Factorization in some ideals of Lau algebras with applications to semigroup algebras

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Abstract

Let A be a Lau algebra with a bounded approximate identity, and let $I_0(A)$ denote the kernel of the identity element in the dual W^*-algebra of A. Here, we show that $I_0(A) = I_0(A)^2$. We also give some applications of this result to closed ideals of codimension one in Banach algebras related to locally compact semigroups.

Introduction

Recall that a *Lau algebra* (Same as F-algebra in Lau [6]) is a complex Banach algebra A which is the (unique) predual of a W^*-algebra M as a Banach space and that the identity element u of M (which always exists) is a multiplicative linear functional on A (see [8], p. 82). In general, there is no connection between the multiplication in A and the multiplication in M, except for the fact that the identity element u of M is multiplicative on A. For every closed (two-sided) ideal I of a Lau algebra A, define I^2 to be the set of all elements of I which are of the form $\sum_{k=1}^{n} b_k c_k$ for some $n \in \mathbf{N}$ and $b_k, c_k \in I$ $(k = 1, ..., n)$, and write $I_0(I) := I \cap I_0(A)$, where $I_0(A)$ denotes the maximal closed ideal $\{a \in A : u(a) = 0\}$ of A.

Let G be a locally compact group. The measure algebra $M(G)$ and its closed ideal $M_a(G)$, the isomorphic image of the group algebra $L^1(G)$ in $M(G)$, are examples of Lau algebras. It is shown by Willis in [9] that $I_0(L^1(G)) = I_0(L^1(G))^2$ and $I_0(M(G)) = I_0(M(G))^2$. In this work, we prove that if I is a closed ideal of a

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Lau algebra A and each of A and I has a bounded approximate identity, then $I_0(I) = I_0(I)^2$; in particular, $I_0(A) = I_0(A)^2$. Finally, we offer an application of this result to the closed ideals of codimension one in semigroup algebras of a large class of locally compact semigroups.

1 The main result

We commence this section with the following proposition.

Proposition 1.1. *Let A be a Lau algebra such that $I_0(A) = I_0(A)^2$ and let I be a closed ideal of A with a bounded approximate identity. Then $I_0(I) = I_0(I)^2$.*

Proof. We must show that $I_0(I) \subseteq I_0(I)^2$. To observe this, let a be an arbitrary element of $I_0(I)$. Then, by Cohen's factorization theorem [2, Theorem 11.10], there exist $x, z \in I$ and $y \in I_0(I)$ such that $a = xyz$. Now, by assumption, y is of the form $\sum_{k=1}^n b_k c_k$, where $b_k, c_k \in I_0(A)$ for $k = 1, ..., n$. We therefore have $a = \sum_{k=1}^n (xb_k)(c_k z)$ which shows that $a \in I_0(I)^2$ as required. ∎

Before we present the main result of this paper, for each Lau algebra A, let $P(A)$ denote the set of all elements in A that induce positive linear functionals on the dual W^*-algebra of A, and $P_1(A)$ denote the set of all $a \in P(A)$ with $\| a \| = 1$; note that $P(A)$ coincides with the set of all $a \in A$ such that $u(a) = \| a \|$.

Theorem 1.2. *Let A be a Lau algebra with a bounded approximate identity. Then $I_0(A) = I_0(A)^2$.*

Proof. We consider two cases:

Case (i). Suppose that A has an identity. We follow an idea contained in the proof of [9, Theorem]. We first recall that since A is the predual of a W^*-algebra M, any element a in A that induces a self-adjoint linear functional on M can be expressed in exactly one way as the difference of two elements a^+, a^- in $P(A)$ with $\| a \| = \| a^+ \| + \| a^- \|$ (the orthogonal decomposition of a). This implies that $P_1(A)$ spans A as a vector space.

