Characterization of translation planes by orbit lengths ii. even order

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Abstract

The translation planes of even order q^2 that admit a collineation group with point orbits at infinity of lengths $q+1$ and q^2-q are classified as either Desarguesian or Hall. Furthermore, the translation planes with spreads in $PG(3,q)$, for q even, admitting a linear collineation group with one point orbit at infinity of length $q+1$ and i point orbits at infinity of lengths $(q^2-q)/i$ for $i=1,2$ are classified as either Desarguesian, Hall, or Ott-Schaeffer.

1 Introduction.

Several years ago, the second author proposed a series of problems involving translation planes (see [15]). We mention the most difficult of these problems.

Determine the translation planes π of order q^n which admit a collineation group having an infinite point orbit of length q^n-q.

To illustrate the complexity of this problem, we note the following classes of examples satisfying the hypothesis.

First of all, when $n=2$, we, of course, have the Desarguesian and Hall planes.

Futhermore, there are tremendous varieties of generalized Hall and related planes (see Jha [16]) .

When $n=3$, there are infinitely many classes of examples including the generalized Desarguesian planes (see e.g. Jha and Johnson [17]) where the collineation group contains $GL(2,q)$.

When $n=4$, translation planes admitting $SL(2,q)\times Z_{1+q+q^2}$ correspond to Desarguesian parallelisms in $PG(3,q)$. Recently, there is an infinite class of such

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parallelisms due to Penttila and Williams [26] which contains the previously known Desarguesian parallelisms. Hence, there are infinitely many associated translation planes.

So, there is quite a variety of translation planes of order q^n admitting a collineation group with an infinite point orbit of length $q^n - q$.

It would appear that some additional assumption must be made in order to make some progress on the given problem.

The translation planes of order q^2 that admit a collineation group isomorphic to $SL(2, q)$ are completely determined by Foulser and Johnson. In this instance, the planes admit a group with an infinite point orbit of length $q + 1$ and i infinite point orbits of lengths $(q^2 - q)/i$ where $i = 1, 2$. So, it might be conceivable that a converse of sorts is possible based merely on the length of the point orbits on the line at infinity of the translation planes.

We note that with the exception of the Dempwolff planes of order 16, all such known planes with the required orbit lengths have spreads in $PG(3, q)$. Furthermore, two of these known planes admit a Baer subplane which is left invariant by the associated collineation group.

Recently, the authors have considered the classification of translation planes π of order q^n that have an infinite point orbit of length $q + 1$ and i orbits of lengths $(q^n - q)/i$ for $i = 1, 2$. It is possible but is not necessarily the case that there is an invariant subplane π_o of order q. If so, and $i = 1$ then there is an infinite point orbit of length $(q^n - q)$ and π is said to be an extension of a flag-transitive plane.

For odd order planes of order q^2 with spreads in $PG(3, q)$, a complete classification can be given for the general problem.

Theorem 1. *(Hiramine, Jha, Johnson [10])*

Let π be a translation plane of odd order q^2 with spread in $PG(3, q)$ which admits a linear collineation group G with infinite point orbits one of length $q + 1$ and i of length $(q^2 - q)/i$ for $i = 1, 2$.

Then, one of the following situations hold:

(i) The plane is Desarguesian, the group G (modulo the kernel) is reducible, there exists an elation in G and $i = 1$.

(ii) The plane is Hall, the group G is reducible, there exists a Baer p-element, $q = p^r$ and $i = 1$.

(iii) The plane is Hering, the group G is irreducible, $q = p^r$ for r odd and $i = 2$.

(iv) The plane is the derived likeable Walker plane of order 25.

For translation planes for which no particular assumption is made on the corresponding spreads but when there is an invariant subplane of order q and there are two infinite point orbits, it is possible to completely classify the planes; the quadratic extensions of flag-transitive planes can also be determined without further assumptions on the spread.

Theorem 2. *(Hiramine, Jha, Johnson [9]).*

Let π be a finite translation plane which is a quadratic extension of a flag-transitive plane π_o.

Then π is either Desarguesian, Hall or the derived likeable Walker plane of order 25.

The first result mentioned above involved translation planes of odd order q^2 with spread in $PG(3,q)$ where a classification is possible. The methods for planes of even order are quite different so the problem is open in the even order case.

However, for even order planes, there are group theoretic results available which provide a framework for the solution of the planes for $n = 2$ provided we assume that the collineation group is also transitive on the remaining $q + 1$ infinite points.

Hence, in this article, we consider translation planes of even order q^2 that admit a collineation group with two infinite point orbits; one orbit of length $q+1$ and one orbit of length $q^2 - q$. Without further assumption on the nature of the spread, we may give a complete classification.

As a corollary, we may complete the analysis of translation planes with spreads in $PG(3,q)$ which admit a linear collineation group with an infinite point orbit of length $q+1$ and i infinite point orbits of lengths $(q^2-q)/i$ for $i = 1, 2$ to include the even order case. The result is similar to the above mentioned result for odd order planes with spreads in $PG(3,q)$.

Thus, our results are, in some sense, an extension of the results of Hiramine, Jha and Johnson [9] and an even order companion to the paper of Hiramine, Jha and Johnson [10].

Our main results are:

Theorem 3. *Let π be a translation plane of even order q^2 that admits a collineation group G in the translation complement with infinite points orbits of lengths $q + 1$ and $q^2 - q$.*

Then one of the following situations occur:

(1) π is Desarguesian or

(2) π is Hall.

As an application of this result, we have

Theorem 4. *Let π denote a translation plane of order q^2, q even, with spread in $PG(3,q)$.*

If G is a linear collineation group with an infinite point orbit of length $q+1$ and i infinite point orbits of length $(q^2-q)/i$ for $i = 1$ or 2 then π is one of the following types of planes:

(1) Desarguesian and $i = 1$,

(2) Hall and $i = 1$, or

(3) Ott-Schaeffer and $i = 2$.

2 Background.

The proof of the main results revolves about the combinatorics of groups acting on translation planes. There are results in three loosely connected areas which we shall group together for convenience; results in translation planes and their collineation groups, combinatorial results on translation planes and purely group theoretic results.

For the most part, we shall list the main results that we shall be using in the proofs.

2.1 Translation Planes and their Groups.

Theorem 5. *(Hiramine, Jha and Johnson [9], (2.2))*

Assume that a translation plane π of order q^2 admits a collineation group G with an infinite orbit Δ of length $q+1$ and all other infinite orbits of lengths λ such that $(\lambda, q+1) \leq 2$.

If G contains an elation then π is Desarguesian or Hall or the order is 81 or 64.

Theorem 6. *(Hering [11])*

Let π be a translation plane of even order 2^m and let S be a 2-group of collineations in the translation complement. Then the exponent e of S divides $2m$. If S does not contain any non-trivial elation with affine axis, then e divides m.

Theorem 7. *(Hering and Ostrom [12], [23], and [24])*

Let π denote a finite translation plane of order p^n and let E denote the collineation group in the translation complement which is generated by all affine elations.

Then one of the following holds:

(1) E is an elementary Abelian p-group,

(2) $p = 2$, and the order of E is $2t$ where t is odd,

(3) E is isomorphic to $SL(2, p^b)$,

(4) E is isomorphic to $S_z(2^c)$ and $p = 2$ or

(5) E is isomorphic to $SL(2, 5)$ and $p = 3$.

Theorem 8. *(Johnson-Ostrom [21])*

In the context of the previous result of Hering-Ostrom, if the translation plane of order q^2 has a spread in $PG(3, q)$ then in case (2), the group is dihedral of order $2t$ where t is the number of elation axes.

Theorem 9. *(Foulser-Johnson [6])*

Let π denote a translation plane of even order q^2 that admits a collineation group isomorphic to $SL(2, q)$.

Then π is one of the following planes:

(1) Desarguesian,

(2) Hall,

(3) Ott-Schaeffer,

(4) the Dempwolff plane of order 16.

Theorem 10. *(Johnson and Ostrom [21] (3.1))*

Let π be a translation plane of order 2^{2r} and of dimension 2 over its kernel. Let G be a collineation group in the linear translation complement and assume that the involutions in G are Baer. Then the Sylow 2-subgroups of G are elementary Abelian.

Theorem 11. *(Johnson and Ostrom [21] part of (3.27))*

Let π be a translation plane of dimension 2 over $GF(q)$, where q is a power of 2. Let G be any subgroup of the translation complement.

If the involutions of G are all Baer, let G_1 denote the subgroup of G generated by the Baer involutions in the linear translation complement. If G is nonsolvable then G_1 is isomorphic to $SL(2, 2^s)$ for some s and is normal in G.

If G_1 is irreducible, π has an Ott-Schaeffer subplane of order 2^{2s}.

If G_1 is reducible then π is derived from a plane also admitting $SL(2, 2^s)$ and the involutions in the derived plane are elations.

