On Solutions to Formal Equations

Tejinder S. Neelon

Abstract

Let $\overline{k}$ be a field of characteristic zero equipped with an absolute value $|\cdot|$. Let $\phi_1(\mathbf{x},\mathbf{y}) = \phi_2(\mathbf{x},\mathbf{y}) = \ldots = \phi_l(\mathbf{x},\mathbf{y}) = 0$ be a system of formal power series equations in variables $\mathbf{x} = (x_{1,} \ldots x_n)$, $\mathbf{y} = (y_{1,} \ldots y_m)$ with coefficients in $\overline{k}$. The notion of $\{M_k\}$-summability of formal power series is defined relative to a sequence $\{M_k\}_{k=0}^{\infty}$ of positive real numbers. Under certain Jacobian conditions on the ϕ_i's, it is shown the $\{M_k\}$-summability of the ϕ_i's implies $\{M_k\}$-summability of any of its formal power series solutions $\mathbf{y} = \mathbf{f}(\mathbf{x})$. In particular, if the ϕ_i's are convergent, then so are its formal solutions. This result generalizes the author's earlier work on formal solutions of systems of analytic equations.

1 Introduction

It is well known that a formal power series solution of a nonzero convergent power series equation is convergent. In [11], the author proved a generalization of this result to systems of equations. A natural question, then, is what kind of properties of formal equations are preserved in their formal solutions? In this note we consider properties of systems of equations which are more general than convergence.

Let $\overline{k}$ be field of characteristic zero equipped with an absolute value $|\cdot|$. Let $\mathfrak{F}_n = \mathfrak{F}(\mathbf{x})$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, denote the ring of formal power series in n variables with coefficients in $\overline{k}$. Let $\{M_k\}$ be a sequence of positive real numbers satisfying

$$M_k^2 \leq M_{k-1}M_{k+1}, \forall k, \quad \text{(logarithmic convexity)} \tag{1}$$

Received by the editors June 1999.
Communicated by R. Delanghe.
1991 *Mathematics Subject Classification :* Primary 13B40, 32A05; Secondary 30G30, 40G99 .
Key words and phrases : Formal solutions, $\{M_k\}$-summable series, ultradifferentiable functions, the implicit function theorem, Gevrey expansions.

$$\exists\rho > 1, M_k \leq \rho^k M_{k-1}, \forall k. \ \text{(differentiablity)} \tag{2}$$

We say that a formal power series $f(\mathbf{x}) = \sum_\alpha f_\alpha x^\alpha \in \mathfrak{F}_n$ is $\{M_k\}$ -summable if there are constants $C = C_f > 0, R = R_f > 0$ such that

$$|f_\alpha| \leq CR^{|\alpha|}M_{|\alpha|}, \forall\alpha \in \mathbb{N}^n, |\alpha| = \alpha_1 + ... + \alpha_n. \tag{3}$$

Let $\mathfrak{F}_n\{M_k\}$ or $\mathfrak{F}(\mathbf{x})\{M_k\}$ denote the subset of $\mathfrak{F}_n$ consisting of all $\{M_k\}$-summable series. The condition (1) implies, see [12],

$$M_k M_j \leq M_{j+k}, \forall j, k, \tag{4}$$

which in turn implies that $\mathfrak{F}_n\{M_k\}$ forms a subring of $\mathfrak{F}_n$. (2) makes $\mathfrak{F}_n\{M_k\}$ closed under the formal differentiation of power series. Since $\mathfrak{F}_n\{M_k\} = \mathfrak{F}_n\{\lambda M_k\}$ for any $0 \neq \lambda \in \overline{k}$, we may assume that $M_0 = 1$. Clearly $\mathfrak{F}_n\{M_k\}$ contains the polynomial ring over $\overline{k}$. The elements of $\mathfrak{F}_n\{(k!)^\nu\}, \nu > 0$, are called Gevrey expansions of order ν. The ring $\mathfrak{F}_n\{k!\}$ is precisely the ring of convergent power series. The Gevrey expansions are important in asymptotic analysis and the theory of multisummable series. See ([1],[13]). Another motivation for studying $\{M_k\}$-summable series comes from the ultradifferentiable function theory: A function $f \in C^\infty(\mathbb{R}^n)$ is said to be in the ultradifferentiable class $C\{M_k\}(\mathbb{R}^n)$ if for every compact set $K \subset \mathbb{R}^n$ there are constants $C > 0$, $R > 0$ such that $|\partial^\alpha f(\mathbf{x})| \leq CR^{|\alpha|}M_{|\alpha|}, \forall\alpha \in \mathbb{N}^n, \mathbf{x} \in K$. Ultradifferentiable classes occur naturally in partial differential equations and harmonic analysis. For a detailed study of ultradifferentiable functions see [7]. If the Taylor expansion (at p) map $T_p : C\{M_k\}(\mathbb{R}^n) \to \mathfrak{F}_n\{M_k\}$ is an injective ring homomorphism for every p, then the class $C\{M_k\}(\mathbb{R}^n)$ is called quasi-analytic.. The problem of determining conditions on $\{M_k\}$ under which this Taylor map is surjective for every p is known as the Carlson problem. See e.g.[8].

