Hyperbolic Rotation Surfaces of Constant Mean Curvature in 3-de Sitter Space

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Dedicated to Professor Udo Simon on the Occation of his Sixtieth Birthday

Abstract

In the 4-dimensional Minkowski space $\mathbb{R}^4_1$, a surface is said to be a hyperbolic rotation surface, if it is a orbit of a regular curve under the action of the orthogonal transformations of $\mathbb{R}^4_1$ which leave a spacelike plane pointwise fixed. In this paper, we give the totally classification of the timelike and spacelike hyperbolic rotation surfaces in 3-dimensional de Sitter space $\mathbb{S}^3_1$.

Introduction.

In differential geometry, for the study of the surfaces theory in space forms, it is a very important and interesting problem to contruct or classify the constant mean curvature surfaces. Spacelike constant mean curvature hypersurfaces in arbitrary spacetime have interest in reletivity theory. They are convenient initial surfaces for the Cauchy problem and provide a time guage which is important in the study of singularities, the positivity of mass, and gravitational radiation.

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The surfaces of constant mean curvature in Minkowski space have been studied extensively. For example, K. Akutagawa and S. Nishikawa give a representation formula for spacelike surfaces with prescribed mean curvature in [AK-N]; such a representation formula for timelike surfaces has been give by M. A. Magid in [MA]. In [IN-1] and [IN-2], I. Inoguchi studied the spacelike and timelike surfaces with constant mean curvature or Gaussian curvature in 3-dimensional Minkowski space $\mathbb{R}^3_1$ via the theory of finite-type harmonic maps and the split-quaternion algebra; he also reformulates the fundamental equations and the representation formula. In [AL], [M-1], [M-2] and [M-3], rotational spacelike surfaces in the de Sitter space are considered and determined.

In this paper, we consider the surfaces in 3-dimensional de Sitter space $\mathbb{S}^3_1$. We will give the classification of all timelike and spacelike hyperbolic rotation surfaces with non-zero constant mean curvature in $\mathbb{S}^3_1$.

1 Preliminaries

Let $\mathbb{R}^{n+1}_1$ be the $(n+1)$-dimensional Minkowski space with the natural basis $e_1, ..., e_{n+1}$, its metric $< \ , \ >$ is given by

$$< x, y >= \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}, \quad x, y \in \mathbb{R}^{n+1}_1,$$

$x = (x_1, x_2, ..., x_{n+1})$, $y = (y_1, y_2, ..., y_{n+1})$. The n-dimensional de Sitter space $\mathbb{S}^n_1$ is defined by

$$\mathbb{S}^n_1 = (x \in \mathbb{R}^{n+1}_1 :< x, x >= 1).$$

It is well known that $\mathbb{S}^n_1$ is the complete simply connected Pseudo-Riemannian hypersurface with constant sectional curvature 1 in $\mathbb{R}^{n+1}_1$ ([L-1], [L-2]).

Let $\mathbf{P}^k (k = 2, 3)$ denote an k-dimensional subspace of $\mathbb{R}^4_1$ passing through the origin and $\mathbf{O}(\mathbf{P}^2)$ the group of orthogonal transformations of $\mathbb{R}^4_1$ with positive determinant that leave $\mathbf{P}^2$ pointwise fixed.

Definition.(cf.[DC-D]) *Choose $\mathbf{P}^2$ and $\mathbf{P}^3$ such that $\mathbf{P}^2 \subset \mathbf{P}^3$ and $\mathbf{P}^3 \cap \mathbb{S}^3_1 \neq \emptyset$. Let C be a regular C^2-curve in $\mathbf{P}^3 \cap \mathbb{S}^3_1$ that does not meet $\mathbf{P}^2$. The orbit of C under the action of $\mathbf{O}(\mathbf{P}^2)$ is called a rotation surface $\mathbf{M}$ in $\mathbb{S}^3_1$ generated by C around $\mathbf{P}^2$ if the induced metric G of $\mathbf{M}$ from $\mathbb{R}^4_1$ is nodegenerate. The surface $\mathbf{M}$ is said to be spherical (resp., hyperbolic or parabolic) if the restriction $\bar{G}/\mathbf{P}^2$ (where $\bar{G}$ is the metric of $\mathbb{R}^4_1$) is a pseudo-Riemannian metric (resp., a Riemannian metric or a degenerate quadric form).*

