{OA}(v_0) - d_1 d_2 P_{OA}(\tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0)) - \\ &\quad - d_2 d_3 P_{OA}(\tilde{\Omega}(\gamma(0,\cdot,\cdot))(v_0)) + \\ &\quad d_1 d_3 P_{OA}(\tilde{\Omega}(\gamma(\cdot,0,\cdot))(v_0)) + d_2 d_3 \tilde{\Omega}(\gamma(d_1,\cdot,\cdot)) \\ &\quad (P_{OA}(v_0) - d_1 d_2 P_{OA}(\tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0)) - \\ &\quad d_2 d_3 P_{OA}(\tilde{\Omega}(\gamma(0,\cdot,\cdot))(v_0)) + \\ &\quad d_1 d_3 P_{OA}(\tilde{\Omega}(\gamma(\cdot,0,\cdot))(v_0)))) \quad \text{[Propositions 3.3 and 3.4]} \\ &= v_0 - d_1 d_2 \tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0) - d_2 d_3 \tilde{\Omega}(\gamma(0,\cdot,\cdot))(v_0) \\ &\quad + d_1 d_3 (\tilde{\Omega}(\gamma(\cdot,0,\cdot))(v_0) + d_2 d_3 P_{AO}(\tilde{\Omega}(\gamma(d_1,\cdot,\cdot))(P_{OA}(v_0))) \end{aligned}$$

(5.16)
$$\begin{aligned} v_5 &= v_0 - d_1 d_2 \tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0) - \\ &\quad d_2 d_3 \tilde{\Omega}(\gamma(0,\cdot,\cdot))(v_0) + d_1 d_3 \tilde{\Omega}(\gamma(\cdot,0,\cdot))(v_0) + \\ &\quad d_2 d_3 P_{AO} \tilde{\Omega}(\gamma(d_1,\cdot,\cdot))(P_{OA}(v_0))) + \\ &\quad d_1 d_2 P_{CO}(\tilde{\Omega}(\gamma(\cdot,\cdot,d_3))(P_{OC}(v_0)) \end{aligned}$$

(5.17)
$$\begin{aligned} v_6 &= R_{OBDA} \circ P_{BO} \circ R_{BDGF} \circ P_{OB} \circ R_{OADB}(v_0 - \\ &\quad d_1 d_2 \tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0) - d_2 d_3 \tilde{\Omega}(\gamma(0,\cdot,\cdot))(v_0) + \\ &\quad d_1 d_3 \tilde{\Omega}(\gamma(\cdot,0,\cdot))(v_0) + d_2 d_3 P_{AO}(\tilde{\Omega}(\gamma(d_1,\cdot,\cdot))(P_{OA}(v_0))) + \\ &\quad d_1 d_2 P_{CO}(\tilde{\Omega}(\gamma(\cdot,\cdot,d_3))(P_{OC}(v_0))) \quad \text{[(5.16)]} \\ &= R_{OBDA} \circ P_{BO} \circ R_{BDGF} \circ P_{OB}(v_0 - d_1 d_2 \tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0) \\ &\quad - d_2 d_3 \tilde{\Omega}(\gamma(0,\cdot,\cdot))(v_0) + d_1 d_3 \tilde{\Omega}(\gamma(\cdot,0,\cdot))(v_0) + \\ &\quad d_2 d_3 P_{AO}(\tilde{\Omega}(\gamma(d_1,\cdot,\cdot))(P_{OA}(v_0))) + \end{aligned}$$