In Section 2, the implicit function theorem for the ring $\mathfrak{F}_n\{M_k\}$ is established. The method of proof is constructive and fairly standard, see e.g. [1],[5]. The author is not aware of any constructive proof in the literature of even the formal implicit function theorem, so, for the sake of self containment of the exposition, the detailed proof for the implicit function theorem for $\{M_k\}$-summable series is given here. The main result, Theorem 3.1 is proved in Section 3. As corollaries of this theorem, we obtain generalizations to $\{M_k\}$-summable systems of various known results about analytic systems. The author wishes to thank Professor V. Thilliez for providing an example that shows that Theorem 3.1 does not generalize to systems of C^∞ equations. Section 4 contains this example and also some remarks.

2 Basic Algebra; the Implicit Function Theorem

General references for this section are [3] and [4]. For $\xi = \left(\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(n)}\right) \in \overline{k}^n$, put $\|\xi\| = \max_{1\leq j\leq n}\left|\xi^{(j)}\right|$. For $f \in \mathfrak{F}_n$, we write

$$f(\mathbf{x}) := \sum_{\mu=0}^{\infty} f^{(\mu)}(\mathbf{x}), \text{ where } f^{(\mu)}(\mathbf{x}) := \sum_{\alpha\in\mathbf{N}^n, |\alpha|=\mu} \binom{\mu}{\alpha} f_\alpha \mathbf{x}^\alpha. \tag{5}$$

Let $\widetilde{f}^{(\mu)}(\xi_1, \xi_2, \cdots, \xi_\mu)$ denote the unique symmetric μ-linear form such that $\widetilde{f}^{(\mu)}(\mathbf{x}, \mathbf{x}, \cdots, \mathbf{x}) = f^{(\mu)}(\mathbf{x})$. (The form $\frac{1}{\mu!}\widetilde{f}^{(\mu)}(\xi_1, \xi_2, \cdots, \xi_\mu)$ is called the μ-th formal

Fréchet derivative of f at 0.) It follows that $f \in \mathfrak{F}(\mathbf{x})\{M_k\}$ if and only if there are constants $C > 0$ and $R > 0$ such that

$$\left|f^{(\mu)}(\mathbf{x})\right| \leq CR^{\mu} M_{\mu} \|\mathbf{x}\|^{\mu}, \forall \mu \geq 0, \forall \mathbf{x} \in \overline{k}^{n}, \tag{6}$$

which is equivalent to

$$\left|\widetilde{f^{(\mu)}}(\xi_1, \xi_2, \cdots, \xi_\mu)\right| \leq CR^{\mu} M_{\mu} \|\xi_1\| \cdot \|\xi_2\| \cdot \cdots \cdot \|\xi_\mu\|, \forall \xi_1, \xi_2, \cdots, \xi_\mu \in \overline{k}^{n}. \tag{7}$$

Units in $\mathfrak{F}_n\{M_k\}$ are precisely the units in $\mathfrak{F}_n$:

Lemma 2.1. *$f \in \mathfrak{F}_n\{M_k\}$ is a unit $\Leftrightarrow f(0) \neq 0$.*

Proof. The necessity part is obvious. So we need only to prove the sufficiency part.

Let $g(\mathbf{x}) = \sum_{\mu=0}^{\infty} g_{(\mu)}(\mathbf{x})$ be a formal power series such that $g(\mathbf{x}) \cdot f(\mathbf{x}) = 1$. We will show that $g \in \mathfrak{F}\{M_k\}(\mathbf{x})$. We have

$$g^{(0)} \cdot f^{(0)} = 1, \text{ and } \sum_{i=0}^{k} g^{(i)}(\mathbf{x}) \cdot f^{(k-i)}(\mathbf{x}) = 0, k \geq 1.$$

Assume without loss of generality that $f^{(0)} = 1$. Let $C > 1, R > 0$ be such that (6) is satisfied. We claim that

$$\left|g^{(k)}(\mathbf{x})\right| \leq C\left(\left(C+1\right)R\right)^{k} M_k \|\mathbf{x}\|^{k}, \forall k. \tag{8}$$

(8) clearly holds for $k = 0$ since $g^{(0)} = 1$ and $M_0 = 1$. By induction on k, we have

$$\begin{aligned}
\left|g^{(k)}(\mathbf{x})\right| &\leq \sum_{i=0}^{k-1} \left|g^{(i)}(\mathbf{x})\right| \left|f^{(k-i)}(\mathbf{x})\right| \\
&\leq \sum_{i=0}^{k-1} C\left(\left(C+1\right)R\right)^{i} M_i C R^{k-i} M_{k-i} \|\mathbf{x}\|^{k} \\
&\leq C^2 R^k M_k \|\mathbf{x}\|^{k} \sum_{i=0}^{k-1} \left(C+1\right)^{i} \quad \text{(by (4))} \\
&\leq C\left(\left(C+1\right)R\right)^{k} M_k \|\mathbf{x}\|^{k}.
\end{aligned}$$