In the 4-dimensional Minkowski space $\mathbb{R}^4_1$, let $C_1 : c_1(u) = (x(u), y(u), 0, w(u))$ or $C_2 : c_2(u) = (x(u), y(u), w(u), 0)$, $u \in I$, be any C^2-curve in $\mathbf{P}^3 \cap \mathbb{S}^3_1$ which is parameterized by arc length, whose domain of definition I is an open interval of real numbers including zero, and for which the following equations are satisfied

$$x(u)^2 + y(u)^2 - w(u)^2 = 1, \tag{1.1.i}$$

$$x'(u)^2 + y'(u)^2 - w'(u)^2 = \varepsilon, \tag{1.2.i}$$

for the curve C_1;

$$x(u)^2 + y(u)^2 + w(u)^2 = 1, \tag{1.1.ii}$$
$$x'(u)^2 + y'(u)^2 + w'(u)^2 = 1, \tag{1.2.ii}$$

for the curve C_2; where $\varepsilon = \pm 1$.

The surfaces

$$\mathbf{M}_1: \quad r(u,v) = (x(u), y(u), w(u)\sinh(v), w(u)\cosh(v)), u \in I, v \in \mathbb{R}, \tag{1.3.i}$$
$$\mathbf{M}_2: \quad r(u,v) = (x(u), y(u), w(u)\cosh(v), w(u)\sinh(v)), u \in I, v \in \mathbb{R}, \tag{1.3.ii}$$

are by definition the hyperbolic rotation surfaces in the 3-dimensional de Sitter space $\mathbb{S}^3_1$ obtained by rotating the curve $c_1(u)$ or $c_2(u)$ (this rotation of $\mathbb{R}^4_1$ fixes a spacelike plane i.e. $x \circ y$ plane). The first fundamental form of $\mathbf{M}_1$ is $\varepsilon \mathrm{du}^2 + \mathrm{w(u)^2 dv^2}$; when $\varepsilon = 1$, the surface is spacelike; when $\varepsilon = -1$, the surface is timelike. The first fundamental form of $\mathbf{M}_2$ is $\mathrm{du}^2 - \mathrm{w(u)^2 dv^2}$ and it is always timelike. Let

$$\begin{aligned}\xi_1(u,v) =& (y'(u)w(u) - w'(u)y(u), w'(u)x(u) - x'(u)w(u),\\ & (y'(u)x(u) - x'(u)y(u))\sinh(v), (y'(u)x(u) - x'(u)y(u))\cosh(v))\end{aligned} \tag{1.4.i}$$

$$\begin{aligned}\xi_2(u,v) =& (y'(u)w(u) - w'(u)y(u), w'(u)x(u) - x'(u)w(u),\\ & (x'(u)y(u) - y'(u)x(u))\cosh(v), (x'(u)y(u) - y'(u)x(u))\sinh(v)).\end{aligned} \tag{1.4.ii}$$

Then for the surface $\mathbf{M}_1$, we have

$$< \xi_1, r_u > = < \xi_1, r_v > = < \xi_1, r > = 0, \ < \xi_1, \xi_1 > = -\varepsilon;$$

for the surface $\mathbf{M}_2$, we have

$$< \xi_2, r_u > = < \xi_2, r_v > = < \xi_2, r > = 0, \ < \xi_2, \xi_2 > = 1;$$

where

$$r_u = \frac{\mathrm{dr(u,v)}}{\mathrm{du}}, \quad r_v = \frac{\mathrm{dr(u,v)}}{\mathrm{dv}}.$$

So we know that $\xi_1(u,v)$ is a field of unit normal vectors on $\mathbf{M}_1$ in $\mathbb{S}^3_1$ and $\xi_2(u,v)$ is a field of unit normal vectors on $\mathbf{M}_2$ in $\mathbb{S}^3_1$.