Now, let $a \in I_0(A)$. Then we can write $a = \sum_{k=1}^n t_k a_k$, where $a_k \in P_1(A)$ and $t_k \in \mathbf{C}$ $(k = 1, ..., n)$. By [4, Lemma 2.1.3], there exist $b_k \in A$ such that $b_k^2 = e - a_k$ $(k = 1, ..., n)$, where e denotes the identity of A. Indeed, for each $k = 1, ..., n$, the sequence with the general term

$$1 + \frac{1}{2}(-a_k) + \frac{1}{2!}\frac{1}{2}(\frac{1}{2} - 1)(-a_k)^2 + ... + \frac{1}{m!}\frac{1}{2}(\frac{1}{2} - 1)...(\frac{1}{2} - m + 1)(-a_k)^m$$

$(m \in \mathbf{N})$ converges to b_k as m increase. Then $b_1, ..., b_n \in I_0(A)$ and, since $u(a) = 0$, we conclude that

$$a = \sum_{k=1}^n (-t_k)(e - a_k) = \sum_{k=1}^n (-t_k) b_k^2.$$

This implies that $a \in I_0(A)^2$. So we have showed that $I_0(A) \subseteq I_0(A)^2$. The other set inclusion is trivial.

Case (ii). Suppose that A has no identity. Let B be the usual unitization $A+\mathbf{C}$ of A; see [2]. Then B is a Lau algebra [6, Proposition 3.6], and hence $I_0(B)=I_0(B)^2$ by Case (i). Now, since B contains A as a closed ideal, it follows from Proposition 1.1 that $I_0(A)=I_0(A)^2$. ∎

A combination of the above results leads us to the following.

Corollary 1.3. *Let A be a Lau algebra and let I be a closed ideal of A. Suppose that each of A and I has a bounded approximate identity. Then $I_0(I)=I_0(I)^2$.*

Remark 1.4. Let A be a Lau algebra. Then $I_0(A)=I_0(A)^2$ if A has the property that for every $a\in I_0(A)$ there exist $x,z\in A$ and $y\in I_0(A)$ such that $a=xyz$. In fact, this is just the property of a Lau algebra A which is needed in the proof of Theorem 1.2 to obtain $I_0(A)=I_0(A)^2$.

In view of Cohen's factorization theorem, the existence of a bounded approximate identity for A is a necessary condition for that A has this property. Following [7], one can give an example of a commutative semisimple Lau algebra which has this property but does not have an (even unbounded) approximate identity. In particular, this example shows that the converse of Theorem 1.2 is not valid in general.

We recall that a Lau algebra A is called *left amenable* if there exists a net (a_α) in $P_1(A)$ such that $\| aa_\alpha - a_\alpha \| \to 0$ for all $a\in P_1(A)$. The concept of left amenability for Lau algebras was introduced and studied by Lau in [6].

Proposition 1.5. *Let A be a left amenable Lau algebra with a bounded right approximate identity. Then $I_0(A)=I_0(A)^2$.*

Proof. This follows from Cohen's factorization theorem and the fact that a Lau algebra A is left amenable and has a bounded right approximate identity if and only if $I_0(A)$ has a bounded right approximate identity [6, Theorem 4.10]. ∎

2 Applications to semigroup algebras

Throughout this section, S denotes a locally compact topological semigroup. The measure algebra $M(S)$ of S defines a Lau algebra which the identity element of the dual of $M(S)$ is the multiplicative linear functional $\mu\mapsto\mu(S)$ $(\mu\in M(S))$; see [10, Theorem 2.2]. Hence, using Theorem 1.2, we have the following generalization of [9, Corollary] from locally compact groups to all locally compact topological semigroups with identity.

Proposition 2.1. *If S has an identity, then $I_0(M(S))=I_0(M(S))^2$.*

Now, let $M_a(S)$ or $\tilde{L}(S)$ denote the set of all measures $\mu\in M(S)$ for which the mappings $x\mapsto\delta_x * |\mu|$ and $x\mapsto |\mu| * \delta_x$ from S into $M(S)$ are weakly continuous, where δ_x denotes the Dirac measure at $x\in X$ and $*$ denotes the convolution product of $M(S)$. Then $M_a(S)$ is a closed ideal of $M(S)$ [1, Theorem 2.6]. S is called a *foundation semigroup* if the closure of $\bigcup\{\mathrm{supp}(\mu) : \mu\in M_a(S)\}$ coincides with S.

