A note on nonexistence of global solutions to a nonlinear integral equation

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Abstract

In this paper we study the Cauchy problem for the integral equation $u_t = -(-\Delta)^{\frac{\beta}{2}}u + h(t)u^{1+\alpha}$ in $\mathbb{R}^N \times (0,T)$, where $0 < \beta \leq 2$. We obtain some extension of results of Fujita who considered the case $\beta = 2$ and $h \equiv 1$.

1 Introduction

THIS article deals with the blow-up of positive solutions to the Cauchy problem for the integrodifferential equation

$$u_t = -(-\Delta)^{\frac{\beta}{2}}u + h(t)u^{1+\alpha} \quad \text{in } \mathbb{R}^N \times (0,T), \tag{1.1}$$

$$u(x,0) = u_0(x) \geq 0 \quad \text{for } x \in \mathbb{R}^N, \tag{1.2}$$

where $(-\Delta)^{\frac{\beta}{2}}$, for $0 < \beta \leq 2$, denote the fractional power of the operator $-\Delta$. It is assumed that u_0 is a continuous function defined on $\mathbb{R}^N$ and α is a positive constant. The function h satisfies

h$_1$) $h \in C[0,\infty), h \geq 0$,

h$_2$) $c_0 t^\sigma \leq h(t) \leq c_1 t^\sigma$ for sufficiently large t, where $c_0, c_1 > 0$ and $\sigma > -1$ are constants.

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When $h \equiv 1$ and $\beta = 2$ the study of (1.1) – (1.2) goes back to the fundamental work of Fujita [2]. It is well known that not all solutions of (1.1) are global. Fujita proved that no positive global solutions exist whenever $N\alpha < 2$. He also showed that equation (1.1) has global solution (i.e. $T = \infty$) for sufficiently small u_0 and $N\alpha > 2$.

The first case is called the blowup case and the second one is called the global existence case. In the critical case $\alpha^* = \frac{2}{N}$, all positive solutions blow up in finite time [5, 6]. When $N\alpha \le \beta$, with $\beta \in (0, 2]$ and $h \equiv 1$, Sugitani [5] proved that the solutions blow up at a finite time under some condition on u_0.

In this paper we shall mainly treat this kind of blowing-up problem for $1+\sigma \ge \frac{\alpha N}{\beta}$ and h satisfies (h_1) and (h_2).

If we look for solution independent of $x, u(x,t) = u(t)$, and $u(0) = a > 0$, we find that

$$u^\alpha(t) = \frac{a^\alpha}{1 - \alpha a^\alpha H(t)}, \tag{1.3}$$

where $H(t) = \displaystyle\int_0^t h(s)ds$, with h satisfying (h_1).

It is clear that if $\lim_{t\to+\infty} H(t) = +\infty$, then for all $a > 0$ there exists $T(a)$ such that

$$\lim_{t\to T(a)} u(t) = +\infty.$$

The way to prove the nonexistence of bounded solutions is to tranform (1.1) into an O.D.E. via the fundamental solution to (1.1).

2 Statement of results

Problem (1.1) – (1.2) is studied via the corresponding Duhamel integral equation

$$u(x,t) = \int_{\mathbb{R}^N} p(x-y,t)u_0(y)dy + \int_0^t ds \int_{\mathbb{R}^N} p(x-y,t-s)h(s)u^{1+\alpha}(y,s)dy, \tag{2.1}$$

where $p(x,t)$ is the fundamental solution to (1.1). It is well known that $p(x,t)$ is given by

$$\int_{\mathbb{R}^N} e^{iz.x}p(x,t)dx = e^{-t|z|^\beta}, \quad 0 < \beta \le 2. \tag{2.2}$$

From [7, pp. 259–263] we have

$$p(x,t) = \int_0^{+\infty} f_{t,\frac{\beta}{2}}(s)T(x,s)ds \text{ for } 0 < \beta \le 2,$$

and

$$p(x,t) = T(x,t) \text{ if } \beta = 2,$$

where

$$f_{t,\frac{\beta}{2}}(s) = \frac{1}{2i\pi}\int_{\tau-i\infty}^{\tau+i\infty} e^{zs - tz^{\frac{\beta}{2}}}dz \ge 0, \quad T(x,s) = (\frac{1}{4\pi s})^{\frac{N}{2}} exp(-\frac{|x|^2}{4s}), \tau > 0, s > 0.$$

For future reference we collect some well known facts about $p(x,t)$.