We denote by $\widetilde{\nabla}$ the covariant differentiation with respect to the indefinite Riemannian metric of $\mathbb{R}^4_1$ and by $\bar{\nabla}$ and ∇ the covariant differentiations with respect to the induced metric of $\mathbb{S}^3_1$ and $\mathbf{M}$, respectively. We denote by $\eta(x)$, $x \in \mathbb{S}^3_1$, the position vector of x with respect to the origin of $\mathbb{R}^4_1$ which is a field of normal vectors of $\mathbb{S}^3_1$ in $\mathbb{R}^4_1$. Then, considering that $\mathbf{M}_1$ (resp. $\mathbf{M}_2$) is locally embedded in $\mathbb{S}^3_1$, we have the following Gauss's and Weingarten's formulas.

$$\widetilde{\nabla}_X Y = \bar{\nabla}_X Y + < X, Y > \eta, \tag{1.5}$$
$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)\xi_i, \tag{1.6}$$
$$\bar{\nabla}_X \xi_i = -A_{\xi_i}(X), \tag{1.7}$$

where X and Y are tangent vector fields on $\mathbf{M}_i$, and A_{ξ_i} is a field of type (1,1) tensor on $\mathbf{M}_i$ corresponding to ξ_i, i.e.,

$$< A_{\xi_i}(X), Y >= h(X, Y),$$

$i = 1, 2$.

From (1.5), (1.6) and (1.7), we get the following equations about the mean curvature of the surfaces $\mathbf{M}_1$ and $\mathbf{M}_2$. For the surface $\mathbf{M}_1$ we have

$$\text{(1.8.i)} \quad \begin{aligned} < A_{\xi_1}(r_u), r_u > &= - < \bar{\nabla}_{r_u}\xi_1, r_u >=< \xi_1, \bar{\nabla}_{r_u} r_u >=< \xi_1, \widetilde{\nabla}_{r_u} r_u > \\ &= x''(y'w - w'y) + y''(w'x - x'w) - w''(y'x - x'y), \end{aligned}$$

$$\text{(1.9.i)} \quad \begin{aligned} < A_{\xi_1}(r_v), r_v > &= - < \bar{\nabla}_{r_v}\xi_1, r_v >=< \xi_1, \bar{\nabla}_{r_v} r_v >=< \xi_1, \widetilde{\nabla}_{r_v} r_v > \\ &= -w(y'x - x'y), \end{aligned}$$

$$\text{(1.10.i)} \quad < A_{\xi_1}(r_u), r_v > = - < \bar{\nabla}_{r_u}\xi_1, r_v >=< \xi_1, \bar{\nabla}_{r_u} r_v >=< \xi_1, \widetilde{\nabla}_{r_u} r_v >= 0;$$

since $< r_u, r_u >= \varepsilon$, $< r_u, r_v >= 0$, $< r_v, r_v >= w^2$, then from (1.8.i), (1.9.i) and (1.10.i), we obtain

$$\text{(1.11. i)} \quad \begin{aligned} 2H &= \mathrm{trace} A_{\xi_1} \\ &= \varepsilon(x''(y'w - w'y) + y''(w'x - x'w) - w''(y'x - x'y)) - (y'x - x'y)/w \\ &= \frac{\varepsilon w^2(x''y' - y''x') + \varepsilon ww'(y''x - x''y) - (\varepsilon ww'' + 1)(y'x - x'y)}{w}. \end{aligned}$$

For the surface $\mathbf{M}_2$ we have

$$\text{(1.8.ii)} \quad \begin{aligned} < A_{\xi_2}(r_u), r_u > &= - < \bar{\nabla}_{r_u}\xi_2, r_u >=< \xi_2, \bar{\nabla}_{r_u} r_u >=< \xi_2, \widetilde{\nabla}_{r_u} r_u > \\ &= x''(y'w - w'y) + y''(w'x - x'w) + w''(x'y - y'x), \end{aligned}$$

$$\text{(1.9.ii)} \quad \begin{aligned} < A_{\xi_2}(r_v), r_v > &= - < \bar{\nabla}_{r_v}\xi_2, r_v >=< \xi_2, \bar{\nabla}_{r_v} r_v >=< \xi_2, \widetilde{\nabla}_{r_v} r_v > \\ &= w(x'y - y'x), \end{aligned}$$

$$\text{(1.10.ii)} \quad < A_{\xi_2}(r_u), r_v > = - < \bar{\nabla}_{r_u}\xi_2, r_v >=< \xi_2, \bar{\nabla}_{r_u} r_v >=< \xi_2, \widetilde{\nabla}_{r_u} r_v >= 0;$$

since $< r_u, r_u >= 1$, $< r_u, r_v >= 0$, $< r_v, r_v >= -w^2$, then from (1.8.ii), (1.9.ii) and (1.10.ii), we obtain

$$\text{(1.11.ii)} \quad \begin{aligned} 2H &= \mathrm{trace} A_{\xi_2} \\ &= (x''(y'w - w'y) + y''(w'x - x'w) + w''(x'y - y'x)) - (x'y - y'x)/w \\ &= \frac{w^2(x''y' - y''x') - ww'(x''y - y''x) + (ww'' - 1)(x'y - y'x)}{w} \end{aligned}$$

