KS-models and symplectic structures on total spaces of bundles

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1 Introduction

In this note we give some applications of the Félix-Thomas theorem on rational models of fibrations [FT] to the problem of constructing symplectic structures on total spaces of bundles.

It is well known that constructing symplectic structures on *closed* manifolds is an important but difficult problem in symplectic topology (see, e.g. [McDS]). One of the methods of creating new symplectic manifolds out of the known ones is based on the use of the given symplectic structures on the base and the fiber of a bundle to get a new one on the total space. There are several ways of doing this, for example, coupling forms of Lerman, Guillemin and Sternberg [GLS], fat bundles of Weinstein [TK, W] and Thurston's method [Th, McDS]. The germs of all these methods can be found in the notion of *symplectic fibration* and a theorem of Thurston which we cite below.

Definition 1.1. Let $M \xrightarrow{\pi} B$ be a locally trivial fibration with symplectic base (B, ω_B) and fibre (F, ω_F). The fibration is called *symplectic* if its structural group acts on the fibre by symplectomorphisms. In this case each fibre $\pi^{-1}(b) = F_b$ carries a symplectic structure ω_{F_b}, [McDS, p.192].

Now the natural question arises when the total space of such a bundle admits a symplectic structure ω_M which is *compatible* with fibration π. Compatibility means

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that $i_b^*[\omega_M] = [\omega_{F_b}]$, where $i_b : F_b \hookrightarrow M$ is an inclusion of the fibre. The answer is given by the following result of Thurston.

Theorem 1.2 (Thurston). *Let $M \xrightarrow{\pi} B$ be a compact symplectic fibration. If there exists a cohomology class*

$$\beta \in H^2(M; \mathbb{R}),$$

such that $i_b^*\beta = [\omega_{F_b}]$, then M admits a symplectic structure ω_M compatible with fibration π. ω_M represents the class $\beta + K[\pi^*\omega_B]$, where $K > 0$ is sufficiently large, [McDS, p.192].

Note that the condition in Theorem 1.2 which guarantees the existence of a symplectic structure is cohomological. Hence, it is quite natural to look for homotopic methods of checking it. Note also that Theorem 1.2 is effective only for compatible symplectic structures (see Remark 3.4). Probably, the first explicit result of this type, developing the idea of Thurston [Th] was obtained in the paper of Geiges [G], in which the author used the Hochschild-Serre spectral sequence.

The novelty of our approach is the use of methods of rational homotopy theory. Our results are inspired by the work of Félix and Thomas on models of fibrations

$$F \to M \to B$$

in which these fibrations are not assumed to be nilpotent (nilpotency appears to be a strong restriction in symplectic geometry, since many important classes of symplectic manifolds do not satisfy this property). Now we are going to present a homotopic part of our method inspired, in a sense, by the works [F, OT, TO]. Recall that to every connected topological space S, there is associated a free differential graded algebra $M_S = (\Lambda X, d_S)$ such that

- there exists a basis $\{x_\alpha\}_{\alpha \in J}$ of X for some well ordered set J such that

  $$d_S(x_\alpha) \in \Lambda X_{<\alpha}$$

  and

  $$|x_\alpha| < |x_\beta| \Rightarrow \alpha < \beta,$$

  where $\Lambda_{<\alpha}$ is a subalgebra in ΛXgenerated by all $x_\beta, \beta < \alpha$,

- there exists a DGA-morphism $M_S \to A(S)$ which induces an isomorphism on cohomology. Here $A(S)$ denotes Sullivan's rational forms if $\mathbb{Q}$-coefficients are used or the De Rham algebra if $\mathbb{R}$-coefficients are used. M_S is called a *minimal model* of S [F].

This idea of minimal models can be used to construct models of mappings and in particular to model fibrations, [H]. Let $F \hookrightarrow M \to B$ be a fibration. There exists the *Koszul-Sullivan model (KS-model)* :

$$\begin{array}{ccccc}
(A(B), d_B) & \xrightarrow{\pi^*} & (A(M), d_M) & \xrightarrow{i^*} & (A(F), d_F) \\
{\scriptstyle \varphi_B}\uparrow & & {\scriptstyle \varphi_M}\uparrow & & {\scriptstyle \varphi_F}\uparrow \\
(\Lambda X, d_X) & \longrightarrow & (\Lambda X \otimes \Lambda Y, d) & \longrightarrow & (\Lambda Y, d_Y)
\end{array}$$

where φ_B and φ_M induce cohomology isomorphisms and:
$d(x) = d_X(x)$ for $x \in X$
$d(y_\alpha) \in \Lambda X \otimes \Lambda Y_{<\alpha}$
$|y_\alpha| < |y_\beta| \Rightarrow \alpha < \beta$, where $\{y_\alpha\}_{\alpha \in J}$ is a basis of Y for a well ordered set J.