Proposition 2.1. *Let $p(x,t)$ be the fundamental solution to* (1.1)*, then*

$$\begin{array}{llll} a) & p(x,ts) & = & t^{-\frac{N}{\beta}} p(t^{-\frac{1}{\beta}} x, s), \\ b) & p(x,t) & \geq & (\frac{s}{t})^{-\frac{N}{\beta}} p(x,s) \text{ for all } t \geq s, \\ c) & \text{if } p(0,t) & \leq & 1 \text{ and } \tau \geq 2, \text{ then } \ p(\frac{1}{\tau}(x-y),t) \geq p(x,t)p(y,t), \\ d) & \|p(.,t)\|_1 & = & 1 \quad \text{for all } t > 0. \end{array}$$

Note that $p(0,t)$ is a decreasing function of t and $p(x,t)$ is a decreasing function of $|x|$.

PROOF. Statements (a) and (d) are obtained from (2.2). Statement (c) follows from (b), in fact since

$$\frac{1}{\tau}|x-y| \leq \frac{2}{\tau} Sup\{|x|,|y|\} \leq Sup\{|x|,|y|\}, \quad \text{if } \tau \geq 2,$$

we have

$$p(\frac{1}{\tau}(x-y),t) \geq p(Sup\{|x|,|y|\},t) \geq Sup\{p(|x|,t);p(|y|,t)\}.$$

So if $p(0,t) \leq 1$, the statement (c) holds. ■

Our maint result gives a condition which guaranties the blowing-up in finite time of solutions to (1.1).

Theorem 2.1. *Let $0 < \beta \leq 2, 0 < \frac{\alpha N}{\beta} \leq 1+\sigma$. Suppose that u_0 is a nontrivial nonnegative and continuous function on $\mathbb{R}^N$. Then the nonnegative solution $u(x,t)$ of the integral equation* (2.1) *blows up for some $T_0 > 0$; $u(x,t) = +\infty$ for every $t \geq T_0$ and $x \in \mathbb{R}^N$.*

3 Proof

The idea of the proof is to show that the function

$$\bar{u}(t) = \int_{\mathbb{R}^N} p(x,t)u(x,t)dx$$

blows up in a finite time. We need first to prove the following lemma.

Lemma 3.1. *Suppose that h satisfies (h_1). Let $u(x,t)$ be a nonnegative solution to* (2.1).
Then the following two conditions are equivalent :
(i) *$u(x,t)$ blows up,*
(ii) *$\bar{u}(t)$ blows up, there exists some $T_1 > 0$ such that $\bar{u}(t) = +\infty$ for all $t \geq T_1$.*

As noticed by J.M. Ball [1], the blow up time of $u(x,t)$ is less than the one of $\bar{u}(t)$.

PROOF. It is enough to show that (ii) implies (i).
We may assume $p(0,T_1) \leq 1$. Then, from (a) of Proposition 2.1 we have $p(0,t) \leq 1$

if $t \geq T_1$.
Let $T_1 \leq t \leq s \leq \frac{6}{2^\beta+1}t$ and $\tau = (\frac{6t-s}{s})^{\frac{1}{\beta}}$. We have

$$p(x-y, 6t-s) = (\frac{s}{6t-s})^{\frac{N}{\beta}} p(\frac{1}{\tau}(x-y), s).$$

Since $\tau \geq 2$, it follows from Proposition 2.1 (c) that

$$p(x-y, 6t-s) \geq (\frac{s}{6t-s})^{\frac{N}{\beta}} p(x,s)p(y,s).$$

Therefore

$$\int_{\mathbb{R}^N} p(x-y, 6t-s)u(y,s)dy \geq (\frac{s}{6t-s})^{\frac{N}{\beta}} p(x,s)\bar{u}(s) = +\infty,$$

by (ii).