2 Constant mean curvature surfaces of type M_1

From the previous argument we see that the surface $\mathbf{M}_1$ has constant mean curvature $H \neq 0$ in $\mathbb{S}^3_1$ if and only if on the interval I the following relations hold:

$$\text{(2.1)} \quad x(u)^2 + y(u)^2 - w(u)^2 = 1$$

$$\text{(2.2)} \quad x'(u)^2 + y'(u)^2 - w'(u)^2 = \varepsilon$$

$$\text{(2.3)} \quad 2Hw = \varepsilon w^2(x''y' - y''x') + \varepsilon ww'(y''x - x''y) - (\varepsilon ww'' + 1)(y'x - x'y).$$

Now we solve the above equations. From (2.1) we may put

$$\begin{cases} x(u) = (w(u)^2+1)^{\frac{1}{2}}\cos\varphi(u), \\ y(u) = (w(u)^2+1)^{\frac{1}{2}}\sin\varphi(u), \end{cases} \tag{2.4}$$

and then determine the function $\varphi(u)$ satisfying (2.2).

Since $x'^2+y'^2-w'^2 = (ww')^2/(w^2+1)+(w^2+1)\varphi'^2-w'^2$, then from (2.2) it follows that

$$\varphi'(u)^2 = \frac{\varepsilon w^2 + w'^2 + \varepsilon}{(w^2+1)^2}. \tag{2.5}$$

We assume that $\varepsilon w^2 + w'^2 + \varepsilon > 0$ on I (when $\varepsilon w^2 + w'^2 + \varepsilon = 0$, φ is constant). Therefore the function $\varphi(u)$ is of the form

$$\varphi(u) = \pm\int_0^u \frac{(\varepsilon w(t)^2 + w'(t)^2 + \varepsilon)^{\frac{1}{2}}}{(w(t)^2+1)}\mathrm{dt}. \tag{2.6}$$

and without loss of generality we may assume that the signature is positive.

From (2.4) and (2.6), we can show that

$$y'x - x'y = (w^2+1)\varphi' = (\varepsilon w^2 + w'^2 + \varepsilon)^{\frac{1}{2}}, \tag{2.7}$$

$$y''x - x''y = (y'x - x'y)' = (\varepsilon ww' + w'w'')/(y'x - x'y). \tag{2.8}$$

Differentiating (2.1) and (2.2) we obtain

$$\begin{aligned} xx' + yy' &= ww', \\ xx'' + yy'' &= ww'' - \varepsilon, \\ x'x'' + y'y'' &= w'w''. \end{aligned}$$

Solving above equations for x'' and y'' we get

$$\begin{aligned} (y'x - x'y)x'' &= y'(ww'' - \varepsilon) - yw'w'', \\ (y'x - x'y)y'' &= -x'(ww'' - \varepsilon) + xw'w''. \end{aligned}$$

So

$$x''y' - y''x' = (\varepsilon ww'' - \varepsilon w'^2 - 1)/(y'x - x'y). \tag{2.9}$$

Putting (2.7), (2.8) and (2.9) into (2.3), then we get

$$ww'' + w'^2 + 2\varepsilon w^2 + \varepsilon = -2Hw(\varepsilon w^2 + w'^2 + \varepsilon)^{\frac{1}{2}}. \tag{2.10}$$

Without loss of generality, we can assume that $w(u) > 0$. When $w(u) \neq$constant, let $\alpha(u) = w^2 + \frac{1}{2}$, then (2.10) becomes

$$\alpha'' + 4\varepsilon\alpha = -2H(\alpha'^2 + 4\varepsilon\alpha^2 - \varepsilon)^{\frac{1}{2}}. \tag{2.11}$$

Since $w(u) \neq$constant, $\alpha' \not\equiv 0$. From (2.11) we have

$$\frac{\frac{1}{2}\mathrm{d}(\alpha'^2 + 4\varepsilon\alpha^2 - \varepsilon)}{(\alpha'^2 + 4\varepsilon\alpha^2 - \varepsilon)^{\frac{1}{2}}} = -2H\alpha'\mathrm{du}.$$