Theorem 12. *(Johnson [20])*

Let π be a translation plane of order q^2 admitting a collineation group isomorphic to $SL(2, p^a)$. If $p^a > \sqrt{q}$ then $SL(2, q)$ is a collineation group of π.

Theorem 13. *(see Lüneburg [22] (50.3) p. 262)*

Let π be an Ott-Schaeffer translation plane of order q^2 and let $S \simeq SL(2, q)$ be a collineation group of π. Then the full translation complement is $N_{\Gamma L(V)}(S)$ where V is the underlying vector space.

Theorem 14. *(André [1])*

Let π^+ be a finite projective plane admitting two homologies with the same center and distinct axes. Then there is an elation in the group generated by the homologies.

2.2 Combinatorial Translation Planes.

Theorem 15. *(Johnson [19])*

Let π be a finite translation plane of order p^r which admits a collineation σ in the translation complement H_0 of order u, a prime p-primitive divisor of $p^r - 1$.

If σ fixes at least three mutually disjoint r-dimensional $GF(p)$-subspaces then there exists a unique Desarguesian spread Σ consisting of the σ-invariant r-dimensional $GF(p)$-subspaces.

Furthermore, $N_{H_0}(\langle\sigma\rangle)$ is a collineation group of Σ.

Theorem 16. *(Foulser [5])*

Let π be a finite translation plane of order q^2 which contains a Baer subplane π_o incident with the zero vector. Let N_{π_o} denote the net of degree $q + 1$ defined by the components of the subplane π_o. Let the kernel of π_o be isomorphic to $GF(p^a)$ where $q = p^r$ and p is a prime and assume that there are at least three Baer subplanes in N_{π_o} which are incident with the zero vector.

Then the number of subplanes of N_{π_o} incident with the zero vector is $1 + p^a$.

Furthermore, the set of Baer subplanes incident with the zero vector is isomorphic to $PG(1, p^a)$.

Theorem 17. *(Ostrom[25])*

Let N be a net of order q^2 and degree $q^2 - q$.

Then N can be extended to at most two non-isomorphic affine planes and if there are two extensions, the planes are related to each other by derivation.

2.3 Group Theoretic.

Theorem 18. *(Hering [13] Theorem 1)*

Let Q be a subgroup of a finite group G with the properties:

(a) $N_G Q \cap Q^x = 1$ for all $x \in G - N_G Q$,

(b) $N_G Q \neq G$, and

(c) $2 \mid |Q|$.

Denote the normal closure of Q in G by S. Then, in general, $S = Q \cdot O(S)$, and Q is a Frobenius complement. An exception is only possible if $S \simeq SL(2, q)$, $S_z(q)$, $SU(3, q)$, or $PSU(3, q)$, where q is a power of 2 and $q \geq 4$.

Theorem 19. *(Holt [18] (stated less generally than in Holt), Theorem 1)*

Denote by $\mathfrak{A}_n$ and $\mathfrak{S}_n$ the alternating and symmetric groups on n letters respectively. If Δ is a set let $\mathfrak{A}^\Delta$ and $\mathfrak{S}^\Delta$ denote the alternating and symmetric groups on Δ respectively.

Let G^Ω be a transitive permutation group on a finite set Ω such that an involution central in a Sylow 2-subgroup fixes a unique point of Ω. Let J denote the involutions which fix a unique point of Ω and let $H = \langle J \rangle$. Let Ψ denote the set of orbits of $O(H)$, and let bars denote images modulo $O(H)$. Then, if $|\Psi| > 1$, we have

$$\overline{H} = \langle \otimes_{i=1}^n M_i, \overline{t} \rangle \text{ for some } n$$

where $t \in J$ and for each i, either we have $M_i \simeq \mathfrak{A}_m$ where $m \geq 5$, or is isomorphic to one of the simple groups $PSL(2,2^k)$ $(k > 1)$, $PSU(3,2^k)$ $(k > 1)$, or $S_z(2^k)$ $(k = 2w+1, w \geq 1)$, in their natural 2-transitive representation. Furthermore, $\overline{t} \in N(M_i)$ for each i and $\overline{t}$ induces an inner automorphism on M_i unless $M_i \simeq \mathfrak{A}_m$ for $m \equiv -1(\mod 4)$ in which case

$$\langle M_i, \overline{t} \rangle \simeq \mathfrak{S}_m.$$

Furthermore, we can write

$$\Psi = \otimes_{i=1}^n \Psi_i$$

where

$$\overline{H}^\Psi \subseteq \otimes_{i=1}^n \mathfrak{S}^{\Psi_i}$$

and M_i acts faithfully on Ψ_i in its natural 2-transitive representation. We have $\overline{t} = (t_1, t_2, ..., t_n)$ where $t_i^{\Psi_i}$ is an involution in $\mathfrak{S}^{\Psi_i}$ and $t_i \in M_i$ unless $M_i \simeq \mathfrak{A}_m$ with $m \equiv -1(\mod 4)$, in which case

$$\langle M_i, t_i \rangle^{\Psi_i} \simeq \mathfrak{S}^{\Psi_i}.$$

If this latter possibility occurs for some i, then $\overline{H}$ acts faithfully on Ψ. Otherwise, we have $\overline{t}^\Psi \in \otimes_{i=1}^n M_i^{\Psi_i}$ and if $O(H) \neq 1$, then the kernel of the action of $\overline{H}$ on Ψ has order 2. Thus $\overline{t} \in \otimes_{i=1}^n M_i$ if and only if $O(H) = 1$ and $|\Psi_i| \equiv 1(\mod 4)$ for all i.

Theorem 20. *(Gleason [7])*

Let G be a finite group operating on a set Ω and let p be a prime. If Ψ is a subset of Ω such that for every $\alpha \in \Psi$, there is a p-subgroup Π_α of G fixing α but no other point of Ω then Ψ is contained in an orbit.

Theorem 21. *(See e.g. Gorenstein [8] (3.16))*

Let H be any finite group of odd order which is normalized by an elementary Abelian 2-group A.

Then $H = \langle C_H(a); a \in A - \{1\} \rangle$.

(Note that 21 refers to p-groups but since a 2-group normalizes some Sylow p-group of order p^a for every odd prime dividing $|H|$, it follows that the group indicated is divisible by p^a for every order p^a of a Sylow p-subgroup.)

Theorem 22. *(See e.g. Aschbacher [2] p. 167 (33.3))*

Let G be a finite perfect group and (H, π) a central extension of G with kernel $ker\pi$. Then $H = ker\pi\ H'$ with H' perfect.

Theorem 23. *(See e.g. Huppert [14] p. 629, 23.3)*

Let G be a finite perfect group with normal subgroup H contained in $Z(G)$. Then H is a subgroup of the Schur multiplier of G/H.

Theorem 24. *(See e.g. Huppert [14] p. 646, 27.5)*

The Schur multiplier of $SL(2, q)$ for q even has order 1 if $q \neq 4$ and 2 if $q = 4$. The outer automorphism group has order r where $q = 2^r$.

Theorem 25. *(See e.g. The Atlas of Finite Groups, table 5 p. xvii)*

The Schur multiplier of $PSU(3, q^{1/3})$ for q even has order 1. The outer automorphism group has order $(3, q^{1/3} + 1)2r/3$ where $q = 2^r$.

2.4 The Planes of the Characterization.

The planes involved in our classification theorem are the Desarguesian, Hall and Ott-Schaeffer planes of order q^2. All of these planes may be regarded as arising from spreads in $PG(3, q)$. Of course the Hall planes are those derived from the Desarguesian spreads by the derivation of a regulus net. Since a Desarguesian plane π of order q^2 admit a collineation group isomorphic to $GL(2, q)$ which fixes a regulus net $\mathcal{R}$ and which acts transitively on the components of $\mathcal{R}$ and the components of $\pi - \mathcal{R}$, we see that the Desarguesian planes of even order q^2 satisfy our hypothesis. We note that the kernel homology group Z_{q^2-1} acting on π also acts on the Hall plane π^* obtained by derivation of $\mathcal{R}$, it follows that $GL(2, q)Z_{q^2-1}$ acts on π^* and has orbits of lengths $q + 1$ and $q^2 - q$.

An Ott-Schaeffer plane of even order q^2 admits a collineation group isomorphic to $SL(2, q)$ such that each Sylow 2-subgroup fixes a unique component of the plane and the group acts irreducibly on the associated 4-dimensional vector space. Hence, there is an orbit of components of length $q+1$. This orbit of components defines a regulus in $PG(3, q)$ whose derivation produces another Ott-Schaeffer plane. Furthermore, the group $SL(2, q)$ has two component orbits of length $(q^2 - q)/2$ both of which are left invariant under the full collineation group.

Our main theorem states that without any assumption on the nature of the spread (or kernel) the only translation planes of even order q^2 admitting collineation groups with component orbit lengths $q+1$ and q^2-q are the Desarguesian and Hall planes. When we assume that the spread is in $PG(3, q)$, only the Ott-Schaeffer planes of order q^2 occur among the possible planes that admit a collineation group with component orbits of lengths $q + 1$ and $(q^2 - q)/2$.

