Two Recurrence Relations for Stirling Factors

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1 Introduction

Let $\Pi(m,n),\ m,n \geq 0$, be the set of partitions of a set of m objects into n mutually disjoint non-empty subsets, and let $P(m,n) = |\Pi(m,n)|$, the number of such partitions. (We systematically denote the cardinality of the set S by $|S|$.) Then it was pointed out in [HPS] that there are certain non-negative integers $\mu_{hk},\ \ h,k \geq 0$,such that[1]

$$P(m,m-k) = \sum_{h=0}^{k} \binom{m}{h+k} \mu_{hk}. \tag{1.1}$$

We called the integers μ_{hk} Stirling factors. Our reason for choosing this terminology was that, in fact,

$$P(m,n) = \mathrm{St}(m,n), \tag{1.2}$$

where

$$\mathrm{St}(m,n) = \frac{1}{n!} \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} r^m, \tag{1.3}$$

the Stirling number of the second kind. For completeness, let us give a proof of (1.2).

[1]Actually, in [HPS], we restricted m,n,h,k to being strictly positive. This slightly reduced the scope of our results. It also obscured from our view the uniqueness theorem (Theorem 3.1) which we state and prove in this paper.

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First observe that $P(m,n)$ clearly satisfies the recurrence relation

$$P(m,n) = nP(m-1,n) + P(m-1,n-1), \quad m,n \geq 1. \tag{1.4}$$

We adopt the reasonable convention that

$$P(0,0) = 1; \tag{1.5}$$

and note that it is, of course, consistent with (1.4). Further,

$$P(m,0) = 0,\ m \geq 1, \quad P(0,n) = 0, \quad n \geq 1. \tag{1.6}$$

Moreover, $P(m,n)$ is entirely determined by (1.4), (1.5), (1.6). Thus, to prove (1.2), it suffices to show that $\mathrm{St}(m,n)$ satisfies the analogs of (1.4), (1.5), (1.6).

Proving that $\mathrm{St}(0,0) = 1$, $\mathrm{St}(0,n) = 0$, $n \geq 1$ is easy, since[2]

$$\mathrm{St}(0,n) = \frac{1}{n!}\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r} = \begin{cases} 1, n = 0 \\ \frac{(-1)^n}{n!}(1-1)^n = 0, n \geq 1 \end{cases}$$

Thus it remains to prove that

$$\mathrm{St}(m,n) = n\mathrm{St}(m-1,n) + \mathrm{St}(m-1,n-1), m,n \geq 1 \tag{1.7}$$

Now

$$\begin{aligned}
\mathrm{RHS} &= \tfrac{1}{(n-1)!}\left(\textstyle\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}r^{m-1} + \sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{n-1}{r}r^{m-1}\right) \\
&= \tfrac{1}{(n-1)!}\left(n^{m-1} + \textstyle\sum_{r=0}^{n-1}(-1)^{n-r}r^{m-1}\left(\binom{n}{r} - \binom{n-1}{r}\right)\right) \\
&= \tfrac{1}{(n-1)!}\left(\textstyle\sum_{r=1}^{n-1}(-1)^{n-r}r^{m-1}\left(\binom{n}{r} - \binom{n-1}{r}\right) + n^{m-1}\right), \\
&\qquad\qquad \text{since } \binom{n}{0} = 1 \quad \text{for all } n, \\
&= \tfrac{1}{(n-1)!}\left(\textstyle\sum_{r=1}^{n-1}(-1)^{n-r}r^{m-1}\binom{n-1}{r-1} + n^{m-1}\right), \text{ by the Pascal Identity.}
\end{aligned}$$

Moreover, $\binom{n-1}{r-1} = \frac{r}{n}\binom{n}{r}, n \geq 1, \ r \geq 1$, so

$$\begin{aligned}
\mathrm{RHS} &= \tfrac{1}{n!}\left(\textstyle\sum_{r=1}^{n-1}(-1)^{n-r}r^{m}\binom{n}{r} + n^m\right) \\
&= \tfrac{1}{n!}\textstyle\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}r^m, \quad \text{since } m \geq 1, \\
&= \mathrm{LHS},
\end{aligned}$$

and (1.7), and hence (1.2), are proved.

The Stirling factors μ_{hk} were simply described arithmetically in [HPS] by means of initial values and a recurrence relation. However, (1.2) allows us to give them an elegant combinatorial interpretation.