We recall that a *semicharacter* on S is a non-zero complex-valued function ρ on S such that $\rho(xy) = \rho(x)\ \rho(y)$ for all $x, y \in S$. Observe that for each bounded and continuous semicharacter ρ on S, $\mid \rho(x) \mid \leq 1$ for all $x \in S$, and the set $I_\rho := \{\mu \in M_a(S) : (\rho\mu)(S) = 0\}$ defines a closed ideal in $M_a(S)$.

Lemma 2.2. *Let S be a foundation semigroup with identity. Then*

(a) $M_a(S)$ is a Lau algebra with a bounded approximate identity contained in $P_1(M_a(S))$.

(b) For any bounded and continuous semicharacters ρ on S, the closed ideal I_ρ has codimension one in $M_a(S)$. Conversely, every closed ideal of codimension one in $M_a(S)$ is of the form I_ρ for some bounded and continuous semicharacters ρ on S.

Proof. (a) This is a spacial case of Lemma 3 in [5].

(b) Consider a bounded and continuous semicharacter ρ on S. Then, since ρ is non-zero and S is a foundation semigroup, it follows easily that $I_\rho \neq M_a(S)$. That is the closed ideal I_ρ has codimension one in $M_a(S)$.

In order to prove the second part, Let I be a closed ideal of codimension one in $M_a(S)$, and recall that I is the kernel of a non-zero multiplicative linear functional h on $M_a(S)$. So if we choose a positive measure $\nu_0 \in M_a(S)$ with $h(\nu_0) \neq 0$, and define the function ρ on S by $\rho(x) = h(\delta_x * \nu_0)/h(\nu_0)$ for all $x \in S$, then ρ is a bounded and continuous function on S by the definition of $M_a(S)$. We also have

$$h(\mu) = (\rho\mu)(S) \quad \text{for all } \ \mu \in M_a(S); \tag{1}$$

this follows from the fact that $f(\mu * \nu) = \int_S f(\delta_x * \nu)\, d\mu(x)$ for all $f \in M_a(S)^*$ and $\mu, \nu \in M_a(S)$ [1, Lemma 2.2]. On the other hand, it is well-known that if ϕ is a bounded and continuous function on S, then ϕ is multiplicative on S if and only if the linear functional $\mu \mapsto (\phi\mu)(S)$ is multiplicative on $M_a(S)$ [5, Proposition 4]. So, invoking (1), we conclude that ρ is also a semicharacter on S and $I = I_\rho$ as required. ∎

Proposition 2.3. *Let S be a foundation semigroup with identity. Suppose that ρ is a bounded and continuous semicharacter on S such that $\mid \rho(x) \mid = 1$ for all $x \in S$. Then $I_\rho = I_\rho^2$. In particular, $I_0(M_a(S)) = I_0(M_a(S))^2$.*

Proof. Since the mapping $\mu \longmapsto \rho\mu$ defines an isomorphism of I_ρ onto $I_0(M_a(S)) = \{\mu \in M_a(S) : \mu(S) = 0\}$, the result follows from Theorem 1.2 and Lemma 2.2 (a). ∎

As a consequence of Lemma 2.2 (b) and Proposition 2.3 we obtain

Corollary 2.4. (Willis [9]) *Let G be a locally compact group, and let I be a closed ideal of codimension one in the group algebra $L^1(G)$. Then $I = I^2$.*

Remark 2.5. If G is a locally compact non-discrete abelian group, then there is a closed ideal I of codimension one in $M(G)$ such that $I \neq I^2$ [3]. So the above corollary is not valid, in general, for an arbitrary Lau algebra instead of $L^1(G)$.

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