The reader is directed to Lüneburg [22], chapter VII for further background on the Ott-Schaeffer planes.

3 The Structure of the Proof:

Assume that conditions of 3. We shall complete the proof for order 2^6 after the main argument.

Furthermore, we assume that the plane is neither Desarguesian nor Hall.

Lemma 1. *The order is not* 16.

Proof: The translation planes with order 16 are completely determined in [4]. The requirement on the group orbits forces the plane to be Desarguesian or Hall.

Lemma 2. *Assume that the order* q^2 *is not* 2^6. *Then we may assume that all involutions are Baer and there is not a* G*-invariant Baer subplane.*

Proof: Apply 2 and 2.1. ∎

Lemma 3. *Since* G *has order divisible by* $(q+1)q(q-1)$, *the Sylow* 2*-subgroups of* G *are not cyclic.*

Proof: Deny! The order of a Sylow 2-subgroup of G is divisible by q. Since there are no elations, if $q = 2^r$ then q divides $2r$ by 6 which implies that $r = 1$ or 2 so $q = 2$ or 4. However, planes of order 4 are Desarguesian and the order is not 16 by the above lemma. ∎

Lemma 4. *Let* σ *be an involution in the center of a Sylow* 2*-subgroup* S *of* G. *Then either* σ *fixes all points of the orbit* Δ *of length* $q+1$ *or fixes exactly one of them.*

Proof: Let Q be in the orbit Γ of length q^2-q. We note that since $S_Q \le G_Q$ then it follows that $[G : S][S : S_Q] = [G : G_Q][G_Q : S_Q]$. Hence, as q divides $[G : G_Q]$, as the index is the orbit length, it follows that q also divides $[S : S_Q]$. That is, if Q is an infinite point of $Fix\sigma$ then the orbit length of Q under S must be q. Hence, either exactly q or 0 infinite points of $Fix\sigma$ are in Γ. ∎

Lemma 5. *If the group* G *admits* $SL(2, q)$ *then the plane cannot be an Ott-Schaeffer plane.*

Proof: In this case, $q = 2^r$ where r is odd and the spread is in $PG(3, q)$. The full collineation group is in $\Gamma L(4, 2^r)$ so the Sylow 2-subgroups are in $GL(4, 2^r)$. Hence, by 10, the Sylow 2-subgroups are elementary Abelian.

The group orbit lengths of the Ott-Schaeffer plane under the group isomorphic to $SL(2, q)$ are $(q+1), (q^2-q)/2$ on the line at infinity and the full translation complement is $N_{\Gamma L(V)}(SL(2, q))$ by 13. Suppose there exists a collineation g which inverts the two $SL(2, q)$-orbits of lengths $(q^2-q)/2$. Hence, g^2 fixes both such orbits as well as the orbit of length $q+1$. It follows that the order of g is even $= 2^e s$ where $(2, s) = 1$.Thus, $\langle g^s\rangle$ is a cyclic subgroup of order 2^e so $e = 1$. Hence, the Sylow 2-subgroup of $\langle g\rangle$ inverts the two orbits of lengths $(q^2-q)/2$.

However, all involutions are Baer so there is a Baer involution τ with fixed point subplane in N_Δ. In the group $SL(2, q)$ acting on the Ott-Schaeffer plane, each Sylow 2-subgroup fixes a unique component of N_Δ. Moreover, τ normalizes $SL(2, q)$ and hence normalizes each Sylow 2-subgroup of $SL(2, q)$. But, each Sylow 2-subgroup of $SL(2, q)$ also leaves invariant a unique Baer subplane upon which it induces an elation group. Hence, τ leaves invariant each Baer subplane of the net N_Δ which is a contradiction. ∎

3.1 Part I of the proof:

In **part I**, we consider the case where there is an involution σ central in a containing Sylow 2-subgroup such that $Fix\sigma$ is a Baer subplane of the net N_Δ defined by the orbit Δ of length $q+1$. Since $Fix\sigma$ is not G-invariant, we have several Baer subplanes of the net N_Δ. Using the structure of a net containing at least three Baer subplanes determined by Foulser, we may show that we must obtain $SL(2,q)$ as a collineation group which forces the plane to be Hall using the theorem of Foulser-Johnson 9. ■

3.2 Part II of the proof:

In **part II**, we assume that some $Fix\sigma$ is not a Baer subplane of the net N_Δ. Hence, an involution in the center fixes a unique point of Δ and G acts transitively on Δ. In this situation, we may use results of Holt 19 to again show that we must obtain $SL(2,q)$ as a collineation group and again the theorem of Foulser-Johnson 9 shows that the plane must be Ott-Schaeffer in this case. However, we have shown that the full collineation group of an Ott-Schaeffer plane does not have such orbit lengths on the line at infinity. ■

4 Part I. Fixσ is in N_Δ.

Let σ be any Baer involution in the center of a Sylow 2-subgroup such that $Fix\sigma$ is in N_Δ. Let T denote the maximal 2-group which fixes $Fix\sigma$ pointwise for a fixed involution σ. We emphasize that T may not necessarily be in the center of a Sylow 2-subgroup.

We also note if there are no elations in the general case for a particular value of q (i.e. when $q = 8$ as well) our arguments do not depend on q and apply also to planes of order 64.

Lemma 6. *Assume that $Fix\sigma$ is a subplane of the net N_Δ defined by Δ. Then the net N_Δ is derivable.*

Proof: We have that there is a 2-group of order at least 2 fixing a Baer subplane pointwise but we know that the subplane is not G-invariant by 2 or Lemma 3.2.

Then there is a second subplane on the net and hence a third since the Baer group must move the second subplane. Hence, by Foulser 16, the lattice of subplanes is $PG(1, \mid K \mid)$ where K is the kernel of one of them. Let S be a Sylow 2-subgroup of G and S_1 the subgroup which is trivial on the set of all Baer subplanes of the net incident with the zero vector. Then S_1 is normal and if not trivial then contains a nontrivial element of the center of S. However, such elements contain only Baer involutions. If such a Baer involution τ fixes Δ pointwise then τ fixes exactly one Baer subplane of N_Δ. If τ fixes exactly one point of Δ then τ cannot induce a Baer involution on any invariant Baer subplane. Thus, τ induces an elation on each Baer involution and again can fix no more than one. Hence, S_1 is trivial.

Let G^* denote the group acting on these Baer subplanes. We see that G^* is a subgroup of $P\Gamma L(2,K)$. A Sylow 2-subgroup S is faithful on this set of Baer subplanes and has order at least q. Let the order of K be 2^a . Then a Sylow 2-subgroup of $P\Gamma L(2,K)$ has order dividing $2^a \mid a \mid_2$. If a is not r then 2^r divides $2^a \mid a \mid_2 \leq 2^{r/2}r/2$. However, this implies that $2^{r/2} \leq r/2$ which can occur only if $r=2$ and the order of the plane is 4 and hence Desarguesian.

Thus, the kernel K is $GF(q)$ so that the net is derivable. ∎

Lemma 7. *Assume that T has order 2 then π is Desarguesian or Hall.*

Proof: A Sylow 2-subgroup S containing T acting on the set of $q+1$ Baer subplanes is faithfully in $\Gamma L(2,q)$. Furthermore, S leaves a Baer subplane $\pi_o = FixT$ invariant. The full translation complement of a derivable net is a group isomorphic to $\Gamma L(2,q)GL(2,q)$ where the first listed group is the group leaving π_o invariant and the second listed group is a group which fixes all components of the net. Note, for example, the group generated by central collineations of the net will leave each Baer subplane incident with the zero vector invariant. The linear part $S \cap GL(2,q) = S'$ of group S sitting in $\Gamma L(2,q)$ as acting on the set of Baer subplanes is a linear subgroup of $\Gamma L(2,q)GL(2,q)$ as a collineation group of the net and/or the translation plane. Considered as a subgroup of the net, we have a 4-dimensional vector space V_4 over a field K isomorphic to $GF(q)$ and the group of the net is a semilinear subgroup of $\Gamma L(4,K)$. Notice that within the first listed $\Gamma L(2,q)$ the subgroup $GL(2,q)$ generated by central collineations of the net acts trivially on the set of Baer subplanes.

We could also represent the group isolating on any component. Let ℓ be any component of the derivable net fixed by the Sylow 2-subgroup S. Then the group induced on ℓ is isomorphic to the group acting on the net which is clearly isomorphic to $\Gamma L(2,q)$. Moreover, S' acting on ℓ must fix a 1-dimensional K-subspace pointwise.

We assert that there can be no central collineations of the translation plane with axis ℓ. We have assumed that there are no elations. Assume that there is a homology τ with axis ℓ. Since there are q remaining points of Δ and the order of τ divides q^2-1, it follows that the center of τ is a point of Δ. Let the center and co-center of τ be denoted by (P,Q). Either there is an elation by André's theorem 14 or for any collineation g of G then $\{P,Q\} \cap \{Pg,Qg\} = \phi$ or the two sets are equal. Hence, the $q+1$ infinite points of Δ are partitioned into pairs which are sets of imprimitivity under G. However, since $q+1$ is odd, this is a contradiction. Thus, there are no central collineations with axis a component of N_{π_o}.