We define a *club* to be a set with at least 2 elements and we then define μ_{hk} to be the number of partitions of a set of $(h+k)$ elements into h clubs. With this definition we can prove (1.1) very much more easily than we proved the corresponding result in [HPS]. For let us partition $\Pi(m, m-k)$ into subsets Π_h, where Π_h consists of

[2]Note that 0^0 is to be interpreted as 1. That $\mathrm{St}(m,0) = 0, m \geq 1$, follows immediately from (1.3)

partitions (of a set with m elements) into $(m-h-k)$ singletons and h clubs. Then $0 \le h \le k$; and

$$|\Pi_h| = \binom{m}{m-h-k}\mu_{hk}. \tag{1.8}$$

Plainly, (1.8) establishes (1.1);and (1.1), (1.2) together establish the identity

$$\mathrm{St}(m, m-k) = \sum_{h=0}^{k} \binom{m}{h+k}\mu_{hk}, \quad m \ge 0, \quad 0 \le k \le m \tag{1.9}$$

justifying the name Stirling factors for the quantities μ_{hk}.

In Section 2 we establish two recurrence relations for the Stirling factors, together with initial conditions which, together with either relation, fully determine these factors. In Section 3 we introduce an entirely different interpretation of the Stirling factors. In this interpretation one recurrence relation is fairly easily proved, thereby validating the interpretation, but we make no attempt to prove the other recurrence relation directly, since, of course, it follows from the validity of the interpretation. We are led by these considerations to a discussion of the relationship between the two recurrence relations, and prove that, essentially, it is only with the particular initial conditions yielding the Stirling factors that the two recurrence relations are compatible.

A table of values of μ_{hk}, $0 \le h, k \le 10$, is given in Figure 1. Various identities satisfied by the Stirling factors are to be found in [HHP].

$k\downarrow \backslash h\rightarrow$	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0
2	0	1	3	0	0	0	0	0	0	0	0
3	0	1	10	15	0	0	0	0	0	0	0
4	0	1	25	105	105	0	0	0	0	0	0
5	0	1	56	490	1260	945	0	0	0	0	0
6	0	1	119	1918	9450	17325	10395	0	0	0	0
7	0	1	246	6825	56980	190575	270270	135135	0	0	0
8	0	1	501	22935	302995	1636635	4099095	4729725	2027025	0	0
9	0	1	1012	74316	1487200	12122110	47507460	94594500	91891800	34459425	0
10	0	1	2035	235092	6914908	84131350	466876410	1422280860	2343240900	1964187225	654729075

The Stirling Factors μ_{hk}.
Figure 1

We close this introduction by pointing out that many nice properties of the Stirling numbers follow from (1.2). For example, it is obvious that $P(m, n) = 0$ if $m < n$.Thus it is true that

$$\mathrm{St}(m, n) = 0, m < n, \tag{1.10}$$

but this does not seem obvious (except if $m = 0$) from the definition (1.3).

2 The recurrence relations for Stirling factors

From the definition of the Stirling factors $\mu_{hk},\ \ h,k \geq 0$, the following initial conditions are straightforward:[3]

$$\mu_{00} = 1; \quad \mu_{0k} = 0, \ \ k \geq 1; \quad \mu_{h0} = 0, \ \ h \geq 1. \tag{2.1}$$

Indeed, it is just as obvious that

$$\mu_{hk} = 0, \ \ h > k, \tag{2.2}$$

since h clubs have, altogether, at least $2h$ members and $h+k < 2h$ if $k < h$.

We now prove our first recurrence relation for μ_{hk}. It will be clear that this relation, together with (2.1), fully determines μ_{hk}.

Theorem 2.1

$$\mu_{h+1,k} = \sum_{j=1}^{k-h} \binom{h+k}{j} \mu_{h,k-j}, \ \ h \geq 0. \tag{2.3}$$

Proof. We note first that (2.3) follows immediately from (2.2) if $h \geq k$, so we may assume $h \leq k-1$. Let $M(h,k)$ be the set of partitions of a set with $(h+k)$ elements into h clubs, so that $\mu_{hk} = |M(h,k)|$. Now partition $M(h+1,k)$ into disjoint subsets M_j, as follows. Distinguish one element of our original set of $(h+k+1)$ elements

$$\bullet\underbrace{\circ\circ\circ\cdots\circ}_{h+k} \tag{$\bigstar$}$$

Then M_j consists of those partitions such that the distinguished elements belongs to a club with $(j+1)$ members. Since $(h+k-j)$ elements must then be assigned to h clubs, we require $h+k-j \geq 2h$, or $j \leq k-h$. Moreover,