In the sequel, we write $H^*(X)$ for a cohomology with rational or real coeficients, if it does not matter which ones are used.

Theorem 1.3 (Félix-Thomas). **[FT]***Let $F \hookrightarrow M \to B$ be a fibration and U denote the largest $\pi_1(B)$-submodule of $H^*(F)$ on which $\pi_1(B)$ acts nilpotently. Assume that the following conditions are satisfied:*

- *$H^*(F)$ is a vector space of finite type*
- *B is a nilpotent space (i.e. $\pi_1(B)$ is a nilpotent group which acts nilpotently on $\pi_k(B)$, $k > 1$).*

Then in the KS-model of the fibration the DGA-morphism

$$\varphi_F : (\Lambda Y, d_Y) \to (A(F), d_F)$$

induces an isomorphism $\varphi_F^* : (\Lambda Y, d_Y) \xrightarrow{\cong} U$.

Remark 1.4 If the action of $\pi_1(B)$ on $H^*(F)$ is nilpotent then $(\Lambda Y, d_Y)$ is a minimal model of the fibre F (Grivel-Halperin-Thomas), [H].

In our paper we use this theorem to construct compatible symplectic forms on symplectic fibrations. In particular, we show that many proofs of existence of symplectic structures become much simpler. We give a new proof of one of Geiges' results [G, Theorem 1] which completely describes the existence of compatible symplectic structures on total spaces of T^2-bundles over T^2 (Theorem 3.3). In fact, Geiges proved the existence of symplectic structure (not necessarily compatible) in any cohomology class $a \in H^2(M)$ (M is the total space of T^2-bundle over T^2) satisfying $a^2 \neq 0$, but it is out of the scope of our work, since we are interested only in structures which are compatible with the fibration. Moreover, in section 4, we give an application to pure fibrations.

Note that our results are applied to constructing *compatible symplectic structures* on symplectic fibrations and probably cannot be used in a more general situation.

2 The method

Let $F \hookrightarrow M \to B$ be a compact symplectic fibration which satisfies the assumptions of Theorem 1.3. We have a KS-model

$$(\Lambda X, d_X) \to (\Lambda X \otimes \Lambda Y, d) \to (\Lambda Y, d_Y).$$

Recall that the action of $\pi_1(B)$ on $H^*(F)$ factors through the component group $\pi_0(G)$, where G denotes the structural group of the fibration, [TO, p.91]. Since G acts on F by symplectomorphisms, then the action of $\pi_1(B)$ preserves the class $[\omega_F]$ of the symplectic form. It means that $[\omega_F] \in U \subset H^*(F)$.

Choose a cocycle $\omega_F' \in (\Lambda Y, d_Y)$ such that $\varphi_F^*[\omega_F'] = [\omega_F]$.

Proposition 2.1. *The symplectic fibration $F \hookrightarrow M \to B$ admits a compatible symplectic structure iff there is a cocycle in $(\Lambda X \otimes \Lambda Y, d)$ of the form $\omega_F' + \chi$, for some $\chi \in \Lambda^+ X \otimes \Lambda Y$.*

Proof. If the fibration admits a compatible symplectic structure, say ω_M, then the class $(\varphi_M^*)^{-1}[\omega_M] \in H^*(\Lambda X \otimes \Lambda Y, d)$ is represented by a cocycle of the form $\omega_F' + \chi$, where $\chi \in \Lambda^+ X \otimes \Lambda Y$.

On the other hand a class $\beta = \varphi_M^*[\omega_F' + \chi]$ satisfies the conditions of Theorem 1.2. ■

It is difficult in general to find an expression of the differential d. In the sequel we analyze cases in which it is possible to answer when ω_F' is closed in KS-model of the fibration and completely solve the case of compatible symplectic structures on T^2-bundles over T^2.

3 T^2-bundles over T^2

A general reference to this section is the work of Sakamoto and Fukuhara [SF], in which they classified T^2-bundles over T^2, see also [G].

Each T^2-bundle over T^2 one can describe by the data: (A, B, m, n), where

$$A, B \in GL(2; \mathbb{Z}),\ \ AB = BA \text{ and } m, n \in \mathbb{Z}.$$

We denote the bundle determined by the above data using the symbol $M(A, B; m, n)$. It is constructed as follows.

$$M(A, B; 0, 0) := T^2 \times \mathbb{R}^2/\sim, \text{ where}$$

$$([x, y], (u + 1, v)) \sim ([A(x, y)], (u, v))$$

$$([x, y], (u, v + 1)) \sim ([B(x, y)], (u, v))$$

Let $D^2 \in T^2$ be a small closed disk. $M(A, B; m, n)$ is defined by cutting out $T^2 \times D^2 \subset M(A, B; 0, 0)$ and glueing it back with the map

$$\begin{gathered} T^2 \times \partial D^2 \to T^2 \times \partial D^2 \\ ([x, y], \alpha) \mapsto \left(\left[x + \frac{m\alpha}{2\pi}, y + \frac{n\alpha}{2\pi} \right] \right). \end{gathered}$$