Let 2^r and $r = 2^c m$ where $(2,m)=1$. Then there is a linear subgroup S' of S of order at least $2^{2^c m}/2^c$. Now $2^{2^c m - c} > 2^{2^{c-1}m}$ if and only if $2^c m - c - 1 \geq 2^{c-1}m$ if and only if $m > c/2^{c-1}$. The latter inequality is valid unless possibly $c=1$ or 2 and $m=1$. Hence, there are possible orders $q=4$ or 16. Since we may assume that q is not 4 as above, we have that S' has order strictly larger than $\sqrt{q}$ or $q=16$ and S' would have order at least 4.

We consider the group generated by the linear parts of the Sylow 2-subgroups. Assume that S' fixes the Baer subplane π_o and fixes $\pi_o \cap \ell$ pointwise. Let π_1 be another Baer subplane incident with the zero vector which is an image of π_o under some collineation of G. Then π_1 admits an involution σ_1 fixing it pointwise. This collineation σ_1 must leave ℓ invariant and map π_o to another subplane π_2 distinct from π_o or π_1. Hence, there is another 2-subgroup S_2' that fixes $\pi_2 \cap \ell$ pointwise. The subgroup generated by 'elations' on ℓ is isomorphic to $SL(2, 2^a)$ for some $2^a > \sqrt{q}$ or $q = 16$. This group is faithful on the translation plane as well and hence by 12, the plane admits a collineation group isomorphic to $SL(2, q)$.

Now assume that $q = 16$ and assume that we obtain a group isomorphic to $SL(2, 4)$ but not one isomorphic to $SL(2, 16)$. The group S of order 16 then contains a subgroup of order 4 which fixes a second subplane of the indicated five subplanes corresponding to the five Sylow 2-subgroups of $SL(2, 4)$. Since an involution fixing at least two subplanes cannot be an elation, this implies that the involution fixes components outside of the net N_{π_o}. Hence, it follows that the Sylow 2-subgroups have order divisible by 2^5. Each Sylow 2-subgroup leaves invariant a Baer subplane and induces upon it a group in $\Gamma L(2, 16)$. The order of a Sylow 2-subgroup faithfully induced is $16 \cdot 4$. Note that since we obtain $SL(2, 4)$, it follows that there is a faithful group induced on π_o of order at least $32/4 = 8$ so there must be an induced linear group of order at least two. It then follows that there is a subgroup of order at least 8 which either fixes π_o pointwise or induces an elation on π_o. However, this means that the group generated as above would be $SL(2, 16)$.

Hence, in any case, by 9, the plane is either Desarguesian, Hall, Ott-Schaeffer or Dempwolff of order 16. The assumptions imply that we must have the Ott-Schaeffer plane.

However, the previous lemma 5 shows that we obtain a contradiction. ∎

Lemma 8. *T does not have order q.*

Proof: Assume that the order of T is q. Then clearly as the net is derivable and we may apply the Hering-Ostrom theorem 7 to the derived plane to show that the group $< T^x$ for x in $G >$ is isomorphic to $SL(2, q)$. Since the Sylow 2-subgroups fix Baer subplanes pointwise, it follows from the classification theorem of Foulser-Johnson 9 that the plane is Hall. ∎

Lemma 9. *π is Desarguesian or Hall.*

Proof: By the previous lemmas, we have $2 < |T| < q$.

Again, we may apply the theorem of Hering and Ostrom 7 to conclude that the group generated by the Baer involutions is isomorphic to $SL(2, 2^b)$ for $b > 1$ and is normal in G and where $|T| = 2^b$.

Hence, G induces an automorphism group on $SL(2, 2^b)$.

Note that $Aut(SL(2, 2^b))/SL(2, 2^b)$ has order b .

Thus, $G/(C_G(SL(2, 2^b) \times SL(2, 2^b))$ has order less than or equal to b. We note that no nontrivial element ρ of a Sylow 2-subgroup S of G commutes with $SL(2, 2^b)$ for if so then ρ must leave invariant each Baer subplane fixed pointwise by a Sylow 2-subgroup of $SL(2, 2^b)$. We have seen above that the subgroup S_1 of S acting trivially on the set of all Baer subplanes is trivial. Similarly, here the subgroup S_1' which acts trivially of the set of $1 + 2^b$ subplanes is normal in G and, hence if non-trivial, contains a nontrivial involution τ of the center of S. Since τ fixes one or all points of Δ, we have a contradiction as before. Hence, S_1' is trivial.

Furthermore, the above remarks then imply that $q/2^b$ divides b_2 (the 2-part of b). Let $b_2 = 2^c$ so that for $q = 2^r$ then $r - b \le c \le b$ so that $b \ge r/2$. If $b = r/2$ then $q/2^{r/2} = 2^{r/2}$ divides $r/2$ which never occurs. Hence, $b > r/2$.

There is a subgroup of the net N_Δ which is isomorphic to $SL(2,q)$ and generated by Baer groups of order q. The group in question, acting on the translation plane, is a subgroup of $SL(2,q)$ and isomorphic to $SL(2,2^b)$ with $b > r/2$. Moreover, we may consider this group acting on the derived plane. Hence, it follows from 12 that the group $SL(2,2^b)$ must be $SL(2,q)$ and we may apply 9 to complete the proof and/or have a contradiction to our assumptions as this forces the order of T to be q. ■

5 Part II. Fixσ is not in N_Δ.

We now assume that some involution σ in the center of a Sylow 2-subgroup has q of its infinite points in Γ. Furthermore, we may assume by the previous section that each involution in the center of a Sylow 2-subgroup fixes exactly one point of Δ.

Lemma 10. *The subgroup $G_{[\Delta]}$ that fixes Δ pointwise has odd order.*

Proof: Let S be any Sylow 2-subgroup. Then $S_{[\Delta]}$ is a normal subgroup and if not trivial contains an element in the center of S which is contrary to our assumptions.■

In addition, we consider the action of G on Δ as $G^* = G/G_{[\Delta]}$ and let $L^* = L/G_{[\Delta]} =$

$$\left\langle z^*; z^{*2} = 1 \text{ and } z^* = zG_{[\Delta]} \in G/G_{[\Delta]} \ \ni \ z^* \text{ fixes exactly one point of } \Delta \right\rangle$$

Note that $\sigma^* = \sigma G_{[\Delta]}$ is in $\overline{L}$. Also, note that $G_{[\Delta]}$ has odd order and is normal in G and hence, normal in L and thus contained in $O(L)$.

Assume that q is not 8 and let u be a 2-primitive divisor of q^2-1. Recall that $q(q-1)(q+1)$ divides the order of G. Let S_u denote a Sylow u-subgroup of G and let Ψ denote the set of $O(L^*)$ orbits on Δ. Note that L is normal in G and $O(L)$ is characteristic in L so normal in G.

Lemma 11. *If R_u is a Sylow u-subgroup of $O(L)$ then $G = N_G(R_u)O(L)$.*

Proof: Apply the Frattini argument. ■

Lemma 12. *A Sylow u-subgroup of $O(L)$ is cyclic or $q = 2$ and the plane is Desarguesian.*

Proof: Let $q = 2^m$ so that $G \leq GL(4m,2)$. Recall that a Sylow u-subgroup of $GL(4m,2)$ is isomorphic to $Z_{u^k} \times Z_{u^k}$ where $u^k = \mid q+1 \mid_u$. Let R_u denote a Sylow u-subgroup of $O(L)$. Then R_u leaves invariant at least two points Q and T of Γ.

Since a Sylow u-subgroup of $GL(2m,2)$ is isomorphic to Z_{u^k}, it follows that either R_u is cyclic or there exists a homology of order u with axis $0Q$ and coaxis $0T$. Hence, $N_G(R_u)$ leaves $\{Q,T\}$ invariant. Thus, a Sylow 2-subgroup of G in $N_G(R_u)$ which fixes Q has index at most 2. Hence, the orbit length of Q is not divisible by 4. This implies that $q = 2$. ■

Lemma 13. $\mid O(L) \mid$ *is not divisible by* u.

Proof: Assume so. Then we have seen previously that R_u is cyclic if q is not 2. Let U_t denote the integers less than and relatively prime to t. Let the order of R_u be u^c and note that U_u is a subgroup of U_{u^c}. Furthermore, the order of U_u is $u-1$ and is cyclic. Furthermore, the order of U_{u^c} is $u^{c-1}(u-1)$. Hence, the Sylow 2-subgroup of U_{u^c} is cyclic.

The automorphism group of R_u induced by G is $N_G(R_u)/C_G(R_u)$ so that the Sylow 2-subgroup $N_G(R_u)/C_G(R_u)$ is cyclic.