$$|M_j| = \binom{h+k}{j} \mu_{h,k-j},$$

so that (2.3) is established. This is essentially the recurrence relation proved in [HPS], modified to include the cases $h = 0,\ \ k = 0$. Notice that, with $h = 0$, (2.3) yields (together with (2.1))

$$\mu_{1k} = \begin{cases} 0, & k = 0 \\ 1, & k \geq 1 \end{cases} \tag{2.4}$$

Of course, (2.4) can easily be deduced from the definition of μ_{hk}, but it will prove important for us that it follows from (2.3) (and (2.1)). It will also be significant that (2.3) actually implies (2.2). ■

We turn now to the second of our recurrence relations. Again, this relation, together with (2.1), plainly determines μ_{hk}.

[3] That $\mu_{00} = 1$ follows exactly the same reasonable convention as (1.5).

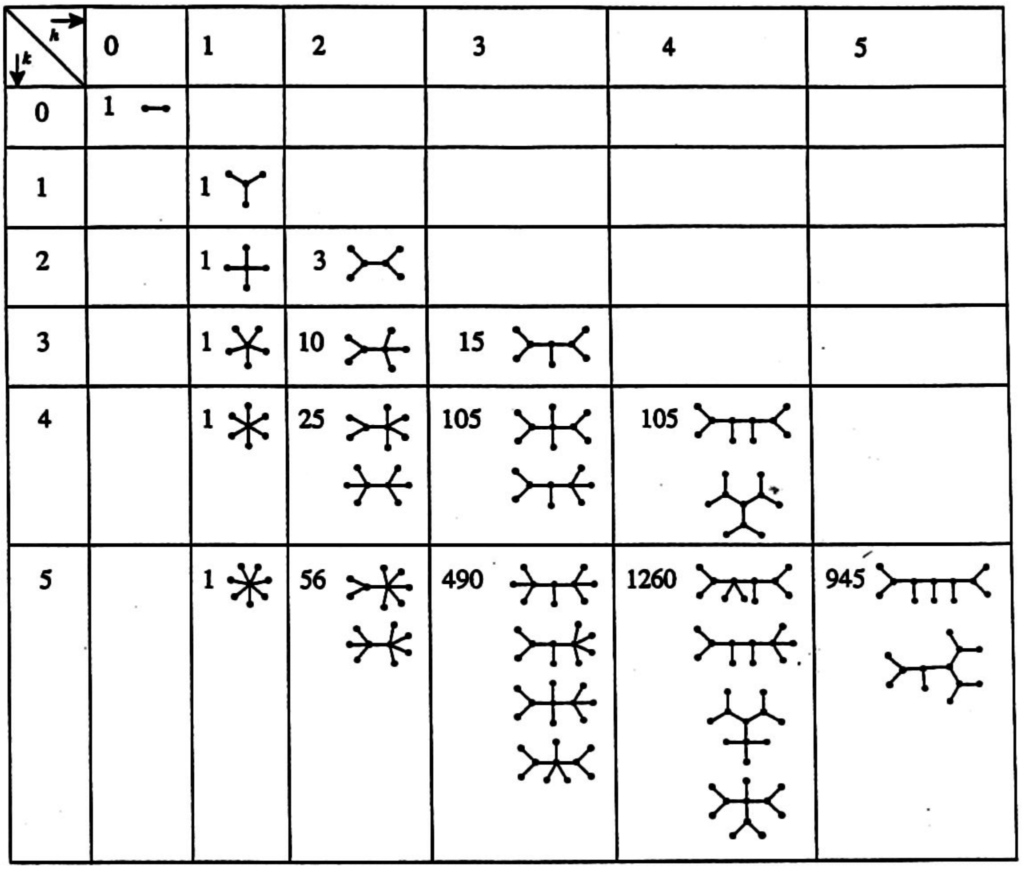
Theorem 2.2

$$\mu_{h,k+1} = h\mu_{hk} + (h+k)\mu_{h-1,k}, \quad h \geq 1, \quad k \geq 0. \tag{2.5}$$

Proof. We note immediately that (2.5) follows from (2.2) if $h > k+1$, so we assume $h \leq k+1$. We are considering partitions of a set with $(h+k+1)$ elements into h clubs and, as in our proof of Theorem 2.1, we distinguish one element of our set. We now consider the remaining $(h+k)$ elements (see (★)). A partition of our original set into h clubs may partition this remaining set into h clubs; it will then assign the distinguished element to any of the clubs. Thus the number of such partitions is $h\mu_{hk}$. The other possibility is that it may partition the remaining set into $(h-1)$ clubs and a singleton — and this could happen in $(h+k)\mu_{h-1,k}$ ways; but then the distinguished element would have to join the singleton to form the h^{th} club. This establishes (2.5). Once again, (2.5), together with (2.1), imply (2.4). ∎

The recurrence relation (2.5) was essentially established in [HHP], but with the two cases $h = 1$, $k = 0$ omitted.

