

Infinitary axiomatizability of slender and cotorsion-free groups

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Abstract

The classes of slender and cotorsion-free abelian groups are axiomatizable in the infinitary logics $L_{\infty\omega_1}$ and $L_{\infty\omega}$ respectively. The Baer-Specker group \mathbb{Z}^ω is not $L_{\infty\omega_1}$ -equivalent to a slender group.

1 Introduction

In 1974, Paul Eklof [4] used infinitary logic to generalize some classical theorems of infinite abelian group theory. He characterized the strongly \aleph_1 -free groups as exactly those abelian groups which are $L_{\infty\omega_1}$ -equivalent to free abelian groups, used his criterion to deduce that the class of free abelian groups is not $L_{\infty\omega_1}$ -definable, and showed that the Baer-Specker group \mathbb{Z}^ω is not $L_{\infty\omega_1}$ -equivalent to a free abelian group, strengthening a theorem of Baer [1] that \mathbb{Z}^ω is not free. This paper continues in the tradition allying infinite abelian group theory with infinitary logic. Its main result is the following:

Theorem 1.1. *The class **SL** of slender abelian groups is axiomatizable in the infinitary logic $L_{\infty\omega_1}$.*

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Corollary 1.2. *The Baer-Specker group \mathbb{Z}^ω is not $L_{\infty\omega_1}$ -equivalent to a slender group.*

Corollary 1.2 improves further the above-mentioned results of Baer [1] and Eklof [4]. Theorem 1.1 contrasts strikingly with another of Eklof's corollaries that the class of free groups is not $L_{\infty\omega_1}$ -definable. The proof of Theorem 1.1 also yields that the class of cotorsion-free abelian groups is $L_{\infty\omega}$ -definable.

It is a tacit assumption of the present paper that all the groups are abelian. Any undefined group-theoretical concepts can be found in Fuchs [8] and Eklof and Mekler [7]; for logical concepts, see Barwise [2] and Dickmann [3]. We recall just the essentials and the notation required to read the paper. The *Baer-Specker group* \mathbb{Z}^ω is the product of countably many copies of \mathbb{Z} (the integers). Recall that a torsion-free group G is *slender* if whenever $h : \mathbb{Z}^\omega \rightarrow G$ is a homomorphism, then $h(e_n) = 0$ for all but finitely many $n \in \mathbb{N}$, where e_n is the standard n -th unit vector $(0, \dots, 0, 1, 0, \dots) \in \mathbb{Z}^\omega$. We shall use **SL** to denote the class of slender groups. Since the proof of Theorem 1.1 uses nothing group-theoretical beyond Nunke's criterion, stated as Theorem 2.1, the reader wishing to have a self-contained proof may take Theorem 2.1(b) as an equivalent definition of slenderness. Examples of slender groups are \mathbb{Z} , and the free groups, since if $\{G_i : i \in I\}$ is a family of slender groups, then the direct sum $\bigoplus_{i \in I} G_i$ is slender. Subgroups of slender groups are slender. Strongly \aleph_1 -free groups are slender, and hence so are the \aleph_2 -free groups.

We use L to denote the vocabulary of groups: L contains a binary function symbol $+$ and a constant **O**. For infinite cardinals $\lambda \leq \kappa$, the infinitary language (or logic) $L_{\kappa\lambda}$ is defined to be the smallest collection of formulas containing the atomic L -formulas and closed under negation, conjunctions (disjunctions) of sets $\{\varphi_\alpha : \alpha < \mu < \kappa\}$ of formulas, and existential (universal) quantifier strings of length less than λ . The language $L_{\infty\lambda}$ is the union of the languages $L_{\kappa\lambda}$ as κ ranges over the class of infinite cardinals. We write $A \equiv_{\kappa\lambda} B$ for groups A and B , to mean that for every $L_{\kappa\lambda}$ -sentence φ , φ holds in A iff φ holds in B , and in this case, we say that A and B are $L_{\kappa\lambda}$ -equivalent. There is a well-known algebraic characterization of $L_{\kappa\lambda}$ -equivalence using systems of partial isomorphisms (see the references [2] or [3] for details). A class **C** of groups is $L_{\infty\lambda}$ -definable (or $L_{\infty\lambda}$ -axiomatizable) if for some $L_{\infty\lambda}$ -sentence φ , for every group G , $G \in \mathbf{C}$ iff φ holds in G .

Many applications of infinitary logic to the study of infinite abelian groups are described in sections 4, 5 and 7 of the survey [5]; the paper [6] contains further results and bibliographical references.

Group-theoretical interest in $L_{\infty\omega_1}$ -axiomatizability stems also in part from a general thesis of classification theory that measures the complexity of a complete system of invariants for isomorphism for a class of abelian groups in terms of whether these invariants can be described by $L_{\infty\omega_1}$ -sentences ([12], p.292; [11]). By Corollary 1.9 of [4], there is a group $G(\aleph_1)$ of cardinality \aleph_1 which is $L_{\infty\omega_1}$ -equivalent, but not isomorphic, to the free group $F(\aleph_1)$ of cardinality \aleph_1 . Since $F(\aleph_1)$ is slender, it follows that $G(\aleph_1)$ is also slender. In the framework of classification theory, the results of this paper establish that the first-order theory of the class **SL** does not have a good structure theorem, since the possible invariants distinguishing the slender groups $F(\aleph_1)$ and $G(\aleph_1)$ cannot have a simple definition. For the same reason, the first-order theory of the cotorsion-free groups has no good structure theorem either.

Finally, let us explain why Theorem 1.1 is the best possible, i.e. why the language $L_{\infty\omega_1}$ is the right one for axiomatizing **SL**. Note first that **SL** is not first-order definable, since if D is any non-principal ultrafilter over ω , then the ultrapower \mathbb{Z}^ω/D is not slender, but, by Łoś Lemma, $\mathbb{Z}^\omega/D \equiv_{\omega\omega} \mathbb{Z}$, and \mathbb{Z} is slender. So **SL** is not $L_{\omega\omega}$ -definable. Nor is **SL** $L_{\infty\omega}$ -definable: Barwise [2] shows that $\mathbb{Z}^\omega \equiv_{\infty\omega} \bigoplus_{n \in \omega} \mathbb{Z}$, and again $\bigoplus_{n \in \omega} \mathbb{Z}$, being free, is slender. On the other hand, it is easy to see that for every cardinal κ greater than the continuum, the class **SL** is $L_{\infty\kappa}$ -definable. So the question of real interest is whether **SL** is $L_{\infty\kappa}$ -definable for $\aleph_1 \leq \kappa \leq 2^{\aleph_0}$. Theorem 1.1 and Corollary 1.2 answer this question in the strongest possible form. All our results are theorems of ZFC (ordinary set theory), and this emphasizes the profound difference, as far as axiomatizability goes, between the classes of slender (and cotorsion-free) groups on the one hand, and the free groups on the other.

2 The proof of Theorem 1.1

The key algebraic component in the proof of Theorem 1.1 is the following well-known characterization of the slender groups, due to Nunke [10] (or see [7] or [8]).

Theorem 2.1. *For an abelian group G , the following conditions are equivalent:*

- (a) G is slender;
- (b) G does not contain a subgroup isomorphic to any of the following groups: \mathbb{Q} (the rationals under addition), \mathbb{Z}^ω , $\mathbb{Z}(p)$ (the cyclic group of order p) or J_p (the p -adic integers) for any prime p .

It will be convenient to split the proof of Theorem 1.1 into some facts and propositions which will all have the general form: for each group B in Theorem 2.1(b) there is a sentence φ such that for every torsion-free group G , φ is true in G iff G has a subgroup isomorphic to B .

Remembering that \mathbb{Q} has a generating system $\{q_n : n \in \omega\}$ with $q_n = (n+1)q_{n+1}$, one verifies easily the following fact.

