On uniform exponential stability of linear skew-product semiflows in Banach spaces

M. Megan A. L. Sasu B. Sasu

Abstract

In this paper we give necessary and sufficient conditions for uniform exponential stability of evolution equations in Banach spaces. This is done by employing a skew-product semiflows technique and Banach function spaces. Generalizations of some well-known results of Datko, Neerven, Rolewicz and Zabczyk are obtained.

1 Introduction

In recent years, an important progress has been made in the study of the asymptotic behaviour of evolution equations in infinite-dimensional Banach spaces. Significant progress has been made in this direction pointing out that an impressive list of classical problems can be treated using the theory of linear skew-product semiflows (see, for example, Sacker and Sell [16], Chow and Leiva [2]-[6], Chicone and Latushkin [1] and Latushkin, Montgomery - Smith and Randolph [11]). There have been obtained results concerning dichotomy of linear skew-product flows over locally compact Banach spaces (see Latushkin, Montgomery-Smith and Randolph [11]) and dichotomy of linear skew-product semiflows over compact Hausdorff spaces, respectively (see Chow and Leiva [3], [4] and [6]). The asymptotic behaviour of the linear skew-product flow has been also characterized in terms of spectral properties of the evolution semigroup associated to the skew-product flow (see Latushkin, Montgomery-Smith and Randolph [11]).

Received by the editors july 2000.

Communicated by F. Bastin.

1991 *Mathematics Subject Classification :* 34D05, 46E30, 93D20.

Key words and phrases : uniform exponential stability, Banach function spaces, linear skew-product semiflows.

In this paper we consider a concept of uniform exponential stability for linear skew-product semiflows which is an extension of the classical concept of exponential stability for time-dependent linear differential equations in Banach spaces (see, for example, Datko [8] and Daleckii and Krein [9]). We give necessary and sufficient conditions for uniform exponential stability of linear skew-product semiflows using a Banach function spaces technique. We not only answer questions concerning stability of linear skew-product semiflows but also obtain generalizations of some well-known results due to Datko ([8]), Zabczyk ([17]), Neerven ([14]) and Rolewicz ([15]).

The theory developed here is applicable for a large class of systems described in Chow and Leiva ([2]-[6]).

2 Notations and preliminaries

In this section we shall present some definitions, notations and results about linear skew-product semiflows and Banach function spaces.

2.1 Linear Skew-Product Semiflows

We begin with the notion of linear skew-product semiflow on the trivial Banach bundle $\mathcal{E} = X \times \Theta$, where X is a fixed Banach space - the state space - and Θ is a compact Hausdorff space. We shall denote by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators from X into itself.

Definition 2.1. A mapping $\sigma : \Theta \times \mathbf{R}_+ \to \Theta$ is called a *semiflow* on Θ, if it has the following properties:

(f_1) $\sigma(\theta, 0) = \theta,$ for all $\theta \in \Theta$;
(f_2) $\sigma(\theta, s+t) = \sigma(\sigma(\theta, s), t),$ for all $(\theta, s, t) \in \Theta \times \mathbf{R}_+^{\mathbf{2}}$;
(f_3) σ is continuous.

Definition 2.2. A pair $\pi = (\Phi, \sigma)$ is called a *linear skew-product semiflow* on $\mathcal{E} = X \times \Theta$ if σ is a semiflow on Θ and $\Phi : \Theta \times \mathbf{R}_+ \to \mathcal{B}(X)$ satisfies the following conditions:

(s_1) $\Phi(\theta, 0) = I$, the identity operator on X, for all $\theta \in \Theta$;
(s_2) $\Phi(\theta, t+s) = \Phi(\sigma(\theta, t), s)\Phi(\theta, t),$ for all $(\theta, t, s) \in \Theta \times \mathbf{R}_+^{\mathbf{2}}$ (*the cocycle identity*);
(s_3) $\lim\limits_{t \to 0_+} \Phi(\theta, t)x = x,$ uniformly in θ. This means that for every $x \in X$ and every $\varepsilon > 0$ there is $\delta = \delta(x, \varepsilon) > 0$ such that $||\Phi(\theta, t)x - x|| < \varepsilon$, for all $\theta \in \Theta$ and $0 \leq t \leq \delta$.