On the other hand we have from (2.1)

$$u(x, 6t) \geq \int_0^{6t} h(s)\bigg(\int_{\mathbb{R}^N} p(x-y, 6t-s)u^{1+\alpha}(y,s)dy\bigg)ds.$$

Finally, applying Jensen's inequality to the above integral, we get

$$u(x, 6t) \geq \int_0^{\frac{6t}{2^\beta+1}} h(s)\bigg(\int_{\mathbb{R}^N} p(x-y, 6t-s)u(y,s)dy\bigg)^{1+\alpha} ds = +\infty,$$

so that $u(x,t) = +\infty$ for any $t \geq 6T_1$ and $x \in \mathbb{R}^N$. ■

Lemma 3.2. *Let u(x,t) be a nonnegative solution to* (2.1), *then there exist some* $t_0 > 0$, $c > 0$ *and* $\delta > 0$ *such that*

$$u(x, t_0) \geq cp(x, \delta) \quad \text{for all } x \in \mathbb{R}^N. \tag{3.3}$$

Proof. Let $t_0 > 0$ such that $p(0, t_0) \leq 1$. We have $p(x-y, t_0) = p(\frac{1}{2}(2x-2y), t_0)$, and from Proposition 2.1

$$p(x-y, t_0) \geq 2^{-N} p(x, \frac{t_0}{2^\beta}) p(2y, t_0).$$

Therefore

$$u(x, t_0) \geq \int_{\mathbb{R}^N} 2^{-N} p(x, \frac{t_0}{2^\beta}) p(2y, t_0) u_0(y) dy,$$

hence

$$u(x, t_0) \geq cp(x, \delta),$$

where $\delta = \dfrac{t_0}{2^\beta} > 0$ and $c = \displaystyle\int_{\mathbb{R}^N} 2^{-N} p(2y, t_0) u_0(y) dy$. ■

Now we present the proof of Theorem 2.1.

As it was mentioned we study the behaviour of $\bar{u}(t)$ for large t. Let t_0 be such that (3.1) holds true, we have from (2.1)

$$u(x,t+t_0) = \int_{\mathbb{R}^N} p(x-y,t)u(y,t_0)dy + \int_0^t ds \int_{\mathbb{R}^N} p(x-y,t-s)h(s+t_0)u^{1+\alpha}(y,s+t_0)dy,$$

for $t > 0, x \in \mathbb{R}^N$.
It follows from Lemma 3.2 that

$$u(x,t+t_0) \geq c\int_{\mathbb{R}^N} p(x-y,t)p(y,\delta)dy + \int_0^t h(s+t_0)\int_{\mathbb{R}^N} p(x-y,t-s)u^{1+\alpha}(y,s+t_0)dyds,$$

so that

$$u(x,t+t_0) \geq cp(x,t+\delta) + \int_0^t h(s+t_0)\int_{\mathbb{R}^N} p(x-y,t-s)u^{1+\alpha}(y,s+t_0)dyds.$$

By comparison it is enough to show that the solution $v(x,t)$ of the following equation

$$v(x,t) = cp(x,t+\delta) + \int_0^t h(s)\int_{\mathbb{R}^N} p(x-y,t-s)v^{1+\alpha}(y,s)dyds, \tag{3.4}$$

blows up or by Lemma 3.1, that $\bar{v}(t) = \int_{\mathbb{R}^N} p(x,t)v(x,t)dx$ blows up in a finite time.