Fact 2.2. *For any torsion-free group G , G contains an isomorphic copy of \mathbb{Q} iff G satisfies the $L_{\omega_1\omega}$ -sentence $\psi^{\mathbb{Q}} \equiv (\exists x)(x \neq 0 \wedge \bigwedge_{n \in \omega}(n! \text{ divides } x))$.*

Next we deal with the problem of expressing the assertion " G contains an isomorphic copy of \mathbb{Z}^ω " by an $L_{\infty\omega_1}$ -sentence. Let $S^* = \{a \in \mathbb{Z}^\omega : n! \text{ divides } a(n)\}$. Note that S^* is a subgroup of \mathbb{Z}^ω . For $a \in S^*$, let $\varphi_a(y, x_0, x_1, \dots, x_n, \dots)$ be the following $L_{\omega_1\omega_1}$ -formula:

$$\bigwedge_{n \in \omega} (n! \text{ divides } (y - \sum_{l < n} a(l)x_l)).$$

Fact 2.3. *Suppose that G is torsion-free and contains no isomorphic copy of \mathbb{Q} . Then:*

$$G \models \forall (x_0, x_1, \dots, x_n, \dots) \exists^{\leq 1} y \varphi_a(y, x_0, x_1, \dots, x_n, \dots); \tag{1}$$

if h is an embedding of \mathbb{Z}^ω into G , then for every $a \in S^*$

$$G \models \varphi_a[h(a), h(e_0), \dots, h(e_n), \dots] \tag{2}$$

Proof. To see that claim (1) is true, suppose otherwise, and let c_n ($n \in \omega$) and $d_1 \neq d_2$ in G witness its failure. Then for every $n \in \omega$, $\bigwedge_{i=1,2} G \models (n! \text{ divides } (d_i - \sum_{l < n} a(l)c_l))$. Subtracting, it follows that $G \models (d_1 - d_2 \neq 0) \wedge \bigwedge_{n \in \omega} (n! \text{ divides } (d_1 - d_2))$, and hence G satisfies $\psi^{\mathbb{Q}}$. By Fact 2.2, this contradicts the hypothesis that G contains no isomorphic copy of \mathbb{Q} .

Claim (2) is immediate because h is an embedding, and the elements $a, e_0, e_1, \dots, e_n, \dots$ satisfy the formula $\varphi_a(y, x_0, x_1, \dots, x_n, \dots)$ in \mathbb{Z}^ω .

Let $\varphi^*(x_0, x_1, \dots, x_n, \dots)$ be the logical conjunction of the following formulas:

- (A) $\bigwedge \left\{ (\exists y) \varphi_a(y, x_0, x_1, \dots, x_n, \dots) : a \in S^* \right\}$
- (B) $\bigwedge \left\{ (\exists y_1 \exists y_2 \exists y_3) \left(\bigwedge_{i=1,2,3} \varphi_{a^i}(y_i, x_0, x_1, \dots, x_n, \dots) \wedge (y_3 = y_1 - y_2) \right) : a^1, a^2, a^3 \in S^*, a^3 = a^1 - a^2 \right\}$
- (C) $\bigwedge_{m \in \omega} (x_0, x_1, \dots, x_m \text{ are independent})$
- (D) $\bigwedge \left\{ (\exists y) \left(\varphi_a(y, x_0, x_1, \dots, x_n, \dots) \wedge y \neq 0 \right) : a \in S^*, a \neq 0 \right\}$.

Observe that $\varphi^*(x_0, x_1, \dots, x_n, \dots)$ is an $L_{(2^{\aleph_0}) + \aleph_1}$ -formula. Let $\varphi^{\mathbb{Z}^\omega}$ be the $L_{(2^{\aleph_0}) + \aleph_1}$ -sentence $\exists(x_0, x_1, \dots, x_n, \dots) \varphi^*(x_0, x_1, \dots, x_n, \dots)$. ■

Proposition 2.4. *Suppose that G is torsion-free and contains no subgroup isomorphic to \mathbb{Q} . The following conditions are equivalent:*

(2.4.1) *there exists an embedding $h : \mathbb{Z}^\omega \rightarrow G$;*

(2.4.2) $G \models \varphi^{\mathbb{Z}^\omega}$.

Proof. The implication (2.4.1) \Rightarrow (2.4.2) is easy, since $\mathbb{Z}^\omega \models \varphi^*[e_0, e_1, \dots, e_n, \dots]$, and one can use the embedding h to find witnesses in G for the existential quantifiers of the conjuncts of φ^* .

Conversely, suppose that (2.4.2) holds, and $c_0, c_1, \dots, c_n, \dots$ are elements of G satisfying $G \models \varphi^*[c_0, c_1, \dots, c_n, \dots]$. The map $h_0 : \mathbb{Z}^\omega \rightarrow S^*$ given by $h_0(a) = \langle \dots, n!a_n, \dots \rangle$ is an embedding, so to establish (2.4.1), it is enough to find an embedding $d : S^* \rightarrow G$ (and then $h = d \circ h_0$ embeds \mathbb{Z}^ω into G). For an element $a \in S^*$, since $G \models \varphi^*[c_0, c_1, \dots, c_n, \dots]$, there exists $d_a \in G$ such that $G \models \varphi_a[d_a, c_0, c_1, \dots, c_n, \dots]$. By Fact 1, the element d_a is uniquely defined. Claim: the map $d : S^* \rightarrow G$ given by $d(a) = d_a$ is an embedding. This is easy to show: to see that d is a homomorphism, note that e.g., for $a, b \in S^*$, for every $n \in \omega$, $G \models (n! \text{ divides } (d_{(a+b)} - (d_a + d_b)))$, and hence $d(a + b) = d(a) + d(b)$ (otherwise Fact 2.2 will reveal a copy of \mathbb{Q} in G). A similar argument using the appropriate conjunct of (D) in φ^* proves that d is one-to-one. ■

The cases $\mathbb{Z}(p)$ are trivial. Since the group table of $\mathbb{Z}(p)$ can be completely described by an $L_{\omega\omega}$ -sentence, for every prime p , there is an $L_{\omega\omega}$ -sentence $\varphi^{\mathbb{Z}(p)}$ such that:

Fact 2.5. *For every group G , $\varphi^{\mathbb{Z}(p)}$ is true in G iff G has a subgroup isomorphic to $\mathbb{Z}(p)$.*

To deal with the p-adic integers J_p , we shall use some elementary properties summarized below (see [F1]).

- (1) every $t \in J_p$ can be represented as a formal sum $\sum_{j < \omega} s_j \cdot p^j$, where the coefficients s_j belong to $\{0, 1, \dots, p - 1\}$.

Since $\{0, 1, \dots, p - 1\}$ can be replaced by any complete set $\{t_0, t_1, \dots, t_{p-1}\}$ of representatives of $\mathbb{Q}_p \bmod p\mathbb{Q}_p$ (where \mathbb{Q}_p is the ring of rational numbers whose denominators are prime to p), it follows that:

- (2) if q is a prime different from p , then for every $t \in J_p$, q divides t in J_p .

Also, J_p has cardinality continuum.

We define, for each $t = \sum_{j < \omega} s_j \cdot p^j \in J_p$, the $L_{\omega_1\omega}$ -formula $\psi_{p, t}(y, x)$ to be the logical conjunction of the following:

- (E) $\bigwedge_{n \in \omega} (p^n \text{ divides } (y - \sum_{j < n} s_j \cdot p^j \cdot x))$
- (F) $\bigwedge \{q \text{ divides } y \text{ and } q \text{ divides } x : q \in \mathbb{Z}, (p, q) = 1\}$.

Since $J_p \models \psi_{p, t}[t, 1]$, one obtains immediately:

Fact 2.6. *If $h : J_p \rightarrow G$ is an embedding, then for every $t \in J_p$, $G \models \psi_{p, t}[h(t), h(1)]$.*

Let $\psi_p(x)$ be the logical conjunction of the formulas:

- (G) $\bigwedge \{(\exists! y)\psi_{p, t}(y, x) : t \in J_p\}$
- (H) $\bigwedge \{(\exists y_1 \exists y_2 \exists y_3) (\bigwedge_{i=1,2,3} \psi_{p, t^i}(y_i, x) \wedge (y_3 = y_1 - y_2)) : t^1, t^2, t^3 \in J_p, t^3 = t^1 - t^2\}$
- (K) $\bigwedge \{\exists y (\psi_{p, t}(y, x) \wedge y \neq 0) : t \in J_p, t \neq 0\}$.

Note that $\psi_p(x)$ is an $L_{(2^{\aleph_0}) + \aleph_0}$ -formula.

Proposition 2.7. *Suppose G is torsion-free and does not contain an isomorphic copy of \mathbb{Q} ; let $g \in G \setminus \{0\}$. The following conditions are equivalent:*

(2.7.1) *there is an embedding $h : J_p \rightarrow G$ such that $h(1) = g$;*

(2.7.2) $G \models \psi_p[g]$.

Proof. The implication (2.7.2) \Rightarrow (2.7.1) is analogous to (2.4.2) \Rightarrow (2.4.1).

As regards the direction (2.7.1) \Rightarrow (2.7.2), we sketch only the slightly different points, arising from J_p , concerning the formula $(\exists! y)\psi_{p, t}(y, x)$. By Fact 2.6, $G \models \psi_{p, t}[h(t), h(1)]$. If for some $d_1 \neq d_2 \in G$, $\bigwedge_{i=1,2} G \models \psi_{p, t}[d_i, h(1)]$, then, for every $n \in \omega$, $G \models (p^n \text{ divides } d_1 - d_2)$, since $\bigwedge_{i=1,2} G \models (p^n \text{ divides } (d_i - \sum_{j < n} s_j \cdot p^j \cdot h(1)))$. But also, if $q \in \mathbb{Z}, (p, q) = 1$, then $\bigwedge_{i=1,2} G \models (q \text{ divides } d_i)$. Hence G satisfies $\psi_p^{\mathbb{Q}}$, contradicting the hypothesis that G does not contain a copy of \mathbb{Q} . Thus $h(t) \in G$ is unique such that $G \models \psi_{p, t}[h(t), g]$. The remaining conjuncts of (H) and (K) hold in G because h is an embedding. So $G \models \psi_p[g]$. ■

Let ψ^{J_p} be the $L_{(2^{\aleph_0})+\aleph_0}$ -sentence $(\exists x)\psi_p(x)$.

Theorem 2.8. *There is an $L_{(2^{\aleph_0})+\aleph_1}$ -sentence Φ such that for any group G , the following conditions are equivalent:*

(2.8.1) G is slender;

(2.8.2) $G \models \Phi$.

Proof. By Theorem 2.1, together with Facts 2.2 and 2.5, and Propositions 2.4 and 2.7, the following sentence Φ is as required:

$$\neg\left(\psi^{\mathbb{Q}} \vee (\varphi^{\mathbb{Z}^\omega}) \vee \left(\bigvee_{p \text{ prime}} \varphi^{\mathbb{Z}(p)}\right) \vee \left(\bigvee_{p \text{ prime}} \psi^{J_p}\right)\right).$$

Theorem 1.1 and Corollary 1.2 follow immediately from Theorem 2.8. ■

For the definitions and basic properties of Whitehead and Shelah groups, see [7].

Corollary 2.9. *If G is a W -group, then \mathbb{Z}^ω is not $L_{(2^{\aleph_0})+\aleph_1}$ -equivalent to G .*

Proof. G is slender [7]. ■

Since every Shelah group is slender, one also obtains:

Corollary 2.10. *If G is a Shelah group, then \mathbb{Z}^ω is not $L_{(2^{\aleph_0})+\aleph_1}$ -equivalent to G .*

From Theorem 1.1 it follows too that \mathbb{Z}^ω is not $L_{(2^{\aleph_0})+\aleph_1}$ -equivalent to an \aleph_2 -free group (this is already a consequence of Eklof [4]).

3 Cotorsion-free groups

Recall that a group A is *cotorsion* if $\text{Ext}(J, A) = 0$ for all torsion-free groups J . A group G is *cotorsion-free* if G does not contain any non-zero subgroup which is cotorsion. The following theorem characterizes the cotorsion-free groups ([9]; see [7]):

Theorem 3.1. *For any group G , the following are equivalent:*

(3.1.1) G is cotorsion-free;

(3.1.2) G does not contain a copy of \mathbb{Q} , $\mathbb{Z}(p)$, or J_p for any prime p .

The sentence $\neg\left(\psi^{\mathbb{Q}} \vee \left(\bigvee_{p \text{ prime}} \varphi^{\mathbb{Z}(p)}\right) \vee \left(\bigvee_{p \text{ prime}} \psi^{J_p}\right)\right)$ belongs to the logic $L_{(2^{\aleph_0})+\aleph_0}$, and so the following corollary is evident:

Corollary 3.2. *The class of cotorsion-free groups is $L_{\infty\omega}$ -definable.*

Since the $L_{\infty\omega_1}$ -equivalent, non-isomorphic groups $F(\aleph_1)$ and $G(\aleph_1)$ of cardinality \aleph_1 mentioned in the introductory section of this paper are cotorsion-free, the first-order theory of the cotorsion-free groups has no good structure theorem either.

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