Remark 2.1. The mapping $t \to \Phi(\theta, t)x$ is right continuous, for all $(x, \theta) \in \mathcal{E}$.

Example 2.1. Let Θ be a compact Hausdorff space and let $\mathbf{S} = \{S(t)\}_{t \geq 0}$ be a C_0 - semigroup on X. Then for every semiflow $\sigma : \Theta \times \mathbf{R}_+ \to \Theta$ on Θ the pair $\pi_S = (\Phi_S, \sigma)$, where

$$\Phi_S(\theta, t) = S(t), \qquad (\theta, t) \in \Theta \times \mathbf{R}_+$$

is a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$, which is called *the linear skew-product semiflow generated by the* C_0 *- semigroup* $\mathbf{S}$ *and the semiflow* σ.

The following example can be found in Chow and Leiva ([2]):

Example 2.2. Let σ be a semiflow on the compact Hausdorff space Θ and let $\mathbf{S} = \{S(t)\}_{t\geq 0}$ be a C_0 -semigroup on the Banach space X. For every strongly continuous mapping $D : \Theta \to \mathcal{B}(X)$ there is a linear skew-product semiflow $\pi_D = (\Phi_D, \sigma)$ on $\mathcal{E} = X \times \Theta$ such that

$$\Phi_D(\theta, t)x = S(t)x + \int_0^t S(t-s)D(\sigma(\theta, s))\Phi_D(\theta, s)x\, ds$$

for all $(x, \theta, t) \in X \times \Theta \times \mathbf{R}_+$.

The linear skew-product semiflow $\pi_D = (\Phi_D, \sigma)$ is called *the linear skew-product semiflow generated by the triplet* $(\mathbf{S}, D, \sigma)$.

Remark 2.2. As a consequence of condition (s_2) from Definition 2.2. it follows that if $\pi = (\Phi, \sigma)$ is a linear skew product semiflow on $\mathcal{E} = X \times \Theta$, then

$$\Phi(\theta, nt) = \Phi(\sigma(\theta, (n-1)t), t) \ldots \Phi(\sigma(\theta, 2t), t)\Phi(\sigma(\theta, t), t)\Phi(\theta, t)$$

for all $(\theta, n, t) \in \Theta \times \mathbf{N} \times \mathbf{R}_+$.

The following result can be found in Chow and Leiva [3].

Proposition 2.1. *Let* $\pi = (\Phi, \sigma)$ *be a linear skew-product semiflow on* $\mathcal{E} = X \times \Theta$. *Then there exist constants* $M \geq 1$ *and* $\omega > 0$ *such that*

$$||\Phi(\theta, t)|| \leq Me^{\omega t}, \quad (\theta, t) \in \Theta \times \mathbf{R}_+.$$

Definition 2.3. A linear skew-product semiflow $\pi = (\Phi, \sigma)$ on $\mathcal{E} = X \times \Theta$ is called *uniformly exponentially stable* if there are $N \geq 1$ and $\nu > 0$ such that

$$||\Phi(\theta, t)|| \leq Ne^{-\nu t}, \quad (\theta, t) \in \Theta \times \mathbf{R}_+.$$

A sufficient condition for uniform exponential stability of a linear skew-product semiflow is given by

Proposition 2.2. *Let* $\pi = (\Phi, \sigma)$ *be a linear skew-product semiflow on* $\mathcal{E} = X \times \Theta$. *If there are* $t_0 > 0$ *and* $c \in (0, 1)$ *such that*

$$||\Phi(\theta, t_0)|| \leq c, \qquad \theta \in \Theta,$$

then π *is uniformly exponentially stable.*

Proof: Let $M \geq 1$ and $\omega > 0$ given by Proposition 2.1. Let ν be a positive number such that $c = e^{-\nu t_0}$.