Lemma 2.2. *If $\Phi(\mathbf{x}) = (\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \cdots \phi_m(\mathbf{x})) \in \mathfrak{F}^m(\mathbf{x})\{M_k\}$ and $h(\mathbf{y}) \in \mathfrak{F}(\mathbf{y})\{M_k\}$, then $\Psi(\mathbf{x}) = h(\Phi(\mathbf{x})) \in \mathfrak{F}(\mathbf{x})\{M_k\}$.*

Proof. The homogenous polynomials $\psi^{(\nu)}$'s can computed as follows. See [2], [4].

$$\psi^{(1)}(\mathbf{x}) = \widetilde{h^{(1)}} \circ \phi^{(1)}(\mathbf{x}), \text{ and for } \nu > 1,$$

$$\psi^{(\nu)}(\mathbf{x}) = \sum_{\mu=1}^{\nu} \sum_{\alpha \in \mathbf{N}^{\mu}, |\alpha|=\nu} \widetilde{h}^{(\mu)}\left(\phi^{(\alpha_1)}(\mathbf{x}), \phi^{(\alpha_2)}(\mathbf{x}), \cdots, \phi^{(\alpha_\mu)}(\mathbf{x})\right). \tag{9}$$

By using (2), we may choose constants $C>0$ and $R>0$ such that (7) is satisfied for h and $\left|\phi_j^{(k)}(\mathbf{x})\right| \le CR^{k-1}M_{k-1}\left\|\mathbf{x}\right\|^k$, $\forall j, \forall k \ge 1$. By (9), we have

$$\begin{aligned}
\left|\psi^{(\nu)}(\mathbf{x})\right| &\le \sum_{\mu=1}^{\nu}\sum_{\alpha\in\mathbf{N}^{\mu},|\alpha|=\nu} CR^{\mu}M_{\mu}\prod_{1\le i\le\mu}\max_{1\le j\le m}\left|\phi_j^{(\alpha_i)}(\mathbf{x})\right| \\
&\le \sum_{\mu=1}^{\nu}\sum_{\alpha\in\mathbf{N}^{\mu},|\alpha|=\nu} CR^{\mu}M_{\mu}\prod_{1\le i\le\mu} CR^{\alpha_i-1}M_{\alpha_i-1}\left\|\mathbf{x}\right\|^{\alpha_i} \\
&\le CR^{\nu}M_{\nu}\left\|\mathbf{x}\right\|^{\nu}\sum_{\mu=1}^{\nu}\binom{\nu+\mu-1}{\mu-1}C^{\mu} \\
&\le C\left(CR\left\|\mathbf{x}\right\|\right)^{\nu}M_{\nu}\binom{2\nu}{\nu-1} \le C\left(4CR\right)^{\nu}M_{\nu}\left\|\mathbf{x}\right\|^{\nu}.
\end{aligned}$$

Now, we prove the Implicit Function Theorem.

Theorem 2.3. *Let* $\mathbf{F}(\mathbf{x},\mathbf{y}) \in \mathfrak{F}^m(\mathbf{x},\mathbf{y})\{M_k\}$, $\mathbf{x}=(x_1,x_2,\cdots,x_n)$, $\mathbf{y}=(y_1,y_2,\cdots,y_m)$. *If* $\mathbf{F}(\mathbf{0},\mathbf{0})=\mathbf{0}$ *and*

$$\det\left[\frac{\partial\mathbf{F}}{\partial\mathbf{y}}\right]_{m\times m}(0,0)\neq 0 \tag{10}$$

then there is a unique $\mathbf{g}(\mathbf{x}) \in \mathfrak{F}^m(\mathbf{x})\{M_k\}$, $\mathbf{g}(\mathbf{0})=\mathbf{0}$, *such that* $\mathbf{F}(\mathbf{x},\mathbf{g}(\mathbf{x}))\equiv\mathbf{0}$.

Proof. By the formal implicit function theorem, there is a unique $\mathbf{g}(\mathbf{x}) \in\mathfrak{F}^m(\mathbf{x})$ such that $\mathbf{F}(\mathbf{x},\mathbf{g}(\mathbf{x}))\equiv\mathbf{0}$. By differentiating $\mathbf{F}(\mathbf{x},\mathbf{g}(\mathbf{x}))\equiv\mathbf{0}$, which we are allowed to because of (2), and by using Lemma 2.1 along with the hypothesis (10), we see that $\mathbf{y}=\mathbf{g}(\mathbf{x})$ satisfies the matrix equation,

$$\left[\frac{\partial\mathbf{y}}{\partial\mathbf{x}}\right]_{\mathbf{m\times n}}\cdot\mathbf{x}=\left[\mathbf{H}(\mathbf{x},\mathbf{y})\right]_{m\times n}\cdot\mathbf{x}:=-\left[\frac{\partial\mathbf{F}}{\partial\mathbf{y}}\right]_{m\times m}^{-1}\cdot\left[\frac{\partial\mathbf{F}}{\partial\mathbf{x}}\right]_{m\times n}\cdot\mathbf{x},$$

where the entries of $[\mathbf{H}(\mathbf{x},\mathbf{y})]$ are in $\mathfrak{F}(\mathbf{x},\mathbf{y})\{M_k\}$.