3 Phylogenetic trees and the compatibility theorem



Phylogenetic trees of type (h,k), $0 \leq h,k \leq 5$
Figure 2

A phylogenetic tree of type (h, k) is a tree with h internal nodes, each of valency ≥ 3, and $(k+2)$ labelled external nodes; and $N(h, k)$ is the set of isomorphism classes of phylogenetic trees of type (h, k). In Figure 2 we display the phylogenetic trees of type (h, k) for $0 \leq h, k \leq 5$, together with the values of $|N(h, k)| = \nu_{hk}$.

The remarkable fact, established in [HHP], is that[4]

$$\mu_{hk} = \nu_{hk}. \tag{3.1}$$

This equality was established in [ES], using a subtle and non-canonical one-one correspondence between the sets $M(h, k)$ and $N(h, k)$. Here we give a simple proof, based on that in [HHP], in which we simply show that the quantities ν_{hk} satisfy the (second) recurrence relation of Theorem 2.2, together with correct initial conditions.

We deal first with the initial conditions; here it suffices to establish $\nu_{00} = 1$, $\nu_{h0} = 0,\ \ h \geq 1$. That $\nu_{00} = 1$ is obvious (see the (0,0)-tree in Figure 2). We will now prove the analog of (2.2), namely,

$$\nu_{hk} = 0, \quad h > k. \tag{3.2}$$

For let τ be a phylogenetic tree (PT) of type (h, k). Then since τ has $(h+k+2)$ nodes it has $h+k+1$ edges. On the other hand, the number R of rays must satisfy, by valency considerations,

$$R \geq 3h + k + 2.$$

Thus $2(h+k+1) \geq 3h+k+2$, so that $k \geq h$, and (3.2) is proved.

It remains to prove that

$$\nu_{h,k+1} = h\nu_{hk} + (h+k)\nu_{h-1,k}, \quad 1 \leq h \leq k+1. \tag{3.3}$$

We will suppose our trees supplied with a fixed labelling of their external nodes. If τ is a PT of type (h, k) we may attach a further edge to an internal node, leading to an external node to be labelled $(k+3)$. This can be done in h ways, resulting in a contribution of $h\nu_{hk}$ to $|N(h, k+1)|$. If τ is a PT of type $(h-1, k)$ we may introduce a further node into any edge of τ and then, as above, attach a further edge to this (internal) node, giving it valency 3, and leading to an external node to be labelled $(k+3)$. This can be done in $(h+k)$ ways, since τ has $(h+k)$ edges, and thus results in a contribution of $(h+k)\nu_{h-1,k}$ to $|N(h, k+1)|$. These two contributions to $N(h, k+1)$ are disjoint, since in the first case the internal node leading to the last external node has valency > 3, and in the second case it has valency 3. Moreover, every PT of type $(h, k+1)$ may be viewed as arising in one or other of these two ways. For given τ of type $(h, k+1)$, consider the edge leading to the external node labelled $(k+3)$. If the internal node at the other end of this edge has valency > 3, delete the external node and the edge, producing a PT of type (h, k). If the internal node at the other end of the edge has valency 3, delete the edge and

[4] The cases $h = 0, k = 0$ were omitted in [HHP]. The significance of phylogenetic trees in evolution theory is to be found in [HLP].

both its nodes, producing a PT of type $(h-1,k)$. This completes the proof that $|N(h,k+1)| = h\nu_{hk} + (h+k)\nu_{h-1,k}$, thus establishing (3.3) and, with it, (3.1). We thus have, as a consequence of (2.3) and (3.1),

Corollary 3.1 The numbers ν_{hk} satisfy

$$\nu_{h+1,k} = \sum_{j=1}^{k-h} \binom{h+k}{j} \nu_{h,k-j}, \quad h \geq 0. \tag{3.4}$$

Remarks (i) We have not attempted to prove (3.4) from the definition of ν_{hk} as $|N(h,k)|$.