Then

$$(\alpha'^2 + 4\varepsilon\alpha^2 - \varepsilon)^{\frac{1}{2}} = a - 2H\alpha, \quad a - 2H\alpha > 0, \tag{2.12}$$

where a is integral constant. From (2.12) we get

$$\alpha'^2 = 4(H^2 - \varepsilon)\alpha^2 - 4Ha\alpha + a^2 + \varepsilon, \tag{2.13}$$

when $H^2 \neq \varepsilon$,

$$\alpha'^2 = 4(H^2 - \varepsilon)\left(\alpha - \frac{aH}{2(H^2 - \varepsilon)}\right)^2 + \frac{\varepsilon H^2 - 1 - \varepsilon a^2}{(H^2 - \varepsilon)}.$$

We solve the equation (2.13) in the following cases.
Case one: $\varepsilon = 1$.
In this case, (2.13) becomes

$$\alpha'^2 = 4(H^2 - 1)\alpha^2 - 4Ha\alpha + a^2 + 1. \tag{2.14}$$

We consider the following subcases respect to the value of H.
(a) When $H^2 = 1$, (2.14) becomes $\alpha'^2 = -4Ha\alpha + a^2 + 1$. Solving this equation we get

$$\begin{cases} \alpha(u) &= \text{constant}, \qquad\qquad \text{when} \quad a = 0 \\ \alpha(u) &= \frac{1}{4Ha}(a^2 + 1 - 4a^2u^2) \\ &= \frac{H}{4a}(a^2 + 1 - 4a^2u^2), \quad \text{when} \quad a \neq 0 \end{cases} \tag{2.15}$$

where, and in the following, we take the integral constant as zero.
(b) When $H^2 > 1$, solving the equation (2.14) we get
(i) if $a^2 > H^2 - 1$,

$$\alpha(u) = \frac{aH}{2(H^2 - 1)} + \frac{\sqrt{a^2 - (H^2 - 1)}}{2(H^2 - 1)} \cosh(2\sqrt{H^2 - 1}u); \tag{2.16}$$

(ii) if $a^2 = H^2 - 1$,

$$\alpha(u) = \frac{H}{2a} + e^{2au}; \tag{2.17}$$

(iii) if $a^2 < H^2 - 1$,

$$\alpha(u) = \frac{aH}{2(H^2 - 1)} + \frac{\sqrt{H^2 - 1 - a^2}}{2(H^2 - 1)} \sinh(2\sqrt{H^2 - 1}u). \tag{2.18}$$

(c) When $H^2 < 1$, solving the equation (2.14) we get

$$\alpha(u) = \frac{aH}{2(H^2 - 1)} + \frac{\sqrt{a^2 + 1 - H^2}}{2(1 - H^2)} \sin(2\sqrt{1 - H^2}u). \tag{2.19}$$

Case two: $\varepsilon = -1$.
In this case, (2.13) becomes

$$\alpha'^2 = 4(H^2+1)\alpha^2 - 4Ha\alpha + a^2 - 1. \tag{2.20}$$

Solving this equation we get
(i) if $a^2 > H^2+1$,

$$\alpha(u) = \frac{aH}{2(H^2+1)} + \frac{\sqrt{a^2-(H^2+1)}}{2(H^2+1)}\sinh(2\sqrt{H^2+1}u); \tag{2.21}$$

(ii) if $a^2 = H^2+1$,

$$\alpha(u) = \frac{H}{2a} + e^{2au}; \tag{2.22}$$

(iii) if $a^2 < H^2+1$,

$$\alpha(u) = \frac{aH}{2(H^2+1)} + \frac{\sqrt{H^2+1-a^2}}{2(H^2+1)}\cosh(2\sqrt{H^2+1}u). \tag{2.23}$$