By substituting $\mathbf{y}=\mathbf{g}(\mathbf{x})$ and by using the notation in (5), we rewrite this equation as

$$\sum_{\nu=1}^{\infty}\nu\mathbf{g}^{(\nu)}(\mathbf{x})=\sum_{\nu=1}^{\infty}\mathbf{H}^{(\nu-1)}(\mathbf{x},\mathbf{g}(\mathbf{x}))\cdot\mathbf{x}. \tag{11}$$

Put $\Phi(\mathbf{x})=(\mathbf{x},\mathbf{g}(\mathbf{x}))$. Let $C>1$ and $R>0$ be such that

$$\begin{aligned}
\left\|\widetilde{\mathbf{H}}^{(\mu)}(\xi_1,\xi_2,\cdots,\xi_\mu)\right\| &\le CR^{\mu}M_{\mu}\prod_{\nu=1}^{\mu}\left\|\xi_\nu\right\|,\ \forall\xi_1,\xi_2,\cdots,\xi_\mu\in\overline{k}^{n+m}, \\
\text{and }\left\|\mathbf{g}^{(1)}(\mathbf{x})\right\| &\le \frac{1}{2}C\left\|\mathbf{x}\right\|,\ \forall\mathbf{x}\in\overline{k}^{n}.
\end{aligned}$$

We claim that

$$\left\|\mathbf{g}^{(k)}(\mathbf{x})\right\|\le C\left[-CR\right]^{k-1}M_{k-1}\binom{\frac{1}{2}}{k}\left\|\mathbf{x}\right\|^k,\forall k. \tag{12}$$

We prove the claim by induction on k. Since (12) clearly holds for $k=1$, assume that it holds for $k \le \nu$. This implies that the coefficients $\phi_j^{(k)}(\mathbf{x})$ in the expansion of j-th component of Φ also satisfy (12) for $k \le \nu$. Now by applying (9), we have[1]

$$
\begin{aligned}
&\left\|(\nu+1)\mathbf{g}^{(\nu+1)}(\mathbf{x})\right\| \\
&\qquad \le \left[\sum_{\mu=1}^{\nu}\sum_{\alpha\in N^{\mu},|\alpha|=\nu} CR^{\mu}M_{\mu}\prod_{1\le i\le\mu}\max_{1\le j\le m}\left\|\phi_j^{(\alpha_i)}(\mathbf{x})\right\|\right]\|\mathbf{x}\| \\
&\le \left[\sum_{\mu=1}^{\nu}\sum_{\alpha\in N^{\mu},|\alpha|=\nu} CR^{\mu}M_{\mu}\prod_{1\le i\le\mu} C\left[-CR\right]^{\alpha_i-1}M_{\alpha_i-1}\binom{\frac12}{\alpha_i}\|\mathbf{x}\|^{\alpha_i}\right]\|\mathbf{x}\| \\
&\le \|\mathbf{x}\|^{\nu+1}\sum_{\mu=1}^{\nu}\sum_{\alpha\in N^{\mu},|\alpha|=\nu} CR^{\mu}M_{\mu}C^{\mu}\left[-CR\right]^{\nu-\mu}M_{\nu-\mu}\prod_{1\le i\le\mu}\binom{\frac12}{\alpha_i} \\
&\le C\left[CR\right]^{\nu}M_{\nu}\|\mathbf{x}\|^{\nu+1}\sum_{\mu=1}^{\nu}\sum_{\alpha\in N^{\mu},|\alpha|=\nu}\left[-1\right]^{\nu-\mu}\prod_{1\le i\le\mu}\binom{\frac12}{\alpha_i} \\
&\qquad = C\left[-CR\right]^{\nu}M_{\nu}(\nu+1)\binom{\frac12}{\nu+1}\|\mathbf{x}\|^{\nu+1}.
\end{aligned}
$$

The last equality follows from the next lemma.