Let $\theta \in \Theta$ be fixed. For $t \in \mathbf{R}_+$ there are $n \in \mathbf{N}$ and $r \in [0, t_0)$ such that $t = nt_0 + r$. Then by Remark 2.2. we obtain

$$||\Phi(\theta, t)|| \leq ||\Phi(\sigma(\theta, nt_0), r)||\, ||\Phi(\theta, nt_0)|| \leq$$

$$\leq Me^{\omega t_0}||\Phi(\sigma(\theta,(n-1)t_0),t_0)||\dots||\Phi(\sigma(\theta,t_0),t_0)||\,||\Phi(\theta,t_0)|| \leq$$

$$\leq M\,e^{\omega t_0}\,e^{-n\nu t_0} \;\leq\; N\,e^{-\nu t},$$

where $N = Me^{(\omega+\nu)t_0}$. So, π is uniformly exponentially stable. ■

2.2 Banach function spaces

Let (Ω,Σ,μ) be a positive σ - finite measure space. By $M(\mu)$ we denote the linear space of μ-measurable functions $f:\Omega\to\mathbf{C}$, identifying the functions which are equal μ - a.e.

Definition 2.4. A *Banach function norm* is a function $N: M(\mu)\to[0,\infty]$ with the following properties:

(n_1) $N(f)=0$ if and only if $f=0\,\mu$ - a.e.;
(n_2) if $|f|\leq|g|\,\mu$ - a.e. then $N(f)\leq N(g)$;
(n_3) $N(af)=|a|N(f)$, for all $a\in\mathbf{C}$ and all $f\in M(\mu)$ with $N(f)<\infty$;
(n_4) $N(f+g)\leq N(f)+N(g)$, for all $f,g\in M(\mu)$.

Let $B=B_N$ be the set defined by:

$$B := \{f\in M(\mu) : |f|_B := N(f) < \infty\}.$$

It is easy to see that $(B,|\cdot|_B)$ is a normed linear space. If B is complete then B is called *Banach function space* over Ω.

Remark 2.3. B is an ideal in $M(\mu)$, i.e. if $|f|\leq|g|\,\mu$ - a.e. and $g\in B$ then also $f\in B$ and $|f|_B\leq|g|_B$.

Remark 2.4. If $f_n\to f$ in norm in B, then there exists a subsequence (f_{k_n}) converging to f pointwise (see [12]).

Let $(\Omega,\Sigma,\mu)=(\mathbf{R}_+,\mathcal{L},m)$ where $\mathcal{L}$ is the σ-algebra of all Lebesgue measurable sets $A\subset\mathbf{R}_+$ and m the Lebesgue measure. For a Banach function space over $\mathbf{R}_+$ we define

$$F_B:\mathbf{R}_+\to\bar{\mathbf{R}}_+,\quad F_B(t):=\begin{cases} |\chi_{[0,t)}|_B &, \text{ if } \chi_{[0,t)}\in B\\ \infty &, \text{ if } \chi_{[0,t)}\notin B\end{cases}$$

where $\chi_{[0,t)}$ denotes the characteristic function of $[0,t)$. The function F_B is called *the fundamental function* of the Banach space B.

In what follows we shall denote by $\mathcal{B}(\mathbf{R}_+)$ the set of all Banach function spaces with the property that $\lim\limits_{t\to\infty}F_B(t)=\infty$ and there exists a strictly increasing sequence (t_n) of positive real numbers with

$$t_n\to\infty,\ \sup_n(t_{n+1}-t_n)<\infty \text{ and } \inf_n|\chi_{[t_n,t_{n+1})}|_B>0.$$

A trivial example of Banach function space over $\mathbf{R}_+$ which belongs to $\mathcal{B}(\mathbf{R}_+)$ is $L^p(\mathbf{R}_+,\mathbf{C})$ with $1\leq p<\infty$.

Similarly, let $(\Omega, \Sigma, \mu) = (\mathbf{N}, \mathcal{P}(\mathbf{N}), \mu_c)$ where μ_c is the countable measure and let B be a Banach function space over $\mathbf{N}$ (in this case B is called *Banach sequence space*). We define

$$F_B : \mathbf{N}^* \to \bar{\mathbf{R}}_+, \quad F_B(n) := \begin{cases} |\chi_{\{0,\dots,n-1\}}|_B & , \quad \text{if } \chi_{\{0,\dots,n-1\}} \in B \\ \infty & , \quad \text{if } \chi_{\{0,\dots,n-1\}} \notin B \end{cases}$$

called *the fundamental function of* B.