Therefore we obtain

Theorem 2.1. *In 3-dimensional de Sitter space* $\mathbb{S}^3_1$,

(i) *the spacelike hyperbolic rotation surface of type* $\mathbf{M}_1$ *is congruent to one (or a part) of the following surfaces:*

$$r(u,v) = (a\sin(u), a\cos(u), b\sinh(v), b\cosh(v)), \quad u \in [0,2\pi], v \in \mathbb{R}, \tag{2.24}$$

where a *and* b *are constants;*

$$\begin{cases} r(u,v) &= (x(u), y(u), w(u)\sinh(v), w(u)\cosh(v)), \quad u \in I, v \in \mathbb{R}, \\ x(u) &= (w(u)^2+1)^{\frac{1}{2}}\cos\varphi(u), \\ y(u) &= (w(u)^2+1)^{\frac{1}{2}}\sin\varphi(u), \\ \varphi(u) &= \int_0^u \frac{(w(t)^2+w'(t)^2+1)^{\frac{1}{2}}}{(w(t)^2+1)}\mathrm{dt}, \\ w(u) &= \left(\frac{H}{4a}(a^2+1-4a^2u^2) - \frac{1}{2}\right)^{\frac{1}{2}}, \end{cases} \tag{2.25}$$

where $a \neq 0$ *is constant and* $H^2 = 1$;

$$\begin{cases} r(u,v) &= (x(u), y(u), w(u)\sinh(v), w(u)\cosh(v)), \quad u \in I, v \in \mathbb{R}, \\ x(u) &= (w(u)^2+1)^{\frac{1}{2}}\cos\varphi(u), \\ y(u) &= (w(u)^2+1)^{\frac{1}{2}}\sin\varphi(u), \\ \varphi(u) &= \int_0^u \frac{(w(t)^2+w'(t)^2+1)^{\frac{1}{2}}}{(w(t)^2+1)}\mathrm{dt}, \\ w(u) &= \left(\frac{aH}{2(H^2-1)} + \frac{\sqrt{a^2-(H^2-1)}}{2(H^2-1)}\cosh(2\sqrt{H^2-1}u) - \frac{1}{2}\right)^{\frac{1}{2}}, \end{cases} \tag{2.26}$$

where a and H are constants, $a^2 > H^2 - 1 > 0$;

$$(2.27)\qquad \begin{cases} r(u,v) & = (x(u), y(u), w(u)\sinh(v), w(u)\cosh(v)), \quad u \in I, v \in \mathbb{R}, \\ x(u) & = (w(u)^2+1)^{\frac{1}{2}} \cos\varphi(u), \\ y(u) & = (w(u)^2+1)^{\frac{1}{2}} \sin\varphi(u), \\ \varphi(u) & = \int_0^u \frac{(w(t)^2+w'(t)^2+1)^{\frac{1}{2}}}{(w(t)^2+1)} \mathrm{dt}, \\ w(u) & = \left(\frac{H}{2a} + e^{2au} - \frac{1}{2}\right)^{\frac{1}{2}}, \end{cases}$$

where a and H are constants, $a^2 = H^2 - 1 > 0$

$$(2.28)\qquad \begin{cases} r(u,v) & = (x(u), y(u), w(u)\sinh(v), w(u)\cosh(v)), \quad u \in I, v \in \mathbb{R}, \\ x(u) & = (w(u)^2+1)^{\frac{1}{2}} \cos\varphi(u), \\ y(u) & = (w(u)^2+1)^{\frac{1}{2}} \sin\varphi(u), \\ \varphi(u) & = \int_0^u \frac{(w(t)^2+w'(t)^2+1)^{\frac{1}{2}}}{(w(t)^2+1)} \mathrm{dt}, \\ w(u) & = \left(\frac{aH}{2(H^2-1)} + \frac{\sqrt{H^2-1-a^2}}{2(H^2-1)} \sinh(2\sqrt{H^2-1}u) - \frac{1}{2}\right)^{\frac{1}{2}}, \end{cases}$$

where a and H are constants, $0 < a^2 < H^2 - 1$;

$$(2.29)\qquad \begin{cases} r(u,v) & = (x(u), y(u), w(u)\sinh(v), w(u)\cosh(v)), \quad u \in I, v \in \mathbb{R}, \\ x(u) & = (w(u)^2+1)^{\frac{1}{2}} \cos\varphi(u), \\ y(u) & = (w(u)^2+1)^{\frac{1}{2}} \sin\varphi(u), \\ \varphi(u) & = \int_0^u \frac{(w(t)^2+w'(t)^2+1)^{\frac{1}{2}}}{(w(t)^2+1)} \mathrm{dt}, \\ w(u) & = \left(\frac{aH}{2(H^2-1)} + \frac{\sqrt{a^2+1-H^2}}{2(1-H^2)} \sin(2\sqrt{1-H^2}u) - \frac{1}{2}\right)^{\frac{1}{2}}, \end{cases}$$

where a and H are constants, $H^2 - 1 < 0$;