Assume that there is an involution z in $C_G(R_u)$ which may be taken in the center of a Sylow 2-subgroup of $N_G(R_u)$. Since z commutes with R_u and fixes exactly one point P of Δ then R_u must fix P and hence must fix a second point of Δ. Since z is not an elation then there exists a set of q points on $0P$ fixed by z which are then permuted by R_u. Let g be an element of R_u of order u. Since u is a 2-primitive divisor, it follows that g must fix this set of points pointwise. It then follows that g must be an affine homology. However, g must fix at least two points of Γ so that we have a contradiction.

Hence, it follows that $C_G(R_u)$ has odd order so that a Sylow 2-subgroup of $N_G(R_u)$ and hence of G is cyclic. Since this cannot occur by a previous lemma then $\mid O(L) \mid$ is not divisible by u. ■

Lemma 14. *The number* Ψ *of orbits on* Δ *of* $O(L)$ *is not* 1.

Proof: Deny! Then $q+1$ divides the order of $O(L)$ and $q+1$ contains a 2-primitive divisor of q^2-1. ■

5.1 The application of Holt's Theorem.

We may now employ the main result of Holt 19.

We let $\widetilde{L} = L/O(L) \simeq ((L/G_{[\Delta]})/(O(L)/G_{[\Delta]})) = L^*/O(L^*) = \overline{L^*}$ and $\widetilde{G} = G/O(L) \simeq (G/G_{[\Delta]})/O(L)/G_{[\Delta]}) = G^*/O(L^*) = \overline{G}^*$ and adopt the notation introduced in 19 but using the isomorphisms developed here.

Lemma 15. $\widetilde{L} =< \otimes_{i=1}^{r} \widetilde{M_i}, \widetilde{\sigma} >$
where $\widetilde{M_i} \simeq A_n$ *for* $n \geq 5$,
$PSL(2, 2^n), n > 1$,
$PSU(3, 2^n), n > 1$,
or $S_z(2^n)$ *for* $n = 2m+1$.

Proof: Apply Holt noting that the number of orbits on Δ is not 1. ■

We note that it is possible that $\widetilde{\sigma}$ is in $\Pi_{i=1}^{r}\widetilde{M_i}$ but if we take $\widetilde{M} = [\widetilde{L}, \widetilde{L}] = \otimes_{i=1}^{r}\widetilde{M_i}$ then $\widetilde{M}$ is a normal subgroup of $\widetilde{G}$.

Lemma 16. $\widetilde{G}$ *acts on* $\{\widetilde{M_i}$ *for* $i = 1, 2, ..., r\}$.

Proof: Note that $\widetilde{M_i}^g \cap \widetilde{M_j}$ for g in $\widetilde{G}$ is normal in $\widetilde{M_j}$ since $g(m_1, m_2, ..., m_r) = (m_1^*, m_2^*, ..., m_r^*)g$ for m_i and m_i^* in $\widetilde{M_i}$ so that
$\widetilde{M_i}^{g(n_1,n_2,...,n_r)}$ for n_j in M_j, is $\widetilde{M_i}^{(n_1^*,n_2^*,...,n_r^*)g} = \widetilde{M_i}^g$. ■

Now reindex, if necessary so that $\{\widetilde{M_1}, \widetilde{M_2}, .., \widetilde{M_s}\}$ is a $\widetilde{G}$-orbit. Hence, $\widetilde{M_o} = \otimes_{i=1}^{s} \widetilde{M_i}$ is a normal subgroup of $\widetilde{G}$.

Lemma 17. *$M_o = \Pi_{i=1}^{s} M_i$ is normal in G, and every Sylow 2-subgroup S_o of M_o contains a central involution z_o and M_o is transitive on Δ.*

Proof: Let S be a Sylow 2-subgroup of G so that $S \cap M_o$ is normal in S and is a Sylow 2-subgroup of M_o. Hence, $Z(S) \cap (S \cap M_o) \neq < 1 >$ so that there exists a central involution z in $Z(S \cap M_o)$. By assumption, z fixes exactly one point of Δ and since M_o is normal, z^x for all x in G is in M_o so that it follows by Gleason's theorem 20 that M_o is transitive on Δ. ■

Lemma 18. *$M_o = M$.*

Proof: If not M is not M_o, let $\widetilde{Y} = \otimes_{i=s+1}^{r} \widetilde{M_i}$ so that $Y = \Pi_{i=s+1}^{r}$is normal in G. Moreover, it follows from the argument of the previous lemma that there is a central involution z_1 in every Sylow 2-subgroup S_1 of Y and that Y is transitive on Δ. Since S_o and S_1 commute modulo $O(L)$, it follows that S_o permutes the Sylow 2-subgroups of $S_1 O(L)$ and since the number of these is odd, it must be that S_o normalizes S_1^g for some g in $O(L)$. This says that S_o fixes some point in the $O(L)$ orbit of P. Choosing P in various orbits implies that S_o fixes a point in each $O(L)$ orbit and since there are > 1 such orbits in Δ, we have a contradiction. ■

Hence, we obtain:

Lemma 19. *L is transitive on Δ and $\widetilde{G}$ acts transitively on $\{\ \widetilde{M_i}$ for $i = 1, 2, ..., r\}$ by conjugation.*

Lemma 20. *$u \mid \mid \widetilde{M_i} \mid$ for all $i = 1, 2, ..., r$.*

Proof: We see that $\widetilde{M}$ has index 1 or 2 in $\widetilde{L}$ so that u divides the order of $\widetilde{M}$ and hence divides the order of one of the $\widetilde{M_i}$ and hence divides the order of all of the $\widetilde{M_i}$. ■

Lemma 21. *$r = 1$ or 2.*

Proof: Note that it follows that $\widetilde{M}$ contains an elementary Abelian u-group of order u^r. However, we have noted that u cannot divide the order of $O(L)$ so that M contains an elementary Abelian u-group of order u^r which is then a subgroup of $GL(4m, 2)$ and hence of $Z_{u^k} \times Z_{u^k}$ where $k = \mid q + 1 \mid_u$. Thus, $r \leq 2$. ■

Lemma 22. *If a Sylow 2-subgroup of M_1 fixes exactly one point of Δ then $r = 1$.*

Proof: Let H denote the preimage normal subgroup of G which is of index 1 or 2 (that is H is the preimage of $N_{\widetilde{G}}(\widetilde{M_1}) = N_{\widetilde{G}}(\widetilde{M_2})$). In any case, M_i are both normal subgroups of H for $i = 1, 2$ and H is transitive on Δ.

Let S_1 be a Sylow 2-subgroup of M_1. Assume that S_1 fixes exactly one point of Δ. Since M_1 is normal and H is transitive then M_1 is transitive on Δ. Since M_1 is congugate to M_2 in G then M_2 is also transitive on Δ. Let S_2 be a Sylow 2-subgroup which fixes P in Δ. Note that S_2 commutes with S_1 modulo $O(L)$ and since S_2 is even and there are an odd number of Sylow 2-subgroups in $S_1O(L)$, it follows that S_2 must leave one invariant. That is, S_1 must normalize S_2^x for some x in $O(L)$ so that $S_1S_2^x$ is a 2-group and hence $[S_2^x, S_1] =< 1 >$. In any case, S_1 must fix the point P^x. Hence, S_1 must fix a point in every $O(L)$-orbit on Δ. Since there is more than one such orbit, we have a contradiction.

Lemma 23. $r = 1$.

Proof: By the previous lemma, we may assume that S_1 fixes at least two points of Δ. It then follows that H must have index two in G. Let S_H be a Sylow 2-subgroup of H. Since M_1 is normal in H, $S_H \cap H = S_1$ is a Sylow 2-subgroup of M_1 which contains a central involution τ of H (this is not necessarily a central involution of G) as the index of S_H in a Sylow 2-subgroup S of G is 1 or 2.

We have noted that if there exists a nontrivial element in $S_{[\Delta]}$ where S is a Sylow 2-group of G then $S \cap S_{[\Delta]}$ is normal and hence contains an element in $Z(S)$. Hence, assuming that no element in the center of a Sylow 2-subgroup acts trivially implies that S acts faithfully on Δ and hence $G_{[\Delta]}$ must be contained in $O(L)$ so that, $L/O(L) \simeq (L/G_{[\Delta]})/(O(L)/G_{[\Delta]})$.

Since S_1 acts faithfully on Δ, τ must fix a point Q of the orbit Γ of length $q^2 - q$. Any orbit length under S of a point of Γ is at least q so any orbit length under S_H is at least $q/2$. It follows since S_1 fixes at least two points of Δ that τ fixes exactly $q/2$ points of Γ and hence exactly $1 + q/2$ points of Δ.