$$d_1 d_2 P_{CO}(\tilde{\Omega}(\gamma(\cdot,\cdot,d_3))(P_{OC}(v_0)))-$$
$$d_1 d_2 \tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0)) \quad \text{[Proposition 3.3]}$$
$$= R_{OBDA} \circ P_{BO} \circ R_{BDGF}(P_{OB}(v_0) - d_1 d_2 P_{OB}(\tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0))-$$
$$- d_2 d_3 P_{OB}(\tilde{\Omega}(\gamma(0,\cdot,\cdot))(v_0)) + d_1 d_3 P_{OB}(\tilde{\Omega}(\gamma(\cdot,0,\cdot))(v_0))+$$
$$d_2 d_3 P_{OB} \circ P_{AO}(\tilde{\Omega}(\gamma(d_1,\cdot,\cdot))(P_{OA}(v_0)))+$$
$$d_1 d_2 P_{OB} \circ P_{CO}(\tilde{\Omega}(\gamma(\cdot,\cdot,d_3))(P_{OC}(v_0)))-$$
$$d_1 d_2 P_{OB}(\tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0)))$$
$$= R_{OBDA} \circ P_{BO}(P_{OB}(v_0) - d_1 d_2 P_{OB}(\tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0))$$
$$- d_2 d_3 P_{OB}(\tilde{\Omega}(\gamma(0,\cdot,\cdot))(v_0)) + d_1 d_3 P_{OB}(\tilde{\Omega}(\gamma(\cdot,0,\cdot))(v_0))+$$
$$d_2 d_3 P_{OB} \circ P_{AO}(\tilde{\Omega}(\gamma(d_1,\cdot,\cdot))(P_{OA}(v_0)))+$$
$$d_1 d_2 P_{OB} \circ P_{CO}(\tilde{\Omega}(\gamma(\cdot,\cdot,d_3))(P_{OC}(v_0)))-$$
$$d_1 d_2 P_{OB}(\tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0))-$$
$$d_1 d_3 \tilde{\Omega}(\gamma(\cdot,d_2,\cdot))(P_{OB}(v_0))) \quad \text{[Proposition 3.3]}$$
$$= R_{OBDA}(v_0 - d_1 d_2 \tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0)-$$
$$d_2 d_3 \tilde{\Omega}(\gamma(0,\cdot,\cdot))(v_0) + d_1 d_3 \tilde{\Omega}(\gamma(\cdot,0,\cdot))(v_0)+$$
$$d_2 d_3 P_{AO}(\tilde{\Omega}(\gamma(d_1,\cdot,\cdot))(P_{OA}(v_0)))+$$
$$d_1 d_2 P_{CO}(\tilde{\Omega}(\gamma(\cdot,\cdot,d_3))(P_{OC}(v_0)))-$$
$$d_1 d_2 \tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0)-$$
$$d_1 d_3 P_{BO}(\tilde{\Omega}(\gamma(\cdot,d_2,\cdot))(P_{OB}(v_0))))$$
$$= v_0 - d_1 d_2 \tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0)-$$
$$d_2 d_3 \tilde{\Omega}(\gamma(0,\cdot,\cdot))(v_0) + d_1 d_3 \tilde{\Omega}(\gamma(\cdot,0,\cdot))(v_0)+$$
$$d_2 d_3 P_{AO}(\tilde{\Omega}(\gamma(d_1,\cdot,\cdot))(P_{OA}(v_0)))+$$
$$d_1 d_2 P_{CO}(\tilde{\Omega}(\gamma(\cdot,\cdot,d_3))(P_{OC}(v_0)))-$$
$$d_1 d_3 P_{BO}(\tilde{\Omega}(\gamma(\cdot,d_2,\cdot))(P_{OB}(v_0)))) \quad \text{[Propositions 3.3 and 3.4]}.$$

It should be the case by Theorem 5.1 that $v_6 = v_0$. Therefore

$$\begin{aligned} &d_1 d_2 \tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0) + d_2 d_3 \tilde{\Omega}(\gamma(0,\cdot,\cdot))(v_0)- \\ &d_1 d_3 \tilde{\Omega}(\gamma(\cdot,0,\cdot))(v_0)- \\ &d_2 d_3 P_{AO}(\tilde{\Omega}(\gamma(d_1,\cdot,\cdot))(P_{OA}(v_0)))- \\ &d_1 d_2 P_{CO}(\tilde{\Omega}(\gamma(\cdot,\cdot,d_3))(P_{OC}(v_0)))+ \\ &d_1 d_3 P_{BO}(\tilde{\Omega}(\gamma(\cdot,d_2,\cdot))(P_{OB}(v_0)))) = 0. \end{aligned} \tag{5.18}$$

Since $v_0 \in E_{\gamma(0,0,0)}$ was chosen arbitrarily, the proof is complete. ■

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