Using (3.2) we can write

$$\begin{aligned}\int_{\mathbb{R}^N} p(x,t)v(x,t)dx &= c\int_{\mathbb{R}^N} p(x,t)p(x,t+\delta)dx \\ &+ \int_{\mathbb{R}^N}\int_0^t h(s)\int_{\mathbb{R}^N} p(x-y,t-s)p(x,t)v^{1+\alpha}(y,s)dydsdx.\end{aligned}$$

Whence

$$\bar{v}(t) = cp(0,2t+\delta) + \int_0^t h(s)\int_{\mathbb{R}^N} p(y,2t-s)v^{1+\alpha}(y,s)dyds.$$

So

$$\bar{v}(t) \geq cp(0,1)(2t+\delta)^{-\frac{N}{\beta}} + \int_0^t \left(\frac{s}{2t-s}\right)^{\frac{N}{\beta}} h(s)\int_{\mathbb{R}^N} p(y,s)v^{1+\alpha}(y,s)dyds.$$

By application of the Jensen inequality, we get

$$\bar{v}(t) \geq cp(0,1)(2t+\delta)^{-\frac{N}{\beta}} + \int_0^t \left(\frac{s}{2t}\right)^{\frac{N}{\beta}} h(s)\bar{v}^{1+\alpha}(s)ds. \tag{3.5}$$

Let $\theta > 0$ be a fixed positive constant.
If we set

$$f_1(t) = t^{\frac{N}{\beta}}\bar{v}(t),$$

for $t \geq \theta$, then we get

$$f_1(t) \geq cp(0,1)(\frac{\theta}{2\theta+\delta})^{\frac{N}{\beta}} + (\frac{1}{2})^{\frac{N}{\beta}} \int_\theta^t s^{-\frac{\alpha N}{\beta}} h(s) f_1^{1+\alpha}(s) ds,$$

thanks to (3.3).
Let f_2 be the solution to

$$f_2(t) = cp(0,1)(\frac{\theta}{2\theta+\delta})^{\frac{N}{\beta}} + (\frac{1}{2})^{\frac{N}{\beta}} \int_\theta^t s^{-\frac{\alpha N}{\beta}} h(s) f_2^{1+\alpha}(s) ds,$$

which is equivalent to

$$\begin{cases} f_2'(t) &= (\frac{1}{2})^{\frac{N}{\beta}} t^{-\frac{\alpha N}{\beta}} h(t) f_2^{1+\alpha}(t) \text{ for } t > \theta, \\ \\ f_2(\theta) &= cp(0,1)(\frac{\theta}{2\theta+\delta})^{\frac{N}{\beta}}. \end{cases}$$

We clearly have

$$f_2^\alpha(t) = \frac{f_2^\alpha(\theta)}{1 - \alpha f_2^\alpha(\theta)(\frac{1}{2})^{\frac{N}{\beta}} H(t)} \ ,$$

where

$$H(t) = \int_\theta^t s^{-\frac{\alpha N}{\beta}} h(s) ds.$$

Since $\lim_{t\to+\infty} H(t) = +\infty$, by (h_2), there exists T_0 such that

$$f_2(t) = +\infty \quad \text{for } t = T_0.$$

By comparison, we have

$$t^{\frac{N}{\beta}} \bar{v}(t) = f_1(t) \geq f_2(t) = +\infty, \quad \text{for } t = T_0,$$

and then $u(x,t)$ blows up in a finite time. ∎

Corollary 3.1. *Assume h has the property (h_1). If*

$$\limsup_{t\to+\infty} \int_1^t s^{-\frac{\alpha N}{\beta}} h(s) ds = +\infty,$$

then every nontrivial solution to (1.1) *blows up in a finite time.*

Remark 3.1. It is interesting to note that if instead of (1.1) we consider

$$u_t = -(-\Delta)^{\frac{\beta}{2}} u - h(t) u^{1+\alpha}, \tag{3.6}$$

then a result on global existence with some decay at infinity can be given. We suppose that h satisfies $(h_1), (h_2), 1+\sigma > 0$ and $\alpha > 0$.
If $u_0(x) \leq a_0 p(x,0)$, then the corresponding solution to $(3.4) - (1.2)$ satisfies

$$\limsup_{t\to+\infty} t^{\frac{1+\sigma}{\alpha}} \int_{\mathbb{R}^N} p(x,t) u(x,t) dx < \infty.$$

In particular

$$\lim_{t\to+\infty} t^{\frac{1+\sigma}{\alpha}} \int_{\mathbb{R}^N} p(x,t) u(x,t) dx = 0 \quad \text{if } 1+\sigma \leq \alpha \frac{N}{\beta}.$$

The proof is similar as above.

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