In what follows we denote by $\mathcal{B}(\mathbf{N})$ the set of all Banach sequence spaces B with $\lim\limits_{n\to\infty} F_B(n) = \infty$ and

$$\inf_n |\chi_{\{n\}}|_B > 0.$$

Remark 2.5. If B is a Banach function space over $\mathbf{R}_+$ which belongs to $\mathcal{B}(\mathbf{R}_+)$ then

$$S_B := \{(\alpha_n)_n : \sum_{n=0}^{\infty} \alpha_n \chi_{[t_n, t_{n+1})} \in B\}$$

with respect to the norm

$$|(\alpha_n)_n|_{S_B} := |\sum_{n=0}^{\infty} \alpha_n \chi_{[t_n, t_{n+1})}|_B,$$

is a Banach sequence space which belongs to $\mathcal{B}(\mathbf{N})$.

Indeed, this assertion follows by observing that

$$|\chi_{\{n\}}|_{S_B} = |\chi_{[t_n, t_{n+1})}|_B \text{ and } F_{S_B}(n) = F_B(t_n), \quad n \in \mathbf{N}.$$

In what follows we shall give some examples of Banach sequence spaces.

Example 2.4. If $p \in [1, \infty)$ then $B = l^p$ with

$$|s|_p = \left(\sum_{n=0}^{\infty} |s(n)|^p\right)^{\frac{1}{p}}$$

is a Banach sequence space which belongs to $\mathcal{B}(\mathbf{N})$.

Example 2.5. (Orlicz sequence spaces) Let $g : \mathbf{R}_+ \to \bar{\mathbf{R}}_+$ be a nondecreasing, left continuous function which is not identically 0 or ∞ on $(0, \infty)$. We define the function:

$$Y_g(t) = \int_0^t g(s)\, ds$$

which is called *the Young function associated to* g.

For every $s : \mathbf{N} \to \mathbf{C}$ we consider

$$M_g(s) := \sum_{n=0}^{\infty} Y_g(|s(n)|).$$

The set O_g of all sequences with the property that there exists $k > 0$ such that $M_g(ks) < \infty$ is easily checked to be a linear space. With respect to the norm

$$|s|_g := inf\{k > 0 : M_g(\frac{1}{k}s) \leq 1\}$$

it is a Banach sequence space called *Orlicz sequence space*. Trivial examples of Orlicz sequence spaces are $l^p, 1 \leq p \leq \infty$ which are obtained for

$$g(t) = p\,t^{p-1},\ 1 \leq p < \infty \quad \text{and} \quad g(t) = \begin{cases} 0, & 0 \leq t \leq 1 \\ \infty, & t > 1 \end{cases} \quad \text{for} \quad p = \infty.$$

Remark 2.6. If $g : \mathbf{R}_+ \to \mathbf{R}_+$ is a nondecreasing left continuous function with $g(t) > 0$, for all $t > 0$ and $g(0) = 0$ then the Orlicz sequence space O_g associated to g belongs to $\mathcal{B}(\mathbf{N})$.

3 The main results

In this section we shall give necessary and sufficient conditions for uniform exponential stability of linear skew-product semiflows in Banach spaces.

Our main result is

Theorem 3.1. *The linear skew-product semiflow $\pi = (\Phi, \sigma)$ on $\mathcal{E} = X \times \Theta$ is uniformly exponentially stable if and only if there are a Banach sequence space $B \in \mathcal{B}(\mathbf{N})$ and a sequence (t_n) of positive real numbers with the following properties:*

(i) $\sup_n |t_{n+1} - t_n| < \infty$;

(ii) for every $(x, \theta) \in \mathcal{E}$ the function

$$\varphi_{x,\theta} : \mathbf{N} \to \mathbf{R}_+, \quad \varphi_{x,\theta}(n) := ||\Phi(\theta, t_n)x||$$

belongs to B;

(iii) there exists $K : X \to (0, \infty)$ such that

$$|\varphi_{x,\theta}|_B \leq K(x), \qquad (x, \theta) \in \mathcal{E}. \tag{3.1}$$

Proof: Necessity. It is immediate by taking $B = l^1$ and $t_n = n$.

Sufficiency. We have two possible situations.

Case 1. If $T = \sup_n t_n < \infty$ then we have

$$||\Phi(\theta, T)x|| \leq ||\Phi(\sigma(\theta, t_n), T - t_n)||\, ||\Phi(\theta, t_n)x|| \leq$$

$$\leq Me^{\omega T}||\Phi(\theta, t_n)x|| = \varphi_{\theta,\tilde{x}}(n), \quad n \in \mathbf{N}, (x, \theta) \in \mathcal{E},$$

where $\tilde{x} = Me^{\omega T}x$ and $M \geq 1, \omega > 0$ are given by Proposition 2.1. Thus we have

$$||\Phi(\theta, T)x||\, \chi_{\{0,\ldots,n-1\}} \leq \varphi_{\tilde{x},\theta}, \quad n \in \mathbf{N}^*.$$

Using (3.1) it follows that

$$F_B(n)||\Phi(\theta, T)x|| \leq |\varphi_{\tilde{x},\theta}|_B \leq K(\tilde{x}), \quad n \in \mathbf{N}^*.$$

Because $B \in \mathcal{B}(\mathbf{N})$ it results

$$\Phi(\theta, T)x = 0, \qquad (x, \theta) \in \mathcal{E}$$

and hence π is uniformly exponentially stable.

Case 2. Suppose that (t_n) is unbounded. Since $B \in \mathcal{B}(\mathbf{N})$ there exists $c > 0$ such that

$$|\chi_{\{n\}}|_B \geq c, \qquad n \in \mathbf{N}.$$

From

$$\varphi_{x,\theta}(n)\chi_{\{n\}} \leq \varphi_{x,\theta}, \qquad n \in \mathbf{N}, (x, \theta) \in \mathcal{E}$$

we have

$$c\,||\Phi(\theta, t_n)x|| \leq |\varphi_{x,\theta}|_B \leq K(x), \qquad n \in \mathbf{N}, (x, \theta) \in \mathcal{E}.$$

By applying the uniform boundedness principle there exists $N > 0$ such that

$$||\Phi(\theta, t_n)|| \leq N, \qquad n \in \mathbf{N},\ \theta \in \Theta.$$

Let $\theta \in \Theta$. If $s \geq t_0$ then using the fact that (t_n) is unbounded and the hypothesis (i) it follows that there exists $n(s) \in \mathbf{N}$ such that

$$t_{n(s)} \leq s \leq t_{n(s)} + \delta$$

where $\delta = \sup_n |t_{n+1} - t_n|$. Then

$$||\Phi(\theta, s)|| \leq ||\Phi(\sigma(\theta, t_{n(s)}), s - t_{n(s)})||\,||\Phi(\theta, t_{n(s)})|| \leq MNe^{\omega\delta}, \quad s \geq t_0, \theta \in \Theta.$$

It follows that

$$||\Phi(\theta, s)|| \leq L := \max\{Me^{\omega t_0}, MNe^{\omega\delta}\}, \qquad s \in \mathbf{R}_+, \theta \in \Theta.$$

We consider the sequence (k_n) defined by $k_0 = 0, k_{n+1} = \min\{j : t_j \geq t_{k_n}\}$. Then $k_n \to \infty$ and

$$t_j \leq t_{k_n}, \qquad j \in \{0, \ldots, k_n\}, n \in \mathbf{N}.$$

From

$$||\Phi(\theta, t_{k_n})x|| \leq ||\Phi(\sigma(\theta, t_j), t_{k_n} - t_j)||\,||\Phi(\theta, t_j)x|| \leq$$

$$\leq L||\Phi(\theta, t_j)x||,\ j \in \{0, \ldots, k_n\},\ n \in \mathbf{N}$$

it results

$$||\Phi(\theta, t_{k_n})x||\,\chi_{\{0,\ldots,k_n\}} \leq L\varphi_{x,\theta}, \quad n \in \mathbf{N}, (x, \theta) \in \mathcal{E}$$

and hence

$$||\Phi(\theta, t_{k_n})x||\, F_B(k_n+1) \leq LK(x), \quad n \in \mathbf{N}, (x,\theta) \in \mathcal{E}$$

By uniform boundedness principle there exists $K \geq 1$ such that

$$||\Phi(\theta, t_{k_n})||\, F_B(k_n+1) \leq K, \quad n \in \mathbf{N}, \theta \in \Theta.$$

This inequality together with $B \in \mathcal{B}(\mathbf{N})$ implies that there is $m \in \mathbf{N}$ such that

$$||\Phi(\theta, t_{k_m})|| \leq \frac{1}{2}, \qquad \theta \in \Theta.$$

By Proposition 2.2. we conclude that π is uniformly exponentially stable. ∎

Corollary 3.1. *The linear skew-product semiflow $\pi = (\Phi, \sigma)$ on $\mathcal{E} = X \times \Theta$ is uniformly exponentially stable if and only if there are $p \in [1, \infty)$ and $K : X \to (0, \infty)$ such that*

$$\sum_{n=0}^{\infty} ||\Phi(\theta, n)x||^p \leq K(x), \qquad (x,\theta) \in \mathcal{E}.$$

Proof: Necessity It is immediate.
Sufficiency. It results from Theorem 3.1. for $B = l^p$ and $t_n = n$. ∎

Theorem 3.2. *The linear skew-product semiflow $\pi = (\Phi, \sigma)$ on $\mathcal{E} = X \times \Theta$ is uniformly exponentially stable if and only if there exist a non-decreasing function $N : \mathbf{R}_+ \to \mathbf{R}_+$, a sequence $(t_n) \subset \mathbf{R}_+$ and a constant $K > 0$ with the following properties:*

(i) $N(0) = 0$ and $N(t) > 0$, for all $t > 0$;
(ii) $\sup_n |t_{n+1} - t_n| < \infty$;
(iii) for every $x \in X$ there exists $\alpha(x) > 0$ such that

$$\sum_{n=0}^{\infty} N(\alpha(x)\, ||\Phi(\theta, t_n)x||) \leq K, \qquad \theta \in \Theta.$$

Proof: Necessity. It results for $N(t) = t$ and $t_n = n$.
Sufficiency. Case 1. If (t_n) is bounded let $T = \sup_n t_n$ and $M \geq 1, \omega > 0$ given by Proposition 2.1. Let $x \in X$ and $\tilde{x} = [\alpha(x)/Me^{\omega T}]x$. Then

$$nN(||\Phi(\theta, T)\tilde{x}||) \leq \sum_{k=1}^{n} N(Me^{\omega T}||\Phi(\theta, t_n)\tilde{x}||) =$$

$$= \sum_{k=1}^{n} N(\alpha(x)||\Phi(\theta, t_n)x||) \leq K, \quad n \in \mathbf{N}, \theta \in \Theta.$$

It follows that $\Phi(\theta, T)\tilde{x} = 0$, for all $\theta \in \Theta$ and hence $\Phi(\theta, T)x = 0$, for all $(x, \theta) \in \mathcal{E}$. So π is uniformly exponentially stable.

Case 2. If $\sup_n t_n = \infty$ without lost of generality we may suppose that (t_n) is a non-decreasing sequence (if not we shall consider a subsequence with this property and the proof is analogous).

Let $r = \sup_n (t_{n+1} - t_n)$ and $n_0 \in \mathbf{N}^*$ with $K < n_0 N(1)$. Then

$$n_0 N(||\Phi(\theta, t_n)\tilde{x}||) \leq \sum_{j=n-n_0+1}^{n} N(\alpha(x)\,||\Phi(\theta, t_j)x||) \leq K,\, n \geq n_0, (x, \theta) \in \mathcal{E}$$

where $\tilde{x} = \alpha(x)/Me^{\omega n_0 r}$. From this inequality we obtain that

$$N(||\Phi(\theta, t_n)\tilde{x}|| < N(1)$$

and hence

$$||\Phi(\theta, t_n)\tilde{x}|| = \frac{\alpha(x)}{Me^{\omega n_0 r}}\,||\Phi(\theta, t_n)x|| < 1.$$

If we denote by $L(x) = Me^{\omega n_0 r}/\alpha(x)$ it results that:

$$||\Phi(\theta, t_n)x|| \leq L(x), \qquