On nonlinear hyperbolic problems with nonlinear boundary feedback

Mohammed Aassila Marcelo M. Cavalcanti

Abstract

In this paper we prove the existence, uniqueness and uniform decay of strong and weak solutions of the nonlinear model of the wave equation

$$u_{tt} - \Delta u + f(u) + h(\nabla u) = 0$$

in bounded domains with nonlinear dissipative boundary conditions given by

$$\frac{\partial u}{\partial \nu} + g(u_t) = 0.$$

The existence is proved by means of Faedo-Galerkin method and the asymptotic behavior is obtained making use of the multiplier technique due to Komornik and Zuazua .

1 Introduction

Consider the nonlinear wave equation with a nonlinear boundary dissipative term

$$(*) \qquad \begin{cases} u_{tt} - \Delta u + f(u) + h(\nabla u) = 0 \quad \text{in} \quad \Omega \times (0,\infty), \\ u = 0 \quad \text{on} \quad \Gamma_1 \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} + g(u_t) = 0 \quad \text{on} \quad \Gamma_0 \times (0,\infty), \\ u(x,0) = u^0(x); \quad u_t(x,0) = u^1(x) \quad \text{in} \quad \Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbf{R}^n$, $n \geq 1$, with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, Γ_0 and Γ_1 are closed and disjoint , ν represents the unit outward normal

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to Γ and f, g and h are nonlinear functions satisfying some general properties (see assumptions $(A.1)-(A.3)$ below).

The main goal of this paper is to prove the existence of strong and weak solutions to problem $(*)$ and, moreover, that they decay to zero uniformly when t goes to infinity. Let us remark that in our work, we do not have the a priori estimate $E(t) \leq E(0)$, where $E(t)$ is given by (1.1). This a priori estimate plays a crucial role in establishing the global existence and when studying the asymptotic stability of the solution, as it was considered in the prior literature.

The proof of the existence is based on the Galerkin's approximation. For strong solutions to $(*)$ this approximation requires a change of variables to transform $(*)$ into an equivalent problem with initial value equals zero. The presence of the nonlinearities $h(\nabla u)$ and $g(u_t)$ brings up serious difficulties when passing to the limit, which were overcome combining arguments of compacity and monotonicity.

Controllability and boundary stabilization of distributed systems has attracted considerable attention in the literature and, in recent years, important progress has been obtained in this context. New techniques were developed which allow us to stabilize a system through its boundary or control it from an initial to a final state. There is a large body of literature regarding boundary stabilization with linear feedbacks. Indeed, when $g(s) = s$ we refer the reader the following works: Chen [2,3], Lagnese [8,9], Russell [14], Triggiani [15], Komornik and Zuazua [7] and Cavalcanti et al. [1]. Now when the boundary conditions are nonlinear we can cite the works of Chen and Wong [4], Lagnese and Leugering [10], Zuazua [17] , You [16], Cipolatti et al. [5] and Lasiecka and Tataru [12], among others.

We note that stability of problems with the nonlinear term $h(\nabla u)$ require a careful treatment because we do not have any information about the influence of the integral $\int_\Omega h(\nabla u)\, u_t\, dx$ on the energy

$$E(t) = \frac{1}{2}\int_\Omega \left(|u_t(x,t)|^2 + |\nabla u(x,t)|^2\right) dx \tag{1.1}$$

or about the sign of the derivative $E'(t)$.

We also observe that our problem deals with nonlinearity which involves the gradient combined with a nonlinear feedback acting on the boundary. This situation was not previously considered and leads to new difficulties. In order to overcome these difficulties we make use of the perturbed energy Liapunov functional due to Komornik and Zuazua [7].

Our paper is organized as follows. In section 2 we establish notations and state the main results. In section 3 we prove existence and uniqueness of strong and weak solutions to problem $(*)$ using Galerkin method. In section 4, we prove the exponential decay of solutions.

2 Notations and Main Results

Consider the Hilbert space

$$V = \left\{ v \in H^1(\Omega); v = 0 \ \text{ on } \ \Gamma_1 \right\},$$

and define the following

$$(u,v) = \int_\Omega u(x)v(x)\,dx; \quad (u,v)_{\Gamma_0} = \int_{\Gamma_0} u(x)v(x)\,d\Gamma,$$

$$||u||_p^p = \int_\Omega |u(x)|^p\,dx; \quad ||u||_{p,\Gamma_0}^p = \int_{\Gamma_0} |u(x)|^p\,d\Gamma.$$

Now, we state the general hypotheses:

(A.1) Assumptions on f :

$$\text{Let } \ f : \mathbf{R} \to \mathbf{R} \ \text{ be } \ \text{a } \ W^{1,\infty}_{loc}(\mathbf{R}), \ \text{piecewise } \ C^1(\mathbf{R}) \ \text{ function,} \tag{$H.1$}$$

$$f(s)s \ge 0 \ \text{ for } \ s \in \mathbf{R}. \tag{$H.2$}$$

Assume that there exists $C > 0$ such that

$$|f'(s)| \le C\left(1 + |s|^{p-1}\right), \ \ 1 < p \le \frac{n}{n-2} \ \text{ for all } \ s \in \mathbf{R}. \tag{$H.3$}$$

Defining

$$F(s) = \int_0^s f(\lambda)\,d\lambda$$

there exist $\alpha, C > 0$ verifying

$$C\,|s|^{p+1} \le F(s) \le \alpha\, s f(s) \ \text{ for all } \ s \in \mathbf{R}. \tag{$H.4$}$$

We observe that from assumption $(H.3)$ we deduce that there exists $C > 0$ such that

$$|f(s)| \le C\,(1 + |s|^p) \qquad \text{for all } \ s \in \mathbf{R}. \tag{2.1}$$

Assume that there exists $C > 0$ such that

$$\left| f(\xi) - f(\hat{\xi}) \right| \le C\left(|\xi|^{p-1} + \left|\hat{\xi}\right|^{p-1} \right) \left|\xi - \hat{\xi}\right| \qquad \text{for all } \ \xi, \hat{\xi} \in \mathbf{R}. \tag{$H.5$}$$

(A.2) Assumptions on h

$$\text{Let } \ h : \mathbf{R}^n \to \mathbf{R} \ \text{ be } \ \text{a } \ C^1 \ \text{ function.} \tag{$H.6$}$$

Assume that there exist $\beta, L > 0$ such that

$$|h(\zeta)| \le \beta\,|\zeta|, \quad \text{for all } \quad \zeta \in \mathbf{R}^n, \tag{$H.7$}$$

$$|h'(\zeta)| \le L, \quad \text{for all } \quad \zeta \in \mathbf{R}^n. \tag{$H.8$}$$

From assumption (H.8), we have

$$\left|h(\zeta) - h(\hat{\zeta})\right| \le L\left|\zeta - \hat{\zeta}\right| \quad \text{for all } \zeta, \hat{\zeta} \in \mathbf{R}^n. \tag{2.2}$$

(A.3) Assumptions on g:

$$\text{Let } g : \mathbf{R} \to \mathbf{R} \text{ be a non-decreasing } C^1 \text{ function,} \qquad (H.9)$$
$$g(s)s \ge 0 \text{ for all } s \ne 0.$$

There exist $C_i > 0$; $i = 1, 2, 3, 4$ such that

$$C_1|s| \le |g(s)| \le C_2|s| \quad \text{if} \quad |s| \le 1 \qquad (H.10)$$

$$C_2|s|^q \le |g(s)| \le C_4|s|^q, \quad 1 < q \le \frac{n-1}{n-2} \quad \text{if} \quad |s| > 1. \qquad (H.11)$$

Remark: The functions $f(u) = |u|^{p-1}u$, $g(u) = |u|^{q-1}u$ and $h(u) = \sum_{i=1}^n sin\left(\frac{\partial u}{\partial x_i}\right)$, for instance, verify all the hypotheses above.

In order to obtain the global existence for strong solutions the following assumptions are made on the initial data:

(A.4) Assumptions on the Initial Data:

Assume that

$$\left\{u^0, u^1\right\} \in \left(V \cap H^2(\Omega)\right)^2 \qquad (H.12)$$

verifying the compatibility condition

$$\frac{\partial u^0}{\partial \nu} + g(u^1) = 0 \quad \text{on} \quad \Gamma_0. \qquad (H.13)$$

Now, we are in a position to state our results:

Theorem 2.1 *Under assumptions (A1, A2, A3, A4), Problem* $(*)$ *possesses a unique strong solution, that is a function* $u :]0,\infty[\times\Omega \to \mathbf{R}$, *such that*

$$u \in L^\infty(0,\infty;V), \;\; u' \in L^\infty(0,\infty;V) \quad \text{and} \quad u'' \in L^\infty(0,\infty;L^2(\Omega)).$$

Moreover, assuming that $q = 1$ *in (H.11) and considering* β *given by (H.7) sufficiently small, the energy determined by the strong solution* u *decays exponentially. That is,*

$$E(t) = \frac{1}{2}\int_\Omega |u'(x,t)|^2\,dx + \frac{1}{2}\int_\Omega |\nabla u(x,t)|^2\,dx + \int_\Omega F(u(x,t))\,dx \le C\exp(-\gamma t) \tag{2.3}$$

for some positive constants C *and* γ.

Theorem 2.2 *Suppose that* $\{u^0, u^1\} \in V \times L^2(\Omega)$, *and the assumptions (A.1)-(A.3) hold. Then,* $(*)$ *has at least a weak solution,* $u : \Omega\times]0,\infty[\to \mathbf{R}$, *in the space*

$$C([0,\infty);V) \cap C^1([0,\infty);L^2(\Omega)).$$

Furthermore, if $q = 1$, *then (2.3) holds for the weak solution.*

3 Existence of Strong and Weak Solutions

In this section we prove the existence and uniqueness of strong solutions to Problem $(*)$. First we consider strong solutions, and then using a density argument we extend the same result to weak solutions.

A variational formulation of Problem $(*)$ leads to the equation

$$(u''(t), w) + (\nabla u(t), \nabla w) + (g(u'(t)), w)_{\Gamma_0} + (f(u(t)), w) + (h(\nabla u(t)), w) = 0$$

for all $w \in V$.

Strong solutions to $(*)$ with boundary condition $(g(u'(t)), w)_{\Gamma_0}$ can not be obtained by the method of 'special basis'; therefore basis formed by eigen-functions of $(-\Delta)$ operator can not be used for it. This leads us to differentiate the variational formulation related with $(*)$ with respect to t. But this brings up serious difficulties when estimating $u''(0)$. To avoid this difficulties, we transform $(*)$ into an equivalent problem with initial value equals to zero. Indeed, the change of variables

$$v(x,t) = u(x,t) - \phi(x,t) \tag{3.1}$$

where

$$\phi(x,t) = u^0(x) + tu^1(x), \quad t \in [0,T]$$

leads to the equivalent problem

$$\begin{cases} v_{tt} - \Delta v + f(v+\phi) + h\left(\nabla v + \nabla\phi\right) = \mathcal{F} \quad \text{in} \quad \Omega \times (0,\infty), \\ v = 0 \quad \text{on} \quad \Gamma_1 \times (0,\infty), \\ \dfrac{\partial v}{\partial \nu} + g\left(v_t + \phi_t\right) = \mathcal{G} \quad \text{on} \quad \Gamma_0 \times (0,\infty), \\ v(0) = v_t(0) = 0 \quad \text{in} \quad \Omega, \end{cases} \tag{3.2}$$

where

$$\mathcal{F} = \Delta\phi \quad \text{and} \quad \mathcal{G} = -\frac{\partial\phi}{\partial\nu}. \tag{3.3}$$

Note that if v is a solution of $(*)$ on [0,T]; then $u = v + \phi$ is a solution of $(*)$ in the same interval. From the estimates obtained below, we are able to prove that

$$||\Delta v(t)||_2^2 + ||\nabla v'(t)||_2^2 \le C, \quad \forall t \in [0,T].$$

Thus from (3.1) the above inequality holds for the solution u. Then using standard methods, we extend u to the interval $(0,\infty)$. Hence, it is sufficient to prove that (3.2) has a local solution, which shall be done by using the Galerkin method.

Let $(\omega_\nu)_{\nu\in\mathbf{N}}$ be a basis in $V \cap H^2(\Omega)$ which is orthonormal in $L^2(\Omega)$. Let V_m the space generated by $\omega_1, \cdots, \omega_m$ and let

$$v_m(t) = \sum_{i=1}^{m} \gamma_j(t)\omega_j \tag{3.4}$$

be the solution to the Cauchy problem

$$(v_m''(t), w) + (\nabla v_m(t), \nabla w) + \left(g\left(v_m'(t) + \phi'(t)\right), w\right)_{\Gamma_0} \tag{3.5}$$

$$+\left(f\left(v_m(t)+\phi(t)\right),w\right)+\left(h\left(\nabla v_m(t)+\nabla\phi(t)\right),w\right)$$
$$=(\mathcal{F}(t),w)+(\mathcal{G}(t),w)_{\Gamma_0};\quad \forall w\in V_m,$$
$$v_m(0)=v_m'(0)=0.$$

At this point it is important to observe that since $p\le\frac{n}{n-2}$ then

$$H^1(\Omega)\hookrightarrow L^{2p}(\Omega) \tag{3.6}$$

and since $q\le\frac{n-1}{n-2}$ one has

$$H^1(\Omega)\hookrightarrow L^{2q}(\Gamma). \tag{3.7}$$

Then, considering the embeddings given by (3.6) and (3.7) and (2.1), (H.7), (H.10) and (H.11) it is easy to see that the variational formulation (3.5) is well defined.

By standard methods of differential equations, we can prove the existence of a solution to (3.5) on some interval $[0,t_m)$. Then, this solution can be extended to the closed interval [0,T] by use of the first estimate below.

A Priori Estimates.

The First Estimate:

Taking $w=v_m'(t)$ in (3.5) we obtain

$$\frac{d}{dt}\left\{\frac{1}{2}\|v_m'(t)\|_2^2+\frac{1}{2}\|\nabla v_m(t)\|_2^2+\int_\Omega F\left(v_m+\phi\right)dx\right\} \tag{3.8}$$
$$+\left(g\left(v_m'(t)+\phi'(t)\right),v_m'(t)+\phi'(t)\right)_{\Gamma_0}$$
$$=(\mathcal{F}(t),v_m'(t))+\frac{d}{dt}(\mathcal{G}(t),v_m(t))_{\Gamma_0}-(\mathcal{G}'(t),v_m(t))_{\Gamma_0}$$
$$+\left(f\left(v_m(t)+\phi(t)\right),\phi'(t)\right)+\left(g\left(v_m'(t)+\phi'(t)\right),\phi'(t)\right)_{\Gamma_0}$$
$$-\left(h\left(\nabla v_m(t)+\nabla\phi(t)\right),v_m'(t)\right).$$

Estimate for $I_1:=\left(f\left(v_m(t)+\phi(t)\right),\phi'(t)\right).$

Here and in the sequel C denotes positive constants. From (2.1) and applying Young's inequality we have

$$|I_1|\le C\int_\Omega\left(1+|v_m+\phi|^p\right)|\phi'|\,dx \tag{3.9}$$
$$\le C\left\{\int_\Omega|\phi'|\,dx+\int_\Omega|v_m+\phi|^{p+1}\,dx+\int_\Omega|\phi'|^{p+1}\,dx\right\}$$
$$\le C+C\int_\Omega|v_m+\phi|^{p+1}\,dx.$$

Estimate for $I_2:=\left(g\left(v_m'(t)+\phi'(t)\right),\phi'(t)\right)_{\Gamma_0}.$

The Young's inequality yields

$$|I_2|\le\eta\int_{\Gamma_0}\left|g\left(v_m'+\phi'\right)\right|^{\frac{q+1}{q}}d\Gamma+C(\eta)\int_{\Gamma_0}|\phi'|^{q+1}\,d\Gamma, \tag{3.10}$$

where η is an arbitrary positive constant.

On the other hand, from assumption $(H.11)$ we deduce

$$|g(s)|^{\frac{q+1}{q}} = |g(s)|\,|g(s)|^{\frac{1}{q}} \leq |g(s)|\,|s|\,; \quad |s| > 1. \tag{3.11}$$

Estimate for $I_3 := \left(h\left(\nabla v_m(t) + \nabla\phi(t)\right), v_m'(t)\right).$

From assumption $(H.7)$ we conclude

$$|I_3| \leq C\left\{||\phi(t)||_2^2 + ||v_m'(t)||_2^2 + ||\nabla v_m(t)||_2^2\right\}. \tag{3.12}$$

Estimate for $I_4 := \left(\mathcal{G}'(t), v_m(t)\right)_{\Gamma_0}.$

Observing that

$$||v||_{2,\Gamma_0} \leq C_0\,||\nabla v||_2\,, \quad \forall v \in V \tag{3.13}$$

from Cauchy-Schwarz's inequality we obtain

$$|I_4| \leq C\left\{||\mathcal{G}'(t)||_{2,\Gamma_0}^2 + ||\nabla v_m(t)||_2^2\right\}. \tag{3.14}$$

Combining (3.8)-(3.14) it follows that

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{1}{2}\,||v_m'(t)||_2^2 + \frac{1}{2}\,||\nabla v_m(t)||_2^2 + \int_\Omega F\left(v_m+\phi\right)dx\right\} \\ &\qquad +(1-\eta)\int_{|v_m'+\phi'|>1} |g(v_m'+\phi')|^{\frac{q+1}{q}}\,d\Gamma \\ &\leq C(\eta) + ||\mathcal{F}(t)||_2^2 + ||\mathcal{G}'(t)||_{2,\Gamma_0}^2 + \frac{d}{dt}\left(\mathcal{G}(t), v_m(t)\right)_{\Gamma_0} \\ &\quad +C\left\{\int_\Omega |v_m+\phi|^{p+1}\,dx + ||v_m'(t)||_2^2 + ||\nabla v_m(t)||_2^2\right\}. \end{aligned} \tag{3.15}$$

Integrating (3.15) over (0,t), noting that $v_m(0) = v_m'(0) = 0$ and taking the assumption $(H.4)$ into account it results that

$$\begin{aligned} &\frac{1}{2}\,||v_m'(t)||_2^2 + \frac{1}{2}\,||\nabla v_m(t)||_2^2 + \int_\Omega F\left(v_m+\phi\right)dx \\ &\qquad +(1-\eta)\int_0^t\int_{|v_m'+\phi'|>1} |g(v_m'+\phi')|^{\frac{q+1}{q}}\,d\Gamma ds \\ &\leq C + C\int_0^t\left\{\int_\Omega F\left(v_m+\phi\right)dx + \int_{|v_m'+\phi'|>1} |g(v_m'+\phi')|^{\frac{q+1}{q}}\,d\Gamma\,. \right. \\ &\qquad \left. +\,||v_m'(s)||_2^2 + ||\nabla v_m(s)||_2^2\right\}ds + \left(\mathcal{G}(t), v_m(t)\right)_{\Gamma_0}. \end{aligned} \tag{3.16}$$

For an arbitrary $\eta > 0$ and taking (3.13) into account we have

$$\left(\mathcal{G}(t), v_m(t)\right)_{\Gamma_0} \leq \frac{C_0^2}{4\eta}\,||\mathcal{G}(t)||_{2,\Gamma_0}^2 + \eta\,||\nabla v_m(t)||_2^2. \tag{3.17}$$

Combining (3.16)- (3.17), choosing $\eta > 0$ small enough and employing Gronwall's lemma we obtain the first estimate

$$||v'_m(t)||_2^2 + ||\nabla v_m(t)||_2^2 + \int_\Omega F\,(v_m + \phi)\,dx + \int_0^t \int_{\Gamma_0} |g(v'_m + \phi')|^{\frac{q+1}{q}}\,d\Gamma ds \le L_1, \quad (3.18)$$

where L_1 is a positive constant independent of $t \in [0, T]$ and $m \in \mathbf{N}$.

The Second Estimate.

Firts of all, we are estimating $v''_m(0)$ in L^2-norm. Then, considering $w = v''_m(0)$ in (3.5) and noting that $v_m(0) = v'_m(0) = 0$, one has

$$||v''_m(0)||_2^2 + \left(g(u^1), v''_m(0)\right)_{\Gamma_0} + \left(f(u^0), v''_m(0)\right) + \left(h\left(\nabla u^0\right), v''_m(0)\right) \quad (3.19)$$

$$= \left(\Delta u^0, v''_m(0)\right) + \left(-\frac{\partial u^0}{\partial \nu}, v''_m(0)\right)_{\Gamma_0}.$$

From (3.19) and taking the assumption (H.13) into account, we obtain

$$||v''_m(0)||_2^2 \le \left(||f(u_0||_2 + \left\|h\left(\nabla u^0\right)\right\|_2 + \left\|\Delta u^0(0)\right\|_2\right) ||v''_m(0)||_2 .$$

Considering the last inequality and (2.1) and (H.7) we deduce that

$$||v''_m(0)||_2 \le N; \quad \forall m \in \mathbf{N} \quad (3.20)$$

where N is a positive constant independent of m.

On the other hand, taking the derivative of (3.5) with respect to t and substituting $w = v''_m(t)$ we get

$$\frac{d}{dt}\left\{\frac{1}{2}||v''_m(t)||_2^2 + \frac{1}{2}||\nabla v'_m(t)||_2^2\right\} + \int_{\Gamma_0} g'\left(v'_m + \phi'\right)\left(v''_m\right)^2 d\Gamma \quad (3.21)$$

$$+ \int_\Omega f'\left(v_m + \phi\right)\left(v'_m + \phi'\right) v''_m\,d\Gamma + \int_\Omega h'\left(\nabla v_m + \nabla\phi\right)\left(\nabla v'_m + \nabla\phi'\right) v''_m\,dx$$

$$= \left(\mathcal{F}(t), v''_m(t)\right) + \frac{d}{dt}\left(\mathcal{G}'(t), v'_m(t)\right)_{\Gamma_0}.$$

Next, we are going to estimate some terms of (3.21).

Estimate for $I_4 := \int_\Omega f'\left(v_m + \phi\right)\left(v'_m + \phi'\right) v''_m\,d\Gamma$.

Assuming that (H.3) holds we deduce that

$$|I_4| \le C \int_\Omega \left(1 + |v_m + \phi|^{p-1}\right) |v'_m + \phi'|\,|v''_m|\,dx. \quad (3.22)$$

Now, observing that $\frac{p-1}{2p} + \frac{1}{2p} + \frac{1}{2} = 1$, from (3.22) and considering the generalized Hölder's inequality we infer

$$|I_4| \le C\left(||v'_m(t) + \phi'(t)||_2^2 + ||v''_m(t)||_2^2\right)$$

$$+ C\left(\left\||v_m(t) + \phi(t)|^{p-1}\right\|_{\frac{2p}{p-1}} ||v'_m(t) + \phi'(t)||_{2p}\,||v''_m(t)||_2\right).$$

Considering the Sobolev imbeding in (3.6) and the first estimate from the last inequality we obtain

$$|I_4| \leq C\left(1+||v_m''(t)||_2^2+||\nabla v_m'(t)||_2^2\right). \tag{3..23}$$

Estimate for $I_5 := \int_\Omega h'\left(\nabla v_m+\nabla\phi\right)\left(\nabla v_m'+\nabla\phi'\right)v_m''\,dx.$

Taking the assumption (H.8) into account we infer

$$\begin{aligned}|I_5| &\leq L\int_\Omega |\nabla v_m'+\nabla\phi'|\,|v_m''|\,dx \\ &\leq C\left(1+||v_m''(t)||_2^2+||\nabla v_m'(t)||_2^2\right).\end{aligned} \tag{3.24}$$

Combining (3.21), (3.23) and (3.24) and noting that $g'(s)\geq 0$ it follows that

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}\,||v_m''(t)||_2^2+\frac{1}{2}\,||\nabla v_m'(t)||_2^2\right\} \\ &\leq ||\mathcal{F}'(t)||_2^2+\frac{d}{dt}\left(\mathcal{G}'(t),v_m'(t)\right)_{\Gamma_0}+C\left(1+||v_m''(t)||_2^2+||\nabla v_m'(t)||_2^2\right).\end{aligned} \tag{3.25}$$

Integrating (3.25) over (0,t), considering (3.20) and noting that $v_m'(0)=0$ we get

$$\begin{aligned}&||v_m''(t)||_2^2+||\nabla v_m'(t)||_2^2 \\ &\leq C+\left(\mathcal{G}'(t),v_m'(t)\right)_{\Gamma_0}+C\int_0^t\left\{||v_m''(s)||_2^2+||\nabla v_m'(s)||_2^2\right\}ds.\end{aligned} \tag{3.26}$$

On the other hand, for an arbitrary $\eta>0$ we have

$$\left(\mathcal{G}'(t),v_m'(t)\right)_{\Gamma_0} \leq \frac{C_0^2}{4\eta}\,||\mathcal{G}'(t)||_{2,\Gamma_0}^2+\eta\,||\nabla v_m'(t)||_2^2. \tag{3.27}$$

Combining (3.26)-(3.27), choosing $\eta>0$ sufficiently small and employing Gronwall's lemma we obtain the second estimate

$$||v_m''(t)||_2^2+||\nabla v_m'(t)||_2^2 \leq L_2 \tag{3.28}$$

where L_2 is a positive constant independent of $m\in\mathbf{N}$ and $t\in[0,T]$.

Analysis of the Nonlinear Terms:

Analysis of f.

From (2.1) and the first estimate one has

$$\int_0^t\int_\Omega |f\left(v_m+\phi\right)|^{\frac{p+1}{p}}\,dx\,ds \leq C\int_0^t\int_\Omega\left(1+|v_m+\phi|^{p+1}\right)dx\,ds \leq M \tag{3.29}$$

where M is a positive constant independent of $t\in[0,T]$ and $m\in\mathbf{N}$. The last inequality yields

$$\{f(v_m+\phi)\} \quad \text{is bounded} \quad \text{in } L^{\frac{p+1}{p}}(Q_T);\ Q_T = \Omega \times (0,T). \tag{3.30}$$

From the first estimate and making use of Aubin-Lions Theorem we can find a subsequence $\{v_\mu\}$ of $\{v_m\}$ such that

$$v_\mu \to v \quad \text{strongly in } L^2(Q_T). \tag{3.31}$$

Then,

$$v_\mu \to v \quad \text{a.e. in } Q_T$$

and therefore, from (H.1)

$$f(v_\mu+\phi) \to f(v+\phi) \qquad \text{a.e. in } Q_T. \tag{3.32}$$

Combining (3.30) and (3.32) and making use of Lions's lemma we deduce

$$f(v_\mu+\phi) \to f(v+\phi) \qquad \text{weakly in } L^{\frac{p+1}{p}}(Q_T). \tag{3.33}$$

Remark 1. We note that from (2.1) and since $H^1(\Omega) \hookrightarrow L^{2p}(\Omega)$ we have

$$\int_\Omega |f(v_\mu+\phi)|^2\,dx \le C\int_\Omega \left(1+|v_\mu+\phi|^{2p}\right)dx$$

$$\le C + C\,||\nabla v_\mu(t) + \nabla\phi(t)||_2^{2p} \le C.$$

Thus,

$$\{f(v_\mu+\phi)\} \quad \text{is bounded} \quad \text{in } L^2(Q_T).$$

Consequently, from (3.33) one has

$$f(v_\mu+\phi) \to f(v+\phi) \qquad \text{weakly in } L^2(Q_T). \tag{3.34}$$

Analysis of h **:**

From assumption (H.7) and the first estimate it results that

$$\{h(\nabla v_m + \nabla\phi)\} \quad \text{is bounded in } L^2(Q_T).$$

Then, there exist $\Xi \in L^2(Q_T)$ and $\{v_\mu\} \subset \{v_m\}$ such that

$$h(\nabla v_\mu + \nabla\phi) \to \Xi \qquad \text{weakly in } L^2(Q_T). \tag{3.35}$$

Analysis of g **:**

In the same way, from the first estimate it holds that

$$\{g(v'_m+\phi')\} \quad \text{is bounded} \quad \text{in } L^{\frac{q+1}{q}}(\Sigma_{0,T});\ \Sigma_{0,T} = \Gamma_0 \times (0,T)$$

and therefore there exist $\chi \in L^{\frac{q+1}{q}}(\Sigma_{0,T})$ and $\{v_\mu\} \subset \{v_m\}$ such that

$$g\left(v'_\mu+\phi'\right) \to \chi \quad \text{weakly} \quad \text{in } L^{\frac{q+1}{q}}(\Sigma_{0,T}). \tag{3.36}$$

Remark 2. We also note that from the first estimate and from assumptions (H.10) and (H.11) we infer

$$\int_{\Gamma_0} \left|g\left(v'_\mu + \phi'\right)\right|^2 d\Gamma$$

$$\int_{\left|v'_\mu+\phi'\right|\le 1} \left|g\left(v'_\mu + \phi'\right)\right|^2 d\Gamma + \int_{\left|v'_\mu+\phi'\right|>1} \left|g\left(v'_\mu + \phi'\right)\right|^2 d\Gamma$$

$$\le C + C\left|\left|v'_\mu(t) + \phi'(t)\right|\right|_{2q,\Gamma_0}^{2q}$$

$$\le C + C\left|\left|\nabla v'_\mu(t) + \nabla\phi'(t)\right|\right|_2^{2q} \le C.$$

Then,

$$g\left(v'_\mu + \phi'\right) \to \chi \quad \text{weakly} \quad \text{in } L^2\left(\Sigma_{0,T}\right). \tag{3.37}$$

Returning to (3.5) and using standard arguments we can show, from the convergences above that

$$v'' - \Delta v + f\left(v+\phi\right) + \Xi = \mathcal{F} \quad \text{in} \quad L^2(0,\infty; L^2(\Omega)). \tag{3.38}$$

Also, using the generalized Green formula we deduce that

$$\frac{\partial v}{\partial \nu} + \chi = \mathcal{G} \quad \text{in} \quad L^2(0,\infty; L^2(\Gamma_0)). \tag{3.39}$$

Next, we are going to prove that $\Xi = h\left(\nabla v + \nabla\phi\right)$. Indeed, considering $w = v_m(t)$ in (3.5) and integrating the result over (0,T) it follows that

$$\int_0^T \left(v''_m(t), v_m(t)\right) dt + \int_0^T \left|\left|\nabla v_m(t)\right|\right|_2^2 dt + \int_0^T \left(g\left(v'_m(t) + \phi'(t)\right), v_m(t)\right)_{\Gamma_0} dt \tag{3.40}$$

$$+ \int_0^T \left(f\left(v_m(t) + \phi(t)\right), v_m(t)\right) dt + \int_0^T \left(h\left(\nabla v_m(t) + \nabla\phi(t)\right), v_m(t)\right) dt$$

$$= \int_0^T \left(\mathcal{F}(t), v_m(t)\right) dt + \int_0^T \left(\mathcal{G}(t), v_m(t)\right)_{\Gamma_0} dt.$$

On the other hand, we note that from the first and second estimates one has

$$\left|\left|v_m(t)\right|\right|_{H^{1/2}(\Gamma_0)}^2 \le C\left|\left|\nabla v_m(t)\right|\right|_2^2 \le C$$

$$\left|\left|v'_m(t)\right|\right|_{H^{1/2}(\Gamma_0)}^2 \le C\left|\left|\nabla v'_m(t)\right|\right|_2^2 \le C.$$

Since $H^{1/2}(\Gamma_0) \hookrightarrow L^2(\Gamma_0)$ is compact from Aubin-Lions Theorem we deduce

$$v_m \to v \quad \text{strongly in } L^2(0,T; L^2(\Gamma_0)). \tag{3.41}$$

Considering the convergences (3.31), (3.34), (3.35), (3.37), (3.41) and the weak one

$$v''_m \to v'' \quad \text{weakly} \quad \text{in } L^2(0,T; L^2(\Omega)) \tag{3.42}$$

we can pass to the limit in (3.40) to obtain

$$\lim_{m\to\infty}\int_0^T ||\nabla v_m(t)||_2^2\, dt \tag{3.43}$$
$$= -\int_0^T (v''(t), v(t))\, dt - \int_0^T (\chi(t), v(t))_{\Gamma_0}\, dt - \int_0^T (f(v(t)+\phi(t)), v(t))\, dt$$
$$-\int_0^T (\Xi(t), v(t))\, dt + \int_0^T (\mathcal{F}(t), v(t))\, dt + \int_0^T (\mathcal{G}(t), v(t))_{\Gamma_0}\, dt.$$

Combining (3.38), (3.39), (3.43) and making use of the generalized Green formula we obtain

$$\lim_{m\to\infty}\int_0^T ||\nabla v_m(t)||_2^2\, dt = \int_0^T ||\nabla v(t)||_2^2\, dt \tag{3.44}$$

and consequently

$$\nabla v_m \to \nabla v \quad \text{a.e. in } Q_T$$

which implies in view of (H.7) that

$$h(\nabla v_m + \nabla\phi) \to h(\nabla v + \nabla\phi) \quad \text{a.e. in } Q_T. \tag{3.45}$$

Using Lions's lemma from (3.35) and (3.45) we deduce

$$h(\nabla v_m + \nabla\phi) \to h(\nabla v + \nabla\phi) \quad \text{weakly in } L^2(Q_T). \tag{3..46}$$

Unfortunatelly we can not use compacity arguments in order to show that $\chi = g(v' + \phi')$. For this purpose, we shall use monotonicity arguments. First of all we note that from the first and second estimates and considering Aubin-Lions Theorem one has

$$v'_m \to v' \quad \text{strongly} \quad \text{in} \quad L^2(0,T;L^2(\Omega)). \tag{3.47}$$

Considering $w = v'_m(t)$ in (5), integrating the obtained result over (0,T), considering the convergences above mentioned and the facts

$$v'' - \Delta v + f(v+\phi) + h(\nabla v + \nabla\phi) = \mathcal{F} \quad \text{in} \quad L^2(0,\infty;L^2(\Omega))$$
$$\frac{\partial v}{\partial \nu} + \chi = \mathcal{G} \quad \text{in} \quad L^2(0,\infty;L^2(\Gamma_0))$$

we deduce, in a similar way we have just done before, that

$$\lim_{m\to\infty}\int_0^T (g(v'_m(t)+\phi'(t)), v'_m(t)+\phi'(t))_{\Gamma_0}\, dt = \int_0^T (\chi(t), v'(t)+\phi'(t))_{\Gamma_0}\, dt. \tag{3.48}$$

On the other hand, since g is a non-decreasing monotone function one has

$$\int_0^T \langle g(v'_m+\phi') - g(\psi), (v'_m+\phi') - \psi\rangle\, dt \geq 0; \quad \text{for all } \psi \in L^{q+1}(\Gamma_0)$$

where $\langle .,.\rangle$ means the duality between $L^{\frac{q+1}{q}}(\Gamma_0)$ and $L^{q+1}(\Gamma_0)$. The last inequality yields

$$\int_0^T \langle g(v'_m+\phi'), \psi\rangle\, dt + \int_0^T \langle g(\psi), (v'_m+\phi') - \psi\rangle\, dt \tag{3.49}$$
$$\leq \int_0^T \langle g(v'_m+\phi'), v'_m+\phi'\rangle\, dt.$$

From (3.49) we deduce

$$\liminf_{m\to\infty}\int_0^T \langle g\left(v_m'+\phi'\right),\psi\rangle\,dt+\liminf_{m\to\infty}\int_0^T \langle g(\psi),(v_m'+\phi')-\psi\rangle\,dt \tag{3.50}$$

$$\leq \liminf_{m\to\infty}\int_0^T \langle g\left(v_m'+\phi'\right),v_m'+\phi'\rangle\,dt.$$

Since

$$||v_m'(t)||_{q+1,\Gamma_0}\leq C\,||\nabla v_m'(t)||_2\leq C$$

it follows that

$$v_m'\to v' \quad \text{weakly star in } L^\infty(0,T;L^{q+1}(\Gamma_0)). \tag{3.51}$$

Then from (3.36), (3.48), (3.50) and (3.51) we conclude that

$$\int_0^T \langle \chi(t)-g(\psi),v'(t)+\phi'(t)-\psi\rangle\,dt\geq 0. \tag{3.52}$$

Considering $\psi=(v'+\phi')+\lambda\xi$ in (3.52) where ξ is an arbitrary element of $L^{q+1}(\Gamma_0)$ and $\lambda>0$, we obtain

$$\int_0^T \langle \chi(t)-g\left((v'+\phi')+\lambda\xi\right),(-\lambda\xi)\rangle\,dt\geq 0.$$

Consequently

$$\int_0^T \langle \chi(t)-g\left((v'+\phi')+\lambda\xi\right),\xi\rangle\,dt\leq 0, \quad \text{for all } \xi\in L^{q+1}(\Gamma_0).$$

As the operator

$$g:L^{q+1}(\Gamma_0)\to L^{\frac{q+1}{q}}(\Gamma_0);\ v\mapsto g(v)$$

is hemicontinuous one has

$$\int_0^T \langle \chi(t)-g\left(v'+\phi'\right),\xi\rangle\,dt\leq 0; \quad \text{for all } \xi\in L^{q+1}(\Gamma_0).$$

Hence

$$\int_0^T \langle \chi(t)-g\left(v'+\phi'\right),\xi\rangle\,dt=0; \quad \text{for all } \xi\in L^{q+1}(\Gamma_0)$$

which implies

$$\chi=g\left(v'+\phi'\right). \tag{3.53}$$

Uniqueness:

Let u_1 and u_2 be two smooth solutions to problem $(*)$. Then, $z=u_1-u_2$ verifies

$$(z''(t),w)+(\nabla z(t),\nabla w)+(g(u_1')-g(u_2'),w)_{\Gamma_0} \tag{3.54}$$

$$=(f(u_2)-f(u_1),w)+\left(h\left(\nabla u_1\right)-h\left(\nabla u_2\right),w\right); \quad \text{for all } w\in V.$$

Substituting $w=z'(t)$ in (3.54) it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{||z'(t)||_2^2+||\nabla z(t)||_2^2\right\}+(g(u_1')-g(u_2'),u_1'-u_2')_{\Gamma_0} \qquad (3.55)\\&=(f(u_2)-f(u_1),z'(t))+(h\,(\nabla u_1)-h\,(\nabla u_2)\,,z'(t))\,.\end{aligned}$$

Next, we are estimating the terms in the right hand side of (3.55).

Estimate for $J_1:=(f(u_2)-f(u_1),z'(t))\,.$

From assumption (H.5) we obtain

$$|J_1|\le D_1\int_\Omega\left[|u_2|^{p-1}+|u_1|^{p-1}\right]|z||z'|\,dx \qquad (3.56)$$

$$\le C\left[||u_2(t)||_{2p}+||u_1(t)||_{2p}\right]||z(t)||_{2p}\,||z'(t)||_2$$

$$\le C\left(||\nabla z(t)||_2^2+||z'(t)||_2^2\right).$$

Estimate for $J_2:=(h\,(\nabla u_1)-h\,(\nabla u_2)\,,z'(t))\,.$

Taking (2.2) into account we infer

$$|J_2|\le D_2\int_\Omega|\nabla u_2-\nabla u_1|\,|z'|\,dx \qquad (3.57)$$

$$\le C\left(||\nabla z(t)||_2^2+||z'(t)||_2^2\right).$$

Combining (3.55)-(3.57), observing that g is monotone and employing Gronwall's lemma we deduce $||\nabla z(t)||_2=||z'(t)||_2=0$ and therefore $u_1=u_2$. This concludes the proof of existence and uniqueness of smooth solutions.

Existence of Weak Solutions:

Let us consider

$$\left\{u^0,u^1\right\}\in V\times L^2(\Omega). \qquad (3.58)$$

Since $D(-\Delta)=\left\{v\in V\cap H^2(\Omega);\ \frac{\partial v}{\partial\nu}=0\ \ on\ \ \Gamma_0\right\}$ is dense in V and $H_0^1(\Omega)\cap H^2(\Omega)$ is dense in $L^2(\Omega)$ there exist $\left\{u_\mu^0\right\}\subset D(-\Delta)$ and $\left\{u_\mu^1\right\}\subset H_0^1(\Omega)\cap H^2(\Omega)$ such that

$$u_\mu^0\to u^0\quad\text{strongly in } V, \qquad (3.59)$$

$$u_\mu^1\to u^1\quad\text{strongly in } L^2(\Omega). \qquad (3.60)$$

Moreover, from (H.10) and (H.11) we have for each $\mu\in\mathbf{N}$

$$\frac{\partial u^0}{\partial\nu}+g(u_\mu^1)=0\quad\text{on }\Gamma_0.$$

Then, for each $\mu\in\mathbf{N}$ there exists $u_\mu:Q\to\mathbf{R}$ a smooth solution of problem $(*)$ verifying

$$\begin{cases} u_\mu'' - \Delta u_\mu + f(u_\mu) + h\left(\nabla u_\mu\right) = 0 \quad \text{in } L^2(0,\infty; L^2(\Omega)), \\ u_\mu = 0 \quad \text{on } \Gamma_1, \\ \dfrac{\partial u_\mu}{\partial \nu} + g(u_\mu') = 0 \quad \text{in } L^2(0,\infty; L^2(\Gamma_0)), \\ u_\mu(0) = u_\mu^0; \quad u_\mu'(0) = u_\mu^1. \end{cases} \tag{3.61}$$

Repeating the same arguments used in the first estimate we obtain

$$\left\|u_\mu'(t)\right\|_2^2 + \|\nabla u_\mu(t)\|_2^2 + \int_0^t \int_\Omega |u_\mu|^{p+1}\, dx\, dt + \int_0^t \int_{\Gamma_0} \left|g(u_\mu')\right|^{\frac{q+1}{q}} d\Gamma ds \le C \tag{3.62}$$

and

$$\int_0^t \int_\Omega |f(u_\mu)|^{\frac{p+1}{p}}\, dx\, ds \le C, \tag{3.63}$$

$$\int_0^t \int_{\Gamma_0} \left|u_\mu'\right|^{q+1} d\Gamma ds \le C \tag{3.64}$$

for all $t \in [0,T]$ and $\mu \in \mathbf{N}$.

Putting $z_{\sigma\mu} = u_\mu - u_\sigma$; $\mu, \sigma \in \mathbf{N}$ where u_μ and u_σ are smooth solutions of (3.61), repeating the same arguments used in the uniqueness of strong solutions and taking (3.62) into account we deduce that there exists $u : Q \to \mathbf{R}$ such that

$$u_\mu \to u \quad \text{in } C^0([0,T]; V), \tag{3.65}$$

$$u_\mu' \to u' \quad \text{in } C^0([0,T]; L^2(\Omega)). \tag{3.66}$$

Moreover, from (3.62), (3.63) and (H.7) we deduce

$$u_\mu' \to u' \quad \text{weakly in } L^{q+1}(\Sigma_{0,T}), \tag{3.67}$$

$$f(u_\mu) \to \eta \quad \text{weakly in } L^{\frac{p+1}{p}}(Q_T), \tag{3.68}$$

$$h\left(\nabla u_\mu\right) \to \Xi \quad \text{weakly in } L^2(Q_T), \tag{3.69}$$

$$g(u_\mu') \to \chi \quad \text{weakly in } L^{\frac{q+1}{q}}(\Sigma_{0,T}). \tag{3.70}$$

In view of (3.65) and using Lions's lemma it is easy to conclude that $\eta = f(u)$ and $\Xi = h(\nabla u)$. Morover, we have

$$\begin{cases} u'' - \Delta u + f(u) + h(\nabla u) = 0 \quad \text{in } L^2(0,\infty; V') \\ u(0) = u^0; \quad u'(0) = u^1. \end{cases} \tag{3.71}$$

Our aim is to show that $\chi = g(u')$. Indeed, multiplying the first equation of (3.61) by u_μ' and integrating over Ω we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|u_\mu'(t)\right\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u_\mu(t)\|_2^2 + \left(f(u_\mu(t)), u_\mu'(t)\right)$$
$$+ \left(h\left(\nabla u_\mu(t)\right), u_\mu'(t)\right) + \left(g(u_\mu'(t)), u_\mu'(t)\right)_{\Gamma_0} = 0.$$

Integrating the last equality over (0,t) it holds that

$$\frac{1}{2}\left\|u_\mu'(t)\right\|_2^2 + \frac{1}{2}\|\nabla u_\mu(t)\|_2^2 + \int_0^t \left(f(u_\mu(s)), u_\mu'(s)\right) ds \tag{3.72}$$

$$+\int_0^t \left(h\left(\nabla u_\mu(s)\right), u_\mu'(s)\right) ds + \int_0^t \left(g(u_\mu'(s)), u_\mu'(s)\right)_{\Gamma_0} ds$$
$$= \frac{1}{2}\left\|u_\mu^1\right\|_2^2 + \frac{1}{2}\left\|\nabla u_\mu^0\right\|_2^2.$$

From (3.72) and taking into account the convergences (3.59), (3.60), (3.65), (3.66), (3.68) and (3.69) we deduce

$$\lim_{\mu\to+\infty}\int_0^t \left(g(u_\mu'(s)), u_\mu'(s)\right) ds \tag{3.73}$$
$$= -\frac{1}{2}\|u'(t)\|_2^2 - \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2}\left\|u^1\right\|_2^2 + \frac{1}{2}\left\|\nabla u^0\right\|_2^2$$
$$-\int_0^t (f(u(s)), u'(s))\, ds - \int_0^t \left(h\left(\nabla u(s)\right), u'(s)\right) ds.$$

On the other hand, assuming that w is a weak solution to problem

$$\begin{cases} w'' - \Delta w + f(w) + h(\nabla w) = 0 \quad \text{ in } \Omega\times(0,\infty), \\ w = 0 \quad \text{on } \Gamma_1\times(0,\infty), \\ \dfrac{\partial w}{\partial \nu} + \chi = 0 \quad \text{ on } \Gamma_0\times(0,\infty), \\ w(0) = u^0; \;\; w'(0) = u^1, \end{cases} \tag{3.74}$$

then, considering the same arguments used to prove (3.73) we conclude that

$$\int_0^t \langle \chi(s), w'(s)\rangle_{\Gamma_0} ds \tag{3.75}$$
$$= -\frac{1}{2}\|w'(t)\|_2^2 - \frac{1}{2}\|\nabla w(t)\|_2^2 + \frac{1}{2}\left\|u^1\right\|_2^2 + \frac{1}{2}\left\|\nabla u^0\right\|_2^2$$
$$-\int_0^t (f(w(s)), w'(s))\, ds - \int_0^t \left(h\left(\nabla w(s)\right), w'(s)\right) ds.$$

Since u is a weak solution to problem (3.74) then, from (3.73) and (3.75) it follows that

$$\lim_{\mu\to+\infty}\int_0^t \left\langle g(u_\mu'(s)), u_\mu'(s)\right\rangle_{\Gamma_0} ds = \int_0^t \langle \chi(s), u'(s)\rangle_{\Gamma_0} ds.$$

The convergence above plays an essential role to show that $\chi = g(u')$ by using the same arguments considered before.

4 Uniform Decay.

In this section we prove the exponential decay for strong solutions of $(*)$, and by a density argument we obtain the same results for weak solutions.

For the rest of this section, let x^0 be a fixed point in $\mathbf{R}^n$. Then put

$$m = m(x) = x - x^0, \quad R = \max_{x\in\overline{\Omega}} \|m(x)\|$$

and partition the boundary Γ into two sets:

$$\Gamma_0 = \{x\in\Gamma; m(x)\cdot\nu(x) \geq 0\}, \;\; \Gamma_1 = \{x\in\Gamma; m(x)\cdot\nu(x) < 0\}.$$

A simple computation shows that

$$E'(t) = -\int_{\Omega} h(\nabla u)\, u' dx - \int_{\Gamma_0} g(u')u' d\Gamma. \tag{4.1}$$

Let us introduce the following perturbed energy functional:

$$E_\varepsilon(t) = E(t) + \varepsilon\rho(t) \tag{4.2}$$

where

$$\rho(t) = \int_{\Omega} u' \left(2m \cdot \nabla u + (n-1)u\right) dx. \tag{4.3}$$

It is easy to see that there exists $\theta_1 > 0$ such that

$$|E_\varepsilon(t) - E(t)| \leq \varepsilon\theta_1 E(t), \tag{4.4}$$

for all $t \geq 0$ and for all $\varepsilon > 0$.

Then, in view of (4.4) and in order to prove our main result it is sufficient to prove the following result:

Proposition 4.1. *There exists $\theta_2 > 0$ such that*

$$E'_\varepsilon(t) \leq -\theta_2 E(t),$$

for all $t \geq 0$.

Proof. From (4.3) it holds that

$$\rho'(t) = \int_{\Omega} \left(2u'' (m \cdot \nabla u) + 2u' (m \cdot \nabla u') + (n-1)u''u + (n-1)(u')^2\right) dx. \tag{4.5}$$

Next, we are going to analyse the terms in the right hand side of (4.5).

Estimate for $I_1 = \int_\Omega u'' u\, dx$.