Let R_o denote a Sylow u-subgroup of M_2 so that $< z >$ commutes with R_o modulo $O(L)$. However, since u does not divide $\mid O(L) \mid$, it follows that $< z >$ commutes with R_o^x for some x in $O(L)$. Since u divides $q + 1$, then R_o^x must fix a point P fixed by z. Let S denote a Sylow 2-subgroup of G that fixes P. Then $S \cap M_1M_2$ is a Sylow 2-subgroup of M_1M_2 so equal to $T_1 \times T_2$ where T_i is a Sylow 2-subgroup of M_i for $i = 1, 2$. Since R_o^x and T_1 commute modulo $O(L)_P$ it follows as above that R_o^x commutes with some T_1^y for y in $O(L)_P$. Thus, $R_o^x \times T_1^y$ acts on the component OP containing P in the projective extension. Since R_o^x is a u-group, if there exists an element h of order u which fixes OP pointwise, then h must fix a second point of Δ which implies that u then divides $q - 1$, which is a contradiction. Hence, R_o^x acts fixed-point-free on OP. Let g be an element of R_o^x of order u. Then g fixes at least two points of Δ and at least one point of Γ.

By Johnson 15, there is a Desarguesian spread Σ_g consisting of g-invariant subspaces of dimension $2m$ over $GF(2)$ where 2^{2m} is the order of the plane. Moreover, T_1^y acts as a collineation group in $GL(2, 2^{2m})$ since the group commutes with g and g acts as a kernel homology group of Σ_g. However, this implies that T_1^y acts as an elation group of the Desarguesian plane and since it fixes OP, it has OP as its axis. Since OP is also a component of the translation plane, we have an elation in G contrary to assumption. This completes the proof of the lemma. ∎

Lemma 24. *$\widetilde{M}_o$ cannot be isomorphic to A_m.*

Proof: Under the assumptions and statement of Holt's theorem, it is possible that the group indicated can be isomorphic to an alternating group A_m where m is the number of $O(L^*)$ orbits on Δ. But $O(L^*) = O(L/G_{[\Delta]}) = O(L)/G_{[\Delta]}$ so the number of $O(L^*)$ orbits on Δ is the number of $O(L)/G_{[\Delta]}$ orbits which is the number of $O(L)$ orbits. However, we have noted that $\widetilde{M}_o$ is transitive and since it is simple $O(\widetilde{M}_o) = 1$ so a re-application of Holt's results imply that the group generated by the involutions which fix exactly one point of Δ is as stated in Holt's theorem. However, since we realize that the group is A_n, this forces the group to be isomorphic to A_{q+1}. If so then there is a cyclic group of order $q/2$ generated by the element in cycle notation $(1, 2, 3, ..., q/2)(q, q+1)$ acting on Δ. Since $O(L)$ contains $G_{[\Delta]}$, it follows that there is a cyclic collineation group of order $q/2$. We have seen that this cannot occur by 6. ∎

Lemma 25. *$O(L^*) =< 1 >$. Hence, $O(L)$ fixes Δ pointwise.*

Proof: By Holt 19, it follows that $\widetilde{\sigma}$ is in $\widetilde{M}_o$. This means that for the subgroup L^* of the group G^* acting on Δ has $O(L^*) = \langle 1 \rangle$ by Holt's theorem. ∎

Lemma 26. *$\widetilde{M}_o$ cannot be isomorphic to $S_z(\sqrt{q})$.*

Proof: Since 2 does not divide $\mid Out(S_z(\sqrt{q}) \mid$ it follows that the order of the Sylow 2-subgroup of G is q. However, since a central involution fixes points of Γ and Γ is an orbit of length $q(q-1)$, it follows that a Sylow 2-subgroup has order $\geq 2q$. ∎

Hence, we arrive at the following situation:

Remark 1. *$G^* = G/O(L)$ contains a normal subgroup $Y^* = Y/O(L)$ isomorphic to $SL(2, q)$ or $PSU(3, q^{1/3})$. The centralizer of Y^* fixes Δ pointwise so is trivial as G^* is a permutation group on Δ. Hence, $G^*/Y^* = (G/O(L))/(Y/O(L)) \simeq G/Y$ is a subgroup of the outer automorphism group of Y^*.*

Lemma 27. *$O(G) = O(L) = G_{[\Delta]}$ and u does not divide the order of $O(G)$.*

Proof: Let S be a Sylow 2-subgroup fixing the unique point P of Δ. The order of $Z(S)$ is ≥ 4 by assumption, so we have an elementary Abelian 2-subgroup A of order 4 normalizing $O(G)$ such that each involution in A fixes a unique point P of Δ. Hence, $C_{O(G)}(g)$ for $g \in A - \{1\}$ fixes the unique point P of Δ which implies that $O(G) = \left\langle C_{O(G)}(g); g \in A - \{1\} \right\rangle$ also fixes P. Since S was arbitrary, it follows that $O(G)$ fixes Δ pointwise.

Let R_o denote a Sylow u-subgroup of $O(G)$. It follows that R_o is cyclic since the subgroup fixes Δ pointwise. Since there are an odd number of Sylow u-subgroups in $O(G)$, it follows that any Sylow 2-subgroup S normalizes some R_o^h for $h \in O(G)$ and hence, normalizes the unique cyclic subgroup of order u. So, $R_o^h S$ fixes a point P on Δ. Let g_u be an element of order u. Since R_o^h is cyclic, there exists a Desarguesian spread consisting of g_u-invariant subspaces of line size and the normalizer of $\langle g_u \rangle$ is a collineation group of the Desarguesian affine plane Σ. Hence, the subgroup S is a subgroup of a group isomorphic to $\Gamma L(2, q^2)$ acting on Σ. Since $q = 2^r > 2r$ for $q > 4$, there must be an elation in S as then $S \cap GL(2, q^2)$ is nontrivial. However, this is contrary to our assumptions. ∎

Lemma 28. *(1) A Sylow 2-subgroup of G/Y is cyclic.*
(2) u does not divide $|G/Y|$.

Proof: G/Y is an outer automorphism group of $SL(2, q)$ or $PSU(3, q^{1/3})$. The outer automorphism group has order r and $(3, q^{1/3}+1)2r/3$ where $q = 2^r$, respectively. ∎

Lemma 29. *Let Q be a point of Γ. Then a Sylow 2-subgroup of Y_Q is cyclic or generalized quaternion.*

Proof: Since Γ is an orbit then $q+1$ divides the order of G_Q. Let $K = O(Y)$. Since G/Y is a group of outer automorphisms of Y^*, and then the order of $G_QY/Y \simeq G_Q/Y_Q$ is not divisible by u so that u divides the order of Y_Q. If $Z_2 \times Z_2$ is a subgroup of Y_Q then Y_QK/K is a subgroup of $Y/K = Y^*$ which contains a element whose order is a 2-primitive divisor of q^2-1 and which contains an elementary Abelian subgroup of order at least 4. By the structure of $SL(2,q)$ or $PSU(3,q^{1/3})$, it follows that $Y_QK/K = Y/K$ so that $Y_QK = Y$. Hence, $|Y_Q|_2 = |Y|_2$ since 2 does not divide the order of K.

Since $|G_Q|_2 = |Y_Q|_2\,[G_Q : Y_Q]_2 = |Y|_2\,|G_QY/Y|$ and
$|G|_2 = |G_Q|_2\,[G : G_Q]_2 = |Y|_2\,|G/Y|_2 = |Y|_2\,|G_QY/Y|\,[G : G_Q]_2$, it follows that $q = [G : G_Q]_2$ (the 2-part of the orbit length of Q) $= |G/Y|_2\,/\,|G_QY/Y| \leq r_2$ where $q = 2^r$ since G/Y induces an outer automorphism group on Y^* and a 2-group is cyclic of order the 2-part of r. Hence, $2^r \leq r$ which cannot occur. ∎

Lemma 30. *The central involutions in a Sylow 2-subgroup of Y share a common fixed point set of q points.*

Proof: Let S be a Sylow 2-subgroup of Y fixing the point P of Δ. Then $Z(S)$ is the set of involutions of a Sylow 2-subgroup isomorphic to one of Y^* which is isomorphic to $SL(2,q)$ or $PSU(3,q^{1/3})$.

Let A be any subgroup of $Z(S)$ of order 4. Note that A cannot fix a point of Γ. Hence, if σ_1 and σ_2 are distinct involutions in A then σ_2 induces on $Fix\sigma_1$ either an elation, Baer involution or the identity. However, since $Z(S)$ can fix exactly one point of Δ and no points of Γ, it follows that σ_2 induces an elation on $Fix\sigma_1$. ∎

Lemma 31. *Let $K = O(G) = O(L) = G_{[\Delta]}$. Then K is Z-group of order divisible by $q-1$.*

Proof: let A be an elementary Abelian 2-group of a Sylow 2-subgroup S of Y that fixes the point P of Δ. Each element z of A is Baer and fixes a subspace on OP of q points. Since K is generated by the centralizers of elements of A, and A fixes a set of q points of OP, it follows that K permutes a set of q points of OP. If some element k of K fixes one of these points then k is a Baer collineation and k divides $q-1$. Let $Fixk = \pi_o$ and note that π_o is a Baer subplane of the net N_Δ. Moreover, k fixes a second Baer subplane π_1 of N_Δ and induces a kernel homology on π_1. It is also true that k fixes a unique second Baer subplane of N_Δ. Suppose k commutes with some element τ of A. Then, τ must fix π_o and π_1. However, τ must have fixed points on π_1 which leads immediately to a contradiction.