Lemma 2.4. *(cf. [5])*

$$\sum_{\mu=1}^{\nu}\sum_{\alpha\in N^{\mu},|\alpha|=\nu}(-1)^{\nu-\mu}\prod_{1\le i\le\mu}\binom{\frac12}{\alpha_i} = (-1)^{\nu}(\nu+1)\binom{\frac12}{\nu+1}$$

Proof. Let

$$g(t)=\sum_{j=1}^{\infty}\binom{\frac12}{j}(-1)^{j-1}t^{j} \quad\text{and}\quad f(x)=\sum_{k=0}^{\infty}x^{k}.$$

Since $g(t)$ and $f(x)$ are Taylor expansions at 0 of functions $1-\sqrt{1-t}$ and $\frac{1}{1-x}$, respectively, we have $g'(t)=f[g(t)]$. By using (9), we see by comparing coefficients of t^{ν} that

$$
\begin{aligned}
(\nu+1)\binom{\frac12}{\nu+1}(-1)^{\nu} &= \sum_{\mu=1}^{k}\sum_{\alpha\in\mathbf{N}^{\mu},|\alpha|=\nu}\prod_{r=1}^{\mu}\binom{\frac12}{\alpha_r}(-1)^{\alpha_r-1} \\
&= \sum_{\mu=1}^{\nu}\sum_{\alpha\in\mathbf{N}^{\mu},|\alpha|=\nu}(-1)^{\nu-\mu}\prod_{r=1}^{\mu}\binom{\frac12}{\alpha_r}.
\end{aligned}
$$

3 Solutions of $\{M_k\}$-Summable Equations

Let $\mathfrak{I}\subset\mathfrak{F}(\mathbf{x},\mathbf{y})\{M_k\}$, $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, $\mathbf{y}=(y_1,y_2,\ldots,y_m)$, be a nonzero ideal. Define $\mathrm{Jac}_m(\mathfrak{I})$ to be the ideal in $\mathfrak{F}(\mathbf{x},\mathbf{y})\{M_k\}$ generated by $\mathfrak{I}$ and all $m\times m$ minors of Jacobians $\partial(\phi_1,\phi_2,\ldots,\phi_m)/\partial(\mathbf{x},\mathbf{y})$, where $\phi_1,\phi_2,\ldots,\phi_m\in\mathfrak{I}$. Set $\mathfrak{I}_0=\mathfrak{I}$, $\mathfrak{I}_{k+1}=Jac_m(\mathfrak{I}_k)$ for all $k\ge 0$, and $\mathfrak{I}_\infty=\cup_{k=0}^{\infty}\mathfrak{I}_k$. Since $\mathfrak{I}_k\subseteq\mathfrak{I}_{k+1},\forall k$, $\mathfrak{I}_\infty$ is

[1]We don't need to use the formal implicit function theorem for the existence of $\mathbf{g}$. Instead we can define $\mathbf{g}^{(\nu)}$'s inductively by using (11) and the formula (9).

an ideal. If $\mathbf{f} = (f_1, f_2, ..., f_m) \in \mathfrak{F}^m(\mathbf{x})$, $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, then $\mathbf{y} = \mathbf{f}(\mathbf{x})$ is called a formal solution of the ideal $\mathfrak{I}$ if

$$\phi(\mathbf{x}, \mathbf{f}(\mathbf{x})) \equiv 0, \forall \phi \in \mathfrak{I}. \tag{13}$$

Let $\mathcal{J}$ denote the ideal in $\mathfrak{F} = \mathfrak{F}(\mathbf{x}, \mathbf{y})$ generated by $y_j - f_j(\mathbf{x})$, $1 \le j \le m$.

Theorem 3.1. *If* $\mathbf{y} = \mathbf{f}(\mathbf{x}) \in \mathfrak{F}^m(\mathbf{x})$ *is a formal solution of a nonzero ideal* $\mathfrak{I} \subset \mathfrak{F}(\mathbf{x}, \mathbf{y})\{M_k\}$ *such that* $\mathfrak{I}_\infty \nsubseteq \mathcal{J}$, *then* $\mathbf{f} \in \mathfrak{F}(\mathbf{x})\{M_k\}$.

Proof. Without loss of generality we may assume that $\overline{k}$ is complete. Indeed, suppose that the result is true when $\overline{k}$ is complete. By treating the components of $\mathbf{f}(\mathbf{x})$, and the elements of $\mathfrak{I}$ as power series over the completion of $\overline{k}$, we can conclude that the components of $\mathbf{f}(\mathbf{x})$ satisfy (3). The variable transformation $(\mathbf{x}, \mathbf{y}) \to (\mathbf{x}, \mathbf{y} - \mathbf{f}(\mathbf{x}))$ shows that $\mathcal{J}$ is a prime ideal of height m, and by (13), we have $\mathfrak{I}_0 \subseteq \mathcal{J}$. Let $k \ge 0$ be the integer such that $\mathfrak{I}_k \subseteq \mathcal{J}$ but $\mathfrak{I}_{k+1} \nsubseteq \mathcal{J}$. Let the ideal $\widehat{\mathfrak{I}}_k$ in $\mathfrak{F}$ denote the completion of the ideal $\mathfrak{I}_k$, and let $\mathfrak{F}_{\mathcal{J}}$ denote the localization of the ring $\mathfrak{F} = \mathfrak{F}(\mathbf{x}, \mathbf{y})$ at $\mathcal{J}$. We have, see e.g. [15],

$$\mathfrak{I}_k \subseteq \mathcal{J} \;\Rightarrow\; \widehat{\mathfrak{I}}_k \subseteq \mathcal{J} \;\Rightarrow \mathrm{ht}(\widehat{\mathfrak{I}}_k \ \mathfrak{F}_{\mathcal{J}}) \le \mathrm{ht}(\mathcal{J}\mathfrak{F}_{\mathcal{J}}) = \mathrm{ht}(\mathcal{J}) = m.$$