The projection of the bundle is

$$\begin{gathered} \pi : M(A, B; m, n) \to T^2 \\ [[x, y], (u, v)] \mapsto [u, v]. \end{gathered}$$

We consider only orientable bundles thus if we endow T^2 with the standard symplectic structure then it follows from the definition of $M(A, B; m, n)$ that this bundle is symplectic. Indeed, the structural group of $M(A, B; m, n)$ can be reduced to the group generated by A, B and translations and all these diffeomorphisms preserve the standard symplectic structure. The orientability of the bundle also implies that the action of $\pi_1(B)$ on $H^0(F) = \mathbb{R}$ and $H^2(F) = \mathbb{R}$ is trivial. The action on $H^1(F) = \mathbb{R}^2$ can be described with the use of the matrices A,B. Let g_1, g_2 be standard generators of $\pi_1(B) = \mathbb{Z}^2$. We can describe the $\pi_1(B)$-action on $H^1(F)$ as follows:

$$\pi_1(B) \to Aut(H^1(F))$$

$$g_1 \mapsto A, g_2 \mapsto B$$

Now we divide our computations into two parts with respect to the action of $\pi_1(B)$ on $H^1(T^2)$:

3.1 Nilpotent action

By Remark 1.4. we have the following form of KS-model of $M = M(A, B; m, n)$:

$$(\Lambda(a, b) \otimes \Lambda(x, y), d)$$

In this case ω'_F is, up to scalar, equal to xy. Without loss of generality we assume that this scalar is equal to 1. Notice that we can also assume that $B = Id$ and $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, $k \geq 0$ [SF]. Let us consider two cases depending on the first Betti number $b_1(M)$ (since $M = T^4$ if $b_1(M) = 4$, we shall say nothing about this trivial case):

3.1.a $b_1(M) = 2$. We have

$$M = M(\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \mathbf{Id}; (m, n) \neq (0, 0))$$

and the KS-model is isomorphic to

$$(\Lambda(a, b) \otimes \Lambda(x, y), d),$$

where $da = db = 0, dx = ab, dy = r_1ax + r_2bx, r_1, r_2 \in \mathbb{R}, r_1r_2 \neq 0$. Thus for any $\chi \in \Lambda^+(a, b) \otimes \Lambda(x, y)$ we have

$$d(\omega'_F + \chi) = d(xy + a(s_1x + s_2y) + b(s_3x + s_4y)) = aby + s_2r_2abx + s_4r_1bax \neq 0,$$

where $s_i \in \mathbb{R}, i = 1, ..., 4$. It means, due to Proposition 2.1, that there is no compatible symplectic structure.

3.1.b $b_1(M) = 3$. The bundle is isomorphic to

$$M = M(\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, Id; 0, 0), \text{ where } k > 0 \text{ or}$$

$$M = M(Id, Id; m, n) \text{ with } (m, n) \neq (0, 0).$$

In the first case we have a model

$$(\Lambda(a, b) \otimes \Lambda(x, y), d), \text{ where } da = db = dx = 0, dy = bx$$

and $d\omega'_F = dxy = 0$.

In the second

$$(\Lambda(a, b) \otimes \Lambda(x, y), d), \text{ where } da = db = dx = 0, dy = ab$$

and we have that $d(\omega'_F + \chi) = d(xy + a(s_1 x + s_2 y) + b(s_3 x + s_4 y)) = xab$. As in 3.1.a, it implies that there is no compatible symplectic structure.

3.2 Non-nilpotent action

Since $dim H^1(T^2) = 2$ then we have that $U^1 = U \cap H^1(T^2) = \{0\}$, if the action of $\pi_1(T^2)$ on $H^1(T^2)$ is non-nilpotent. Thus there are no elements of degree 1 in Y because if there were then $H^1(\Lambda Y, d_Y) \neq 0$ and it would give a contradiction to Theorem 1.3. Hence, by Theorem 1.3, the KS-model has the following form

$$(\Lambda(a, b) \otimes \Lambda(Y), d), \text{ where } H^*(\Lambda(Y), d_Y) \cong H^*(S^2).$$

Thus we have the only possibility, namely

$$(\Lambda(a, b) \otimes \Lambda(\sigma, \tau), d), \text{ where } da = db = d\sigma = 0, d\tau = \sigma^2$$

and in this case $\omega'_F = \sigma$.