In any case, $\pi_1\tau$ is a third subplane of the net N_Δ which implies that there are $1 + |\text{kernel of } \pi_o| = 1 + 2^a$ Baer subplanes incident with the zero vector of N_Δ. The group induced on the $1+2^a$ subplanes is a subgroup of $P\Gamma L(2,2^a)$. A Sylow 2-subgroup S that fixes P must also leave π_o invariant and cannot fix a second Baer subplane since if so then an involution in S must induce an elation on each Baer subplane forcing the involution to be an elation. Hence, Y/K is isomorphic to $SL(2,q)$ and the kernel of π_o is $GF(q)$. Note also that K must fix π_o. Furthermore, since this argument is independent upon the Sylow 2-subgroup S, it follows that each Baer subplane is pointwise fixed by some element in K as K is normal and K must leave each such Baer subplane invariant. Note that each Sylow 2-subgroup must leave invariant a unique point of Δ and for any point P^* of Δ, there is a Sylow 2-subgroup fixing P^*. However, this implies that k leaves each subplane invariant which is a contradiction.

Hence, K is fixed-point-free on the sets of q points fixed pointwise by the set of central involutions of Sylow 2-subgroups. It follows that K is a Frobenius complement as it acts on a vector space. Hence, all Sylow p-subgroups are cyclic for p odd and K has odd order so that K is a Z-group of order dividing $q-1$. ∎

Lemma 32. *K commutes with $Z(S)$ for any Sylow 2-subgroup, S of Y.*

Proof: We have the group $Z(S)K$. Let $\sigma \in Z(K)$ and $x \in K$, then consider $\sigma x \sigma x^{-1}$. Since $\sigma = \sigma^{-1}$ and K is normal then $\sigma x \sigma x^{-1} = x^* x^{-1}$ for some element x^* of K. But, σ and σ^x both fix a set of q points on a component OP pointwise for P a point of Δ and K is fixed-point-free on this vector subspace. Hence, $x^* x^{-1} = 1$ so that $x^* = x$ and $\sigma x \sigma = x$. Thus, K commutes with every element $\sigma \in Z(Q)$. ∎

Lemma 33. *$Y = \langle Z(S); S$ is a Sylow 2-subgroup of $Y\rangle K$ and thus $Y = C_Y(K)K$.*
Furthermore, $C_Y(K)$ is a central extension of Y^*.

Proof: Note that $\langle Z(S); S$ is a Sylow 2-subgroup of $Y\rangle$ is a normal subgroup of Y and $\langle Z(S); S$ is a Sylow 2-subgroup of $Y\rangle K/K$ is a normal subgroup of Y^*. It follows that $Y = \langle Z(S); S$ is a Sylow 2-subgroup of $Y\rangle K$.

Since $\langle Z(S); S$ is a Sylow 2-subgroup of $Y\rangle$ centralizes K, we have the proof of the lemma. ∎

Lemma 34. *$C_Y(K)$ is a central extension of Y^*.*
Then $C_Y(K) = Z(K)(C_Y(K))'$ and $(C_Y(K))'$ is perfect.

Proof: Y/K is isomorphic to $C_Y(K)/C_Y(K) \cap K$. But, $C_Y(K) \cap K \subseteq Z(C_Y(K))$ since all elements of K commute with all elements of $C_Y(K)$. Hence, $C_Y(K)$ is a central extension of a perfect group so the result follows by 22. ∎

Lemma 35. *$(C_Y(K))' \cap K = \langle 1\rangle$ so that Y contains a subgroup isomorphic to either $SL(2,q)$ or $PSU(3,q^{1/3})$.*

Proof: Note that $(C_Y(K))' \cap K \subseteq Z(C_Y(K)')$ since every element of $C_Y(K)$ commutes with every element of K. Furthermore,

$$\begin{aligned} Y/K &= C_Y(K)K/K = (C_Y(K))'Z(K)K/K = \\ &= (C_Y(K))'K/K \simeq (C_Y(K))'/(C_Y(K))' \cap K \end{aligned}$$

so that as $(C_Y(K))'$ is perfect and $(C_Y(K))' \cap K \subseteq Z((C_Y(K))')$, it follows that $(C_Y(K))' \cap K$ is a subgroup of the Schur multiplier of Y/K.

Thus, $(C_Y(K))' \cap Z(K)$ is a subgroup of the Schur multiplier of $(C_Y(K))'K/K = Y^*$ by 23. When Y^* is $SL(2,q)$ or $PSU(3,q^{1/3})$ the Schur multiplier is trivial (see 24, 25) so that $(C_Y(K))'$ is isomorphic to $SL(2,q)$ or $PSU(3,q^{1/3})$. ∎

Actually, there is an alternative proof without the use of Schur multipliers that we might mention here. Starting with the lemma 32 so that

$Y = \langle Z(S); S$ is a Sylow 2-subgroup of $Y\rangle\, K$ and thus $Y = C_Y(K)K$, we consider $g \in N_{C_Y(K)}(Z(S)) \cap Z(S)^x$ for $x \in C_Y(K) - N_{C_Y(K)}(Z(S))$. We see that it must be that x fixes OP where P is the unique point of Δ fixed by $Z(S)$. Since K commutes with $Z(S)$, it follows that $S \neq S^x$ and hence modulo K are distinct Sylow 2-subgroups acting in Y/K. But, the action on Δ is the standard 2-transitive in Y/K so that any two Sylow 2-subgroups of Y generate the group Y/K. Thus, x cannot leave OP invariant and hence $N_{C_Y(K)}(Z(S)) \cap Z(S)^x = \langle 1\rangle$. Thus, we may use the results of Hering 18 to show that there is a subgroup isomorphic to $SL(2, 2^a)$, $S_z(2^a)$, $SU(3, 2^a)$, or $PSU(3, 2^a)$ which acts transitively on Δ so there are at least $q+1$ Sylow 2-subgroups. Thus, we must obtain either the case $SL(2, q)$ or $PSU(3, q^{1/3})$. ∎

Lemma 36. *There cannot be a subgroup of Y isomorphic to $PSU(3, q^{1/3})$ or $SL(2, q)$.*

Proof: First assume that the group is isomorphic to $PSU(3, q^{1/3})$. $H = M_o$ is a normal subgroup of G so acts 1/2-transitively on Γ. Hence, $2(q+1)$ divides $(M_o)_Q$. Since $\widetilde{M}_o$ acts on Δ in its normal 2-transitive representation (Holt 19), it follows that $(M_o)_Q$ is transitive on Δ.

First we note that all involutions are Baer as all involutions in $PSU(3, q^{1/3})$ are conjugate or merely note that otherwise we would have an elation and be finished by previous results.

Since H is normal in G and G is transitive on Γ then the number of involutions in H_Q is equal to the number of involutions in H_{Q^x} for all x in G and hence equal to the number of involutions in H_T for T in Γ.

We count the set $\Lambda = \{(Q, z); Q \in \Gamma$ and z an involution in $H_Q\}$ in two ways. We shall call (Q, z) a 'flag'. The number of flags is obtained by counting the number of point Q times the number of involutions fixing Q which is then equal to the number of involutions z in H times the number of points of Γ that z fixes.

Since each involution in H fixes exactly one point of Δ and all involutions in H are conjugate, we have:

$(q^2 - q) \times C = [H : C_H(z)] \times q$ as a Baer involution fixes exactly q points of Γ where C is the number of involutions in H_Q. Let $e = (q^{1/3}+1, 3)$ and $q_1 = q^{1/3}$ so that $[H : C_H(z)] \times q = [((q_1^3+1)q_1^3(q_1^2-1)/e)/(q_1^3(q_1+1))] \times q$

$= (q+1)(q_1-1)q$.

Hence, $C = ((q+1)(q_1-1)q)/(q^2-q) = ((q+1)(q_1-1))/(q-1)$ which is a contradiction as $(q+1, q-1) = 1$ and $q^{1/3} - 1 < q - 1$.

Hence, M_o must be isomorphic to $SL(2, q)$ so that the plane is classified by Foulser-Johnson 9. The structure of the group forces the plane to be Ott-Schaeffer which is contrary to lemma 5.

Hence, the theorem is proven except possibly when the order is 64.

5.2 Order 64.

It remains to consider order 64.

The proof given in general applied to order 64 breaks down in essentially two places. In particular, it might be possible that there is an elation with center in Δ when $q = 8$.

Furthermore, in the argument of part II, it was required that there is a 2-primitive divisor of $q^2 - 1$ which is, of course, not valid when $q = 2^6$.

5.2.1 The elation case.

When there is an elation group of order at least 4, it follows that either $SL(2, 4)$ and hence $SL(2, 8)$ is generated by the elations. Then the plane is Desarguesian by theorem 9.