Since $\mathfrak{I}_{k+1} \nsubseteq \mathcal{J}$, there exist $\phi_1, \phi_2, \ldots, \phi_m \in \mathfrak{I}_k$ such that the rank modulo $\mathcal{J}$ of $\mathrm{Jac}(\phi_1, \phi_2, \ldots, \phi_m)$ is m. Hence by the Jacobian Criterion for simple points, see e.g. [9, Theorem 30.4], we have $m \le \mathrm{ht}(\widehat{\mathfrak{I}}_k \mathfrak{F}_{\mathcal{J}})$. Hence $\mathrm{ht}(\widehat{\mathfrak{I}}_k \mathfrak{F}_{\mathcal{J}}) = \mathrm{ht}(\mathcal{J}\mathfrak{F}_{\mathcal{J}})$.

This implies that $\mathcal{J}$ must be a minimal prime ideal belonging to $\widehat{\mathfrak{I}}_k$. Now, it is a consequence of the Zariski-Nagata Theorem, see e.g. [6, p 89], that every minimal prime ideal belonging to the completion $\widehat{\mathfrak{I}_k}$ is of the form $\hat{\wp} = \wp\mathfrak{F}$ where $\wp$ is a minimal prime ideal belonging to $\mathfrak{I}_k$. Hence, there is a prime ideal $\wp$ in $\mathfrak{F}(\mathbf{x}, \mathbf{y})\{M_k\}$ such that $\mathfrak{I}_k \subseteq \wp$, and $\widehat{\wp} = \mathcal{J}$.

Let $p_1, p_2, \ldots, p_l \in \mathfrak{F}(\mathbf{x}, \mathbf{y})\{M_k\}$ be a set of generators of $\wp$. Then, since $\hat{\wp} = \wp\mathfrak{F} = \mathcal{J}$, there exist $a_{ik} \in \mathfrak{F}$, $1 \le i, k \le m$, such that

$$y_i - f_i(\mathbf{x}) = \sum_{k=1}^{l} a_{ik} p_k, \;\; 1 \le i \le m.$$

By differentiating the above equations with respect to y_j, for each j , $1 \le j \le m$, and by setting $(\mathbf{x}, \mathbf{y}) = 0$, we get the matrix equation

$$I_{m\times m} = (a_{ik}(0,0))_{m\times l} \cdot \left(\frac{\partial p_k}{\partial y_j}(0,0)\right)_{l\times m}.$$

Hence, by reordering the p_i's, if necessary, we may assume that the Jacobian determinant $\mathrm{Jac}_y(p_1, p_2, \ldots, p_m)(0,0) \neq 0$.

Now, Theorem 2.3 yields $\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}), \ldots, g_m(\mathbf{x})) \in \mathfrak{F}^m(\mathbf{x})\{M_k\}$ such that

$$p_i = \sum_{k=1}^{m} \alpha_{ik}(y_k - g_k), \;\; \alpha_{ik} \in \mathfrak{F}(\mathbf{x}, \mathbf{y})\{M_k\}, \;\; 1 \le i \le m.$$

Since $\alpha_{ik}(0,0) = \frac{\partial p_i}{\partial y_k}(0,0)$, the matrix (α_{ik}) is invertible, and hence

$$(p_1, p_2, \ldots, p_m)\,\mathfrak{F}(\mathbf{x}, \mathbf{y})\{M_k\} = (y_1 - g_1, \ldots, y_m - g_m)\,\mathfrak{F}(\mathbf{x}, \mathbf{y})\{M_k\}.$$

Since $(y_1 - g_1, \ldots, y_m - g_m)$ generates a prime ideal of height m contained in $\wp$ and $\mathrm{ht}(\wp) = \mathrm{ht}(\hat{\wp}) = m$ (see [15]), we have $\wp = (y_1 - g_1, \ldots, y_m - g_m)$. Since $\mathfrak{J} = (y - g)\mathfrak{F}$, there exist $b_{ik} \in \mathfrak{F}, 1 \leq i, k \leq m$, such that

$$y - f(\mathbf{x}) = B \cdot (y - g(\mathbf{x})), \;\; B = (b_{ik})_{m \times m}, .$$

By setting $\mathbf{y} = \mathbf{g}(\mathbf{x})$ in the above identity, we get $\mathbf{g} = \mathbf{f}$. ∎