(ii) Notice that (2.3) may be re-expressed as

$$\mu_{h+1,k} = \sum_{j=h}^{k-1} \binom{h+k}{h+j} \mu_{hj}, \quad h \geq 0. \tag{3.5}$$

Similarly, of course, for (3.4).

We now study the relationship between the two recurrence relations established for μ_{hk} (or ν_{hk}).

We consider the recurrence relations (with $h, k \geq 0$)

$$\alpha_{h+1,k} = \sum_{j=1}^{k-h} \binom{h+k}{j} \alpha_{h,k-j}, \quad h \geq 0 \tag{3.6}$$

$$\beta_{h,k+1} = h\beta_{hk} + (h+k)\beta_{h-1,k}, \quad h \geq 1, \quad k \geq 0 \tag{3.7}$$

Since (3.6), by its very form, implies that $\alpha_{h0} = 0, \ h \geq 1,$ we will assume henceforth that

$$\alpha_{h0} = \beta_{h0} = 0, \quad h \geq 1, \tag{3.8}$$

and will regard the initial values as the values of $\alpha_{0k}, \ \beta_{0k}, \ k \geq 0$. Plainly, both $\alpha_{hk}, \ \beta_{hk}$ are completely determined by the recurrence relations, (3.8), and the initial values. We now prove

Theorem 3.1 Suppose $\alpha_{0k} = \beta_{0k}, \ k \geq 0$. Then $\alpha_{hk} = \beta_{hk}$, for all h, k, if and only if

$$\alpha_{0k} = \beta_{0k} = 0, \quad k \geq 1. \tag{3.9}$$

Remark. Thus we may say that the two recurrence relations are compatible if and only if (3.9) holds. In the latter case we may say that each of (3.6), (3.7) implies the other.

Proof of Theorem 3.1 Suppose first that (3.9) holds and set $\alpha_{00} = \beta_{00} = a$. Then α_{hk} and $a\mu_{hk}$ both satisfy (3.6) with the same initial conditions, so

$$\alpha_{hk} = a\mu_{hk}, \quad h, k \geq 0. \tag{3.10}$$

But we know (Theorem 2.2) that $a\mu_{hk}$ satisfies (3.7) with the same initial conditions as β_{hk},

$$a\mu_{0k} = \beta_{0k}, \quad k \geq 0, \tag{3.11}$$

so that $a\mu_{hk} = \beta_{hk}, \quad h, k \geq 0,$(3.12)
and hence

$$\alpha_{hk} = \beta_{hk}, \quad \text{for all } h, k. \tag{3.13}$$

Conversely, suppose there exists $k \geq 1$ such that $\alpha_{0k} = \beta_{0k} \neq 0$. Let ℓ be the minimum such k and set

$$\begin{cases} \alpha_{00} = \beta_{00} = a \\ \alpha_{0\ell} = \beta_{0\ell} = b \neq 0 \qquad (\ell \geq 1) \\ \alpha_{0,\ell+1} = \beta_{0,\ell+1} = c \end{cases} \tag{3.14}$$

Then, by (3.6), $\alpha_{1,\ell+2} = \sum_{j=1}^{\ell+2} \binom{l+2}{j} \alpha_{0,\ell+2-j} = (\ell+2)c + \binom{\ell+2}{2} b + a.$
On the other hand,

$$\begin{aligned} &\beta_{1\ell} = \beta_{1,\ell-1} = \cdots = \beta_{11} = \beta_{00} = a, \\ &\beta_{1,\ell+1} = \beta_{1,\ell} + (\ell+1)\beta_{0\ell} = a + (\ell+1)b, \\ &\beta_{1,\ell+2} = \beta_{1,\ell+1} + (\ell+2)\beta_{0,\ell+1} = a + (\ell+1)b + (\ell+2)c. \end{aligned}$$

Thus $\alpha_{1,\ell+2} = \beta_{1,\ell+2}$ if and only if $\binom{\ell+2}{2} b = (\ell+1)b$.
Since $b \neq 0$ and since $\binom{\ell+2}{2} = \ell + 1$ if and only if $\ell = 0$ (or -1), we see that, since $\ell \geq 1$, $\alpha_{1,\ell+2} \neq \beta_{1,\ell+2}$. This completes the proof of our theorem.

∎

Remark Notice that we only used the fact that $\beta_{10} = 0$ in this last argument (to infer that $\beta_{11} = \beta_{00}$), not the full force of (3.8).

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