Finally, due to Proposition 2.1, we obtain the following

Theorem 3.3. *Let M=M(A,B;m,n) be an orientable T^2-bundle over T^2. Then M*

- *is a symplectic T^2-fibration over T^2*
- *admits a compatible symplectic structure*

with two exceptions:

- *M=M(Id,Id;(m,n)$\neq$(0,0))*
- *$b^1(M) = 2$ and $\pi_1(T^2)$ acts nilpotently on $H^*(T^2)$.*

In these exceptional cases there is no compatible symplectic structure.

Remark 3.4 The total spaces of the exceptional cases are also symplectic manifolds. The first is diffeomorphic to the total space of $M = M(\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, Id; 0, 0)$, where $k > 0$ so we have shown that it admits a symplectic structure. The second is a nilmanifold (as well as the first) and one can easily compute its cohomology and take an invariant symplectic form [G].

Remark 3.5 Let $\Sigma_g \hookrightarrow M \to T^2$ be a symplectic bundle with the largest $\pi_1(T^2)$-submodule $U \subset H^*(\Sigma_g)$, on which $\pi_1(T^2)$ acts nilpotently. Here Σ_g denotes a compact surface of genus g. If this submodule equals to $H^0(\Sigma_g) \oplus H^2(\Sigma_g) \cong H^*(S^2)$ then we compute the KS-model of this fibration which has the same form as that in 3.2:

$$(\Lambda(a,b) \otimes \Lambda(\sigma,\tau), d), \text{ where } da = db = d\sigma = 0, d\tau = \sigma^2.$$

Hence, there is a symplectic form compatible with the fibration.

4 Further results

In [T] Thomas studied so called pure fibratons, see also [L]. In this section we give an application of these results to symplectic fibrations.

Definition 4.1. A fibration $F \to M \to B$ is called *pure* if its KS-model $(\Lambda X \otimes \Lambda Y, d)$ satisfies $d(Y^{even}) = 0$ and $d(Y^{odd}) \subset \Lambda X \otimes \Lambda Y^{even}$.

The next theorem is the main result of Thomas [T] about pure fibrations.

Theorem 4.2. *Suppose that a fibration $F \to M \to B$ satisfies one of the following conditions:*
1.$F = G/H$, where G is a compact connected Lie group, H is its closed connected subgroup and the fibration is associated to a G-principal bundle via the standard action of G on G/H.
2.$dim(Y^{even}) = dim(Y^{odd}) \leq 2$.
Then the fibration is pure.

The following theorem shows, in some cases, the existence of compatible symplectic structures on total spaces of pure fibrations.

Proposition 4.3. *Let $F \to M \to B$ be a compact symplectic fibration. Suppose that this fibration is pure and there exist a cocycle $\omega' \in (\Lambda Y, d_Y)$ such that*
1.$\varphi_F^*[\omega_F'] = [\omega_F]$, where ω_F is the symplectic form on F,
2.$\omega_F' = \sum a_i y_i$, where $a_i \in \mathbb{R}$, $y_i \in Y$ and $|y_i| = 2$.
Then there exist a compatible symplectic form on M.

Proof. Since ω_F' is a sum of generators of degree 2 then, due to assumption of pureness, $d(\omega_F') = 0$ and according to Propositon 2.1. there is a compatible symplectic structure on M. ∎

Example 4.4. Let $T^4 \to M \xrightarrow{\pi} T^2$ be a symplectic bundle constructed as follows. Let $f : T^4 \to T^4$ be a diffeomorphism give by the following matrix

$$A = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

T^4 is endowed with the standard symplectic structure, namely $\omega_{T^4} = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$ which is preserved by f. Then $M = T^4 \times \mathbb{R}^2/\sim$, where

$$(x, s+1, t) \sim (f(x), s, t)$$
$$(x, s, t+1) \sim (x, s, t)$$

and $\pi : M \to T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is given by $\pi[x, s, t] = [s, t]$. The action of $\pi_1(T^2)$ on $H^*(F)$ is given by

$$g_1 \mapsto f^*, g_2 \mapsto Id,$$

where g_1, g_1 are the standard generators of $\pi_1(T^2)$. Recall that U denotes the largest submodule of $H^*(F)$ on which $\pi_1(T^2)$ acts nilpotently. Since $det(A - Id) \neq 0$ then $U^1 = U \cap H^1(F) = \{0\}$. Similarly $U^3 = \{0\}$. It follows from the straightforward computation that $dimU^2 = 2$ and one of the generators is the cohomology class $[\omega_{T^4}]$ of the symplectic form. Obviously $U^0 = U^4 = \mathbb{R}$. Thus $U^* \cong H^*(S^2 \times S^2)$ and we get the following form of the KS-model of the fibration

$$(\Lambda(a, b) \otimes \Lambda(x, y, z, t), d),$$

where $|x| = |y| = 2, |z| = |t| = 3$ and due to Theorem 4.2. we obtain that the fibration is pure. Thus it admits a compatible symplectic structure, according to Proposition 4.3.

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