Hence, if there is an elation, we may assume that there is a unique elation of order 2 per center in Δ. The argument given in Hiramine, Jha and Johnson [9] supporting theorem 5 applies in this case to show that if M is the group generated by the elations then $O(M)$ fixes $\ell_\infty - \Delta$ pointwise. The argument when there exists a prime 2-primitive divisor applies that connects a Desarguesian spread to the components of $\ell_\infty - \Delta$ will apply provided if there exists a collineation of order 9 when is then a 2-primitive divisor of $2^6 - 1$ albeit not a prime divisor. Then there exists a Desarguesian spread sharing the components of $\ell_\infty - \Delta$ and by Ostrom's result on critical deficiency, the plane is Hall or Desarguesian.

Hence, it remains to show that there is a cyclic collineation group of order 9.

We see that $O(M)$ is faithful on any component ℓ of $\ell_\infty - \Delta$ and is semi-regular and is thus fixed point free. Since $O(M)$ then becomes a Frobenius complement, the Sylow 3-subgroups are cyclic. Similarly, if $O(M)$ fixes a proper $GF(2)$-subspace then 9 must divide $2^a - 1$ for $a = 1, 2, 3, 4$ or 5 which it does not. Hence, $O(M)$ acts irreducibly on ℓ and hence the centralizer in $Hom(\ell, \ell)$ is a field. But, this implies that $O(M)$ is cyclic and hence the plane is either Desarguesian or Hall as noted above.

5.2.2 When $Fix\sigma$ is in N_Δ.

We have note previously that the arguments given in the relevant section does not depend on the value for q. Hence, when $q = 8$, we have also completed the analysis.

5.2.3 When $Fix\sigma$ is not in N_Δ.

We note that it might be expected that the Ott-Schaeffer planes would occur here, but the orbit length of $q^2 - q$ for $q = 8$ does not occur in the Ott-Schaeffer planes as we have seen previously.

We may assume that every involution ρ in the center of a Sylow 2-subgroup fixes exactly one point of Δ.

But, this implies that the order of a Sylow 2-subgroup is divisible by $2 \cdot 8$. Furthermore, this also says that there is an involution τ which fixes at least two infinite points of Δ.

Lemma 37. *If an involution τ fixes at least two points of Δ then $Fix\tau$ is a subplane of N_Δ and N_Δ is derivable.*

Proof: If $Fix\tau$ is not a subplane of N_Δ then τ fixes a component, say ℓ, of Γ. But, there exists a central involution σ fixing ℓ. Hence, there exists a Sylow 2-subgroup S such that σ is central in S and τ is also in S. But, S fixes exactly one point of Δ and τ can't induce a Baer involution on $Fix\sigma$ since the order is 8. Hence, τ induces an elation on $Fix\sigma$ but fixes at least two components of $Fix\sigma$, a contradiction. Hence, $Fix\tau$ is a Baer subplane of N_Δ.

Since $Fix\tau$ cannot be G-invariant by 2 and $Fix\tau$ has order 8 then this implies that either N_Δ is derivable or there exist exactly three Baer subplanes of N_Δ incident with the zero vector. In this case, we consider the group $A = \langle \sigma, \tau \rangle$ and note that σ must induce an elation on $Fix\tau$. Hence, if σ fixes a second Baer subplane π_1 of the three then σ must fix points of π_1 which is a contradiction as σ fixes exactly one infinite point P of Δ and σ is not an elation. Similarly, τ cannot fix either of the two remaining Baer supblanes. Hence, it follows that $\sigma\tau$ leaves all three planes invariant which is a similar contradiction as $\sigma\tau$ is an involution which fixes $Fix\tau \cap OP$ pointwise and fixes P but fixes no other points of Δ. Hence, the net N_Δ is a derivable net. ∎

Lemma 38. *Let T denote the subgroup which fixes $Fix\tau$ pointwise.*
Then the order of T is 2.

Proof: If the order of T is strictly larger than 2, then there is a group isomorphic to $SL(2, 2^a \geq 4)$ within a group isomorphic to $SL(2, 8)$ which is generated by $\langle T^x; x \in G \rangle$. Since the plane is derivable, it follows immediately that the group generated must be $SL(2, 8)$ and the plane is Hall by 9. ∎

Now since S acting on the set of 9 Baer subplanes incident with the zero vector is faithful (i.e. there are no elations), it is a subgroup of $P\Gamma L(2, 8)$ which cannot be the case since the order of S is at least $2 \cdot 8$.

Hence, we have:

Lemma 39. *The case that $Fix\sigma$ is not in N_Δ does not occur.*

Hence, we have completed the case when the order is 64 which, in turn, completes the proof of our main theorem. ∎

6 The spread is in $PG(3, q)$, q even.

Theorem 26. *Let π denote a translation plane of order q^2, q even, with spread in $PG(3, q)$.*

If G is a linear collineation group which has an infinite point orbit of length $q+1$ and i infinite point orbits of lengths $(q^2 - q)/i$ and $i = 1$ or 2 then π is one of the following types of planes:

(1) Desarguesian, $i = 1$,

(2) Hall, $i = 1$ or

(3) Ott-Schaeffer and $i = 2$.

Proof:

First we note that if $i = 1$, we may employ the main result of the previous sections.

Assume that the order is 64 and there exist elations. Since the group is linear, then it follows from the result of Johnson and Ostrom 8 that either $SL(2,8)$ is generated or there is a dihedral group of order $2 \cdot 9$ generated. In the former case, the plane is Desarguesian. In the latter case, the cyclic group C_9 of order 9 is normal in the full group. Furthermore, C_9 fixes a component in each of the two orbits at infinity of length 28. It follows that C_9 fixes 56 infinite points. Thus, there is a collineation of order 9 and since 9 is a 2-primitive divisor of $2^6 - 1$ of order 9, it follows similarly as above in the order 64 case for $i = 1$ that, using Ostrom's theorem on critical deficiency, the plane is Desarguesian or Hall.

Thus, we may assume by 5 and the above argument that the involutions are all Baer. Since the group is linear then by Johnson-Ostrom 10, the Sylow 2-subgroups are elementary Abelian.

First assume that the group is nonsolvable. Then, by Johnson-Ostrom 11, we obtain a normal subgroup isomorphic to $SL(2,2^s)$ for some integer s. Since, the Sylow 2-subgroups has order at least $q/2$, it follows from Johnson 12, that if $q/2 > \sqrt{q}$ then $SL(2,q)$ is obtained as a collineation group. Hence, since for $q = 4$, all translation planes are known, we may assume that $q > 4$ and thus $q/2 > \sqrt{q}$. Now, by the results of 9, it follows that the planes listed are the only possible planes of order q^2 admitting $SL(2,q)$.

Thus, assume that the group G is solvable. Let Γ denote an orbit of components of length $(q^2 - q)/i$ for $i = 1$ or 2. The stabilizer of a component L of Γ has order divisible by $(q+1)$. By the general section on planes of even order, we may assume that $i = 2$.

Now a Sylow 2-subgroup S has an orbit in Γ of length divisible by $q/2$. First assume that S has order $q/2$. Since there are two orbits of length $(q^2 - q)/2$ then each involution must fix each infinite points of the orbit Δ of length $q + 1$. Since S must fix a 1-space pointwise on a component of Δ, it follows that S must fix a Baer subplane pointwise. Assume that Fix$S$ is invariant under the group G. The order of the group G is divisible by $(q-1)^2(q+1)q$ and is linear so it follows that $G/Z(G)$ is a solvable subgroup of $PGL(2,q)$ which is a contradiction by order.

Thus, $FixS$ is not G invariant and thus there exist at least two Baer groups of order $q/2$. It follows that the group generated by $SL(2,2^s)$ where $2 \geq q/2$. So, either $q = 4$ or we obtain $SL(2,q)$ as a collineation group. Since all translation planes of order 16 are determined, we again have a contradiction.

So, a Sylow 2-subgroup has order divisible by q and as all involutions are Baer, the order is exactly q. Since there are exactly $q - 1$ involutions in S, it follows that each involution fixes exactly $q/2$ infinite points of each orbit of length $(q^2 - q)/2$.

Hence, each Sylow 2-subgroup fixes exactly one component of Δ and G is doubly transitive on Δ. It follows that S is a subgroup of $\Gamma L(1, q+1)$ and $q+1$ is a prime power v^b since G is solvable. However, $GL(1, q+1)$ is cyclic of order q. When $S \cap GL(1, q+1)$ has order at least $q/(q,b) > 2$, then the subgroup is cyclic and elementary Abelian which is a contradiction. However, $q/(q,b) > 2$ for q larger than 4. To see this, note that otherwise, $2^r + 1 = v^{2^{r-1}z} > 2^{2^{r-1}}$ which implies that $r = 1$ or 2.

So, when $q > 4$, the group must be solvable and this completes the proof of the theorem. ■

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