Corollary 3.2. *Let* $\mathbf{f}(\mathbf{x}) \in \mathfrak{F}^m(\mathbf{x})$, $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, *and* $\Phi(\mathbf{x}, \mathbf{y}) = (\phi_1(\mathbf{x}, \mathbf{y}), \phi_2(\mathbf{x}, \mathbf{y}), \ldots, \phi_m(\mathbf{x}, \mathbf{y})) \in \mathfrak{F}(\mathbf{x}, \mathbf{y}) \{M_k\}$. *If the following two conditions are satisfied,*

$$\text{(i)} \;\; \Phi(\mathbf{x}, \mathbf{f}(\mathbf{x})) \in \mathfrak{F}(\mathbf{x}) \{M_k\},$$

$$\text{(ii)} \;\; \det \left[\frac{\partial \Phi}{\partial y}\right] (\mathbf{x}, \mathbf{f}(\mathbf{x})) \;\; \text{is a nonzero power series,}$$

then $\mathbf{f}(\mathbf{x}) \in \mathfrak{F}(\mathbf{x}) \{M_k\}$.

Proof. If $\mathfrak{J} = (\Phi(\mathbf{x}, \mathbf{y}) - \Phi(\mathbf{x}, \mathbf{f}(\mathbf{x})))$, then (ii) implies that $\mathfrak{J}_1 \not\subseteq (\mathbf{y} - \mathbf{f}(\mathbf{x}))$, so Theorem 3.1 applies. ∎

Corollary 3.3. *Let* $\mathfrak{J} \subset \mathfrak{F}(\mathbf{x}, \mathbf{y}) \{M_k\}$ *be a nonzero ideal. If* $1 \in \mathfrak{J}_\infty$, *then any formal solution of* $\mathfrak{J}$ *is* $\{M_k\}$*-summable.*

Proof. The ideal $\mathcal{J}_f$ is proper for any solution f, since by definition of a solution, $f(0) = 0$. In particular, $1 \notin \mathcal{J}$ and the corollary follows. ∎

Corollary 3.4. *Let* $0 \neq \phi \in \mathfrak{F}(x_1, x_2, \ldots, x_n, t) \{M_k\}$. *If a formal power series* $f(x_1, x_2, \ldots, x_n)$ *is such that* $\phi(x_1, x_2, \ldots, x_n, f(x_1, x_2, \ldots, x_n)) \equiv 0$, *then* $f \in \mathfrak{F}(x_1, x_2, \ldots, x_n) \{M_k\}$.

Proof. $\phi \neq 0$ implies that $\psi(x_1, x_2, \ldots, x_n, t) := \phi(x_1, x_2, \ldots, x_n, t + f(0)) \neq 0$. Then $f(x_1, x_2, \ldots, x_n) - f(0, 0, \ldots 0)$ is a solution to $\psi(x_1, x_2, \ldots, x_n, t) \equiv 0$. If $\mathfrak{J}$ is the ideal generated by ψ, then $\mathfrak{J}_\infty$ is the ideal generated by all the derivative of ψ. Since $\psi \neq 0$, some derivative of ψ has a nonzero constant term. Hence $1 \in \mathfrak{J}_\infty$, Corollary 3.3 applies. ∎

The conditions in Theorem 3.1 and Corollary 3.2 are solution specific but the condition in Corollary 3.3 is not. So, Corollary 3.3 is one of the correct generalizations of Corollary 3.4 to systems of equations. Another result that follows from the proof of Theorem 3.1, in which the condition does not depend on a particular solution is given below.

Corollary 3.5. *(cf.[15]) If* $\mathbf{y} = \mathbf{f}(\mathbf{x}) \in \mathfrak{F}^m(\mathbf{x})$ *is solution of a nonzero ideal* $\mathfrak{J} \subset \mathfrak{F}(\mathbf{x}, \mathbf{y}) \{M_k\}$ *of height* m, *then* $\mathbf{f}(\mathbf{x}) \in \mathfrak{F}^m(\mathbf{x}) \{M_k\}$.

4 Solutions of C^∞ Equations

In the analytic case (i.e. $M_k = k!$, and $\overline{k} = \mathbb{R}$ or $\mathbb{C}$) by using Artin's Approximation Theorem the proof of Theorem 3.1 becomes elementary. See [11]. In the analytic case a much stronger version of Theorem 3.1 holds (see [11]): Any C^∞ solution $\mathbf{y} = \mathbf{f}(\mathbf{x})$ of a nonzero ideal analytic ideal $\mathfrak{I}$ such that $\mathfrak{I}_\infty \nsubseteqq (\mathbf{y} - \mathbf{f}(\mathbf{x}))$ is analytic. A natural question is then: Does every C^∞ solution $t{=}f(\mathbf{x})$ of a nonzero equation $\Phi(\mathbf{x},t) = \mathbf{0}$, $\mathbf{\Phi}(\mathbf{x},t) \in C\{M_k\}(\mathbb{R}^{n+1})$ necessarily belong to $C\{M_k\}(\mathbb{R}^n)$? The answer is no, as shown by the following example, due to Vincent Thilliez[14], in the case $n = 2$.

Example 4.1. Let $\alpha > 0$ be fixed. Define

$$h(u) = \left\{ \begin{array}{c} 0 \ \mathit{if}\ u = 0 \\ \exp\left(-\frac{1}{|u|^{1/\alpha}}\right) \ \mathit{if}\ u \neq 0. \end{array} \right\}$$

It is well known that there is a constant $C > 0$ such that for all $l \in \mathbb{N}$, $\left|h^{(l)}(u)\right| \leq C^{l+1} l!^{1+\alpha}$, $\forall u \in \mathbb{R}$. Hence the function $\Phi(x_1, x_2, t) := (x_1^2 + x_2^4)t - h(x_2) \in C\{l!^{1+\alpha}\}$. Since h is flat at 0, the function $f(x_1, x_2) = \frac{h(x_2)}{x_1^2 + x_2^4} \in C^\infty(\mathbb{R}^2)$, and $t = f(x)$ is a solution to $\Phi(x_1, x_2, t) = 0$. Observe that for $0 \leq x_1 < x_2^2$, we can write

$$f(x) = \frac{1}{x_2^4}\left[\frac{h(x_2)}{1 + (x_1/x_2^2)^2}\right] = \sum_{j=0}^{\infty}(-1)^j x_1^{2j}\left(\frac{h(x_2)}{x_1^{2(2j+2)}}\right).$$

Now put $x_1 = 0$,

$$\frac{\partial^{2j} f}{\partial x_1^{2j}}(0, x_2) = (-1)^j\,(2j)!\frac{h(x_2)}{x_1^{2(2j+2)}}.$$

By Stirling's formula, we have

$$\left|\frac{\partial^{2j} f}{\partial x_1^{2j}}(0, \frac{1}{j!^\alpha})\right| \geq (const.)^{2j+1}(2j)!^{1+2\alpha}.$$

Hence, for $s < 2\alpha$, f does not belong to $C\{l!^{1+s}\}$. in any neighborhood of 0 in $\mathbb{R}^2$.

Remark 4.1. Finally, we remark that one needs additional conditions on the sequence $\{M_k\}$ in order that analogs of Lemma 2.1 and Theorem 2.3 hold in ultradifferentiable class $C\{M_k\}$. See [8],[12].

Addendum After this work was accepted for publication, the author received a paper of A. Mouze[10] that proves Artin's Approximation theorem for $\mathfrak{F}_n\{M_k\}$, where $\{M_k\}$ satisfies conditions (1) and (2). Hence the Theorem 3.1 can be proved in the same way as Theorem 1 of [11].

References

[1] Balser, W., *From Divergent Power Series to Analytic Functions: Theory and Application of Multisummable Power Series*, Springer-Varlag, 1994.

[2] Berger, E., Remarks on the Analytic Implicit Function Theorem. J. Math. Anal. Appl. **209**,435-459 (1997)

[3] Berger, M. S., *Nonlinearity and Functional Analysis*, Academic Press, 1977.

[4] Cartan, H., *Differential Calculus*, Hermann, 1971.

[5] Krantz, S and Parks, H., *A Primer of Real Analytic Functions*, Birkhauser.

[6] Malgrange, B., *Ideals of differentiable functions.* Tata Institute of Fundamental Research, Bombay, Oxford University Press, 1966.

[7] Mandelbrojt, S., *Séries Adhérentes,régularisations des Suites, Applications.* Coll. Borel, Gauthier-Villars,Paris 1952

[8] Matsumoto, W., Theory of Pseudo-differential operators of Ultradifferentiable Class, J. Math. Kyoto Univ. (JMKYAZ) 27-2 (1987) 453-500

[9] Matsumura, H., *Commutative ring theory*, Cambridge University Press. 1986.

[10] Mouze, A., Un Theorem d'Artin Pour des anneaux de Series formaelles a croissance controlee, C. R. Acad. Sci. Paris, t. 330. Serie I. p15-20,2000.

[11] Neelon, T. S., On Solutions of Real Analytic Equations. Proc. AMS **125**, No.9, 2531-2535 (1997).

[12] Rudin,W., Division in Algebra of Infinitely Differentiable Functions, Journal of Math. and Mech. Vol 11, No. 5 (1962) 797-809.

[13] Sibuya, Y., Linear Differential Equations in the Complex Domain: Problems of Analytic Continuations. Trans. Math. AMS 82 (1990).

[14] Thilliez, V., Personal Communication. March 2000.

[15] Tougeron, J. C., *Ideaux de fonctions differnetiables*, Springer-Verlag, 1972..

Department of Mathematical Sciences
California State University
San Marcos, CA 92096-0001
e-mail : neelon@mailhost1.csusm.edu