(ii) *the timelike hyperbolic rotation surface of type $\mathbf{M}_1$ is congruent to one (or a part) of the following surfaces*:

$$(2.30)\qquad \begin{cases} r(u,v) & = (x(u), y(u), w(u)\sinh(v), w(u)\cosh(v)), \quad u \in I, v \in \mathbb{R}, \\ x(u) & = (w(u)^2+1)^{\frac{1}{2}} \cos\varphi(u), \\ y(u) & = (w(u)^2+1)^{\frac{1}{2}} \sin\varphi(u), \\ \varphi(u) & = \int_0^u \frac{(-w(t)^2+w'(t)^2-1)^{\frac{1}{2}}}{(w(t)^2+1)} \mathrm{dt}, \\ w(u) & = \left(\frac{aH}{2(H^2+1)} + \frac{\sqrt{a^2-(H^2+1)}}{2(H^2+1)} \sinh(2\sqrt{H^2+1}u) - \frac{1}{2}\right)^{\frac{1}{2}}, \end{cases}$$

where a and H are constants, $a^2 > H^2 + 1$;

$$(2.31)\qquad \begin{cases} r(u,v) & = (x(u), y(u), w(u)\sinh(v), w(u)\cosh(v)), \quad u \in I, v \in \mathbb{R}, \\ x(u) & = (w(u)^2+1)^{\frac{1}{2}} \cos\varphi(u), \\ y(u) & = (w(u)^2+1)^{\frac{1}{2}} \sin\varphi(u), \\ \varphi(u) & = \int_0^u \frac{(-w(t)^2+w'(t)^2-1)^{\frac{1}{2}}}{(w(t)^2+1)} \mathrm{dt}, \\ w(u) & = \left(\frac{H}{2a} + e^{2au} - \frac{1}{2}\right)^{\frac{1}{2}}, \end{cases}$$

where a and H are constants, $a^2 = H^2 + 1$;

$$\text{(2.32)}\qquad \begin{cases} r(u,v) &= (x(u), y(u), w(u)\sinh(v), w(u)\cosh(v)), \quad u \in I, v \in \mathbb{R}, \\ x(u) &= (w(u)^2+1)^{\frac{1}{2}}\cos\varphi(u), \\ y(u) &= (w(u)^2+1)^{\frac{1}{2}}\sin\varphi(u), \\ \varphi(u) &= \int_0^u \frac{(-w(t)^2+w'(t)^2-1)^{\frac{1}{2}}}{(w(t)^2+1)}\mathrm{dt}, \\ w(u) &= \left(\frac{aH}{2(H^2+1)} + \frac{\sqrt{H^2+1-a^2}}{2(H^2+1)}\cosh(2\sqrt{H^2+1}u) - \frac{1}{2}\right)^{\frac{1}{2}}, \end{cases}$$

where a and H are constants, $a^2 < H^2 + 1$.

3 Constant mean curvature surfaces of type M_2

For the surface $\mathbf{M}_2$, it has constant mean curvature $H \neq 0$ in $\mathbb{S}^3_1$ if and only if on the interval I the following relations hold:

$$\text{(3.1)}\qquad x(u)^2 + y(u)^2 + w(u)^2 = 1$$

$$\text{(3.2)}\qquad x'(u)^2 + y'(u)^2 + w'(u)^2 = 1$$

$$\text{(3.3)}\qquad 2Hw = w^2(x''y' - y''x') - ww'(x''y - y''x) + (ww'' - 1)(x'y - y'x).$$

Now we solve the above equations. From (3.1) we may put

$$\text{(3.4)}\qquad \begin{cases} x(u) = (1-w(u)^2)^{\frac{1}{2}}\cos\varphi(u), \\ y(u) = (1-w(u)^2)^{\frac{1}{2}}\sin\varphi(u), \end{cases} \quad |w(u)| < 1$$

and by (3.2) we get

$$\text{(3.5)}\qquad \varphi'(u)^2 = \frac{1-w^2-w'^2}{(1-w^2)^2}.$$

We assume that $1 - w^2 - w'^2 > 0$ on I (when $1 - w^2 - w'^2 = 0$, φ is constant). Therefore the function $\varphi(u)$ is of the form

$$\text{(3.6)}\qquad \varphi(u) = \pm\int_0^u \frac{(1-w(t)^2-w'(t)^2)^{\frac{1}{2}}}{(1-w(t)^2)}\mathrm{dt}.$$

and without loss of generality we may assume that the signature is positive.