Taking the generalized Green formula we infer

$$\begin{aligned} I_1 &= \int_{\Omega} (\Delta u - f(u) - h(\nabla u))\, u\, dx \\ &= -\int_{\Gamma_0} g(u')u\, d\Gamma - \int_{\Omega} |\nabla u|^2\, dx - \int_{\Omega} f(u)u\, dx - \int_{\Omega} h(\nabla u)\, u\, dx. \end{aligned} \tag{4.6}$$

Estimate for $I_2 = 2\int_\Omega u' (m \cdot \nabla u')\, dx$.

From Gauss theorem we deduce

$$I_2 = \int_{\Omega} m \cdot \nabla (u')^2 dx = -n\int_{\Omega} |u'|^2 dx + \int_{\Gamma_0} |u'|^2 d\Gamma. \tag{4.7}$$

Estimate for $I_3 = 2\int_\Omega u'' (m \cdot \nabla u)\, dx$.

Making use of Green and Gauss theorems we obtain

$$I_3 \leq (n-2)\int_\Omega |\nabla u|^2\,dx - \int_{\Gamma_0}\left[2u'\,(m\cdot\nabla u) + |\nabla u|^2\right]d\Gamma \qquad (4.8)$$
$$+2n\int_\Omega F(u)dx - 2\int_\Omega h\,(\nabla u)\,(m\cdot\nabla u)\,dx.$$

Combining (4.1), (4.2), (4.5)-(4.8) it follows that

$$E'_\varepsilon(t) \leq -\int_\Omega h\,(\nabla u)\,u'dx - \int_{\Gamma_0} g(u')u'd\Gamma \qquad (4.9)$$
$$+\varepsilon\left\{(n-2)\int_\Omega |\nabla u|^2 - \int_{\Gamma_0}\left[2u'\,(m\cdot\nabla u) + |\nabla u|^2\right]d\Gamma + 2n\int_\Omega F(u)dx\right.$$
$$-2\int_\Omega h\,(\nabla u)\,(m\cdot\nabla u)\,dx - n\int_\Omega |u'|^2dx + \int_{\Gamma_0}|u'|^2d\Gamma - (n-1)\int_{\Gamma_0} g(u')u\,d\Gamma$$
$$-(n-1)\int_\Omega |\nabla u|^2\,dx - (n-1)\int_\Omega f(u)u\,dx - (n-1)\int_\Omega h\,(\nabla u)\,u\,dx$$
$$\left.+(n-1)\int_\Omega |u'|^2\,dx\right\}.$$

Since from (H.7) we have

$$\int_\Omega\left(-h\,(\nabla u)\,u' - 2\varepsilon h\,(\nabla u)\,(m\cdot\nabla u) - \varepsilon(n-1)h\,(\nabla u)\,u\right)dx \qquad (4.10)$$
$$\leq \frac{1}{2}\int_\Omega |u'|^2\,dx + \left(\frac{\beta^2}{2} + 2\varepsilon\beta R + \frac{\varepsilon(n-1)}{2}\beta^2 + \frac{\varepsilon(n-1)}{2}C(\Omega)\right)\int_\Omega |\nabla u|^2\,dx$$

where

$$\int_\Omega |u|^2\,dx \leq C(\Omega)\int_\Omega |\nabla u|^2\,dx$$

from (4.9) and (4.10) we deduce

$$E'_\varepsilon(t) \leq -\left(-\frac{n}{2} + \varepsilon\right)\int_\Omega |u'|^2\,dx \qquad (4.11)$$
$$-\left(\varepsilon - \frac{\beta^2}{2} - 2\varepsilon\beta R - \frac{\varepsilon(n-1)}{2}\beta^2 - \frac{\varepsilon(n-1)}{2}C(\Omega)\right)\int_\Omega |\nabla u|^2\,dx$$
$$+2n\varepsilon\int_\Omega F(u)\,dx - \varepsilon(n-1)\int_\Omega f(u)u\,dx - \int_{\Gamma_0} g(u')u'\,d\Gamma$$
$$-\varepsilon\int_{\Gamma_0}\left[2u'\,(m\cdot\nabla u) + |\nabla u|^2\right]d\Gamma + \varepsilon\int_{\Gamma_0}|u'|^2\,d\Gamma - \varepsilon(n-1)\int_{\Gamma_0} g(u')u\,d\Gamma.$$

On the other hand we have

$$\int_{\Gamma_0}|2u'(m\cdot\nabla u)|\,d\Gamma \leq R^2\int_{\Gamma_0}|u'|^2\,d\Gamma + \int_{\Gamma_0}|\nabla u|^2\,d\Gamma, \qquad (4.12)$$

$$\int_{\Gamma_0}|(n-1)g(u')u|\,d\Gamma \leq \frac{1}{2}(n-1)^2\int_{\Gamma_0}|g(u')|^2\,d\Gamma + C(\Omega)E(t). \qquad (4.13)$$

Combining (4.11)-(4.13), taking the assumption (H.4) into account and assuming that there exist positive constants C_1, C_2 such that

$$C_1|s| \leq |g(s)| \leq C_2|s|; \quad \text{for all } s\in\mathbf{R},$$

we deduce

$$E'_\varepsilon(t) \leq -\left(-\frac{1}{2} + \varepsilon - \frac{C(\Omega)\varepsilon}{2}\right) \int_\Omega |u'|^2 \, dx \tag{4.14}$$

$$-\left(\varepsilon - \frac{\beta^2}{2} - 2\varepsilon\beta R - \frac{\varepsilon(n-1)}{2}\beta^2 - \frac{\varepsilon(n-1)}{2}C(\Omega) - \frac{C(\Omega)\varepsilon}{2}\right) \int_\Omega |\nabla u|^2 \, dx$$

$$-\left(-2n\varepsilon + \frac{\varepsilon(n-1)}{\alpha} - \frac{C(\Omega)\varepsilon}{2}\right) \int_\Omega F(u) \, dx$$

$$-\left(1 - \varepsilon\left(\frac{2(1+R^2)}{C_1} + 2(n-1)^2 C_2\right)\right) \int_{\Gamma_0} g(u')u' d\Gamma.$$

Hence, if we choose $\varepsilon, \alpha, \beta, C_1$ and C_2 such that

$$-\frac{1}{2} + \varepsilon - \frac{C(\Omega)\varepsilon}{2} > 0$$

$$\varepsilon - \frac{\beta^2}{2} - 2\varepsilon\beta R - \frac{\varepsilon(n-1)}{2}\beta^2 - \frac{\varepsilon(n-1)}{2}C(\Omega) - \frac{C(\Omega)\varepsilon}{2} > 0$$

$$-2n\varepsilon + \frac{\varepsilon(n-1)}{\alpha} - \frac{C(\Omega)\varepsilon}{2} > 0$$

$$\varepsilon\left(\frac{2(1+R^2)}{C_1} + 2(n-1)^2 C_2\right) \leq 1$$

we conclude the proof of proposition 4.1 and consequently the exponential decay of the energy. ■

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Mohammed Aassila
Intitute de Recherche Mathématique Avancée
Université Louis Pasteur et C. N. R. S.
7 rue René Descartes
67084 - Strasbourg cédex - France
E-mail: aassila@math.u-strasbg.fr

Marcelo M. Cavalcanti
Departamento de Matemática
Universidade Estadual de Maringá
87020-900, Maringá - Brasil
E-mail:marcelo@gauss.dma.uem.br