From (3.4) and (3.6), we can show that

$$\text{(3.7)}\qquad x'y - y'x = -(1-w^2)\varphi' = -(1-w^2-w'^2)^{\frac{1}{2}},$$

$$\text{(3.8)}\qquad x''y - y''x = (x'y-y'x)' = -(ww' + w'w'')/(x'y - y'x).$$

Differentiating (3.1) and (3.2) we obtain

$$\begin{aligned} xx' + yy' &= -ww', \\ xx'' + yy'' &= -ww'' - 1, \\ x'x'' + y'y'' &= -w'w''. \end{aligned}$$

Solving above equations for x'' and y'' we get

$$(x'y - y'x)x'' = y'(ww''+1) - yw'w'',$$
$$(x'y - y'x)y'' = -x'(ww''+1) + xw'w''.$$

So

$$x''y' - y''x' = (1 + ww'' - w'^2)/(x'y - y'x). \tag{3.9}$$

Putting (3.7), (3.8) and (3.9) into (3.3), then we get

$$ww'' + w'^2 + 2w^2 - 1 = -2Hw(1 - w^2 - w'^2)^{\frac{1}{2}}. \tag{3.10}$$

Without loss of generality, we can assume that $w(u) > 0$. When $w(u) \neq$constant, let $\alpha(u) = w^2 - \frac{1}{2}$, then (3.10) becomes

$$\alpha'' + 4\alpha = -2H(1 - \alpha'^2 - 4\alpha^2)^{\frac{1}{2}}. \tag{3.11}$$

Since $w(u) \neq$constant, $\alpha' \not\equiv 0$. From (3.11) we have

$$\frac{-\frac{1}{2}\mathrm{d}(1 - \alpha'^2 - 4\alpha^2)}{(1 - \alpha'^2 - 4\alpha^2)^{\frac{1}{2}}} = -2H\alpha'\mathrm{du}.$$

Then

$$(1 - \alpha'^2 - 4\alpha^2)^{\frac{1}{2}} = a + 2H\alpha, \quad a + 2H\alpha > 0, \tag{3.12}$$

where a is integral constant. From (3.12) we get

$$\begin{aligned} \alpha'^2 &= -4(H^2+1)\alpha^2 - 4Ha\alpha + 1 - a^2 \\ &= 4(H^2+1)\left(\frac{H^2+1-a^2}{4(H^2+1)^2} - \left(\alpha + \frac{aH}{2(H^2+1)}\right)^2\right), \end{aligned} \tag{3.13}$$

therefore, $a^2 < H^2 + 1$. Solving (3.13) we get

$$\alpha(u) = \frac{-aH}{2(H^2+1)} + \frac{\sqrt{H^2+1-a^2}}{2(H^2+1)}\sin(2\sqrt{H^2+1}u), \tag{3.18}$$

where, we take the integral constant as zero. Therefore we obtain

Theorem 3.1. *In 3-dimensional de Sitter space $\mathbb{S}^3_1$, the hyperbolic rotation surface of type $\mathbf{M}_2$ is timelike and congruent to the following surface (or a part):*

$$r(u,v) = (a\sin(u), a\cos(u), b\cosh(v), b\sinh(v)), \quad u \in [0, 2\pi], v \in \mathbb{R}, \tag{3.19}$$

where a and b are constants;

$$\begin{cases} r(u,v) &= (x(u), y(u), w(u)\cosh(v), w(u)\sinh(v)), \quad u \in I, v \in \mathbb{R}, \\ x(u) &= (1 - w(u)^2)^{\frac{1}{2}}\cos\varphi(u), \\ y(u) &= (1 - w(u)^2)^{\frac{1}{2}}\sin\varphi(u), \\ \varphi(u) &= \int_0^u \frac{(1-w(t)^2-w'(t)^2)^{\frac{1}{2}}}{(1-w(t)^2)}\mathrm{dt}, \\ w(u) &= \left(\frac{-aH}{2(H^2+1)} + \frac{\sqrt{H^2+1-a^2}}{2(H^2+1)}\sin(2\sqrt{H^2+1}u) + \frac{1}{2}\right)^{\frac{1}{2}}, \end{cases} \tag{3.20}$$

where a and H are constants, $a^2 < H^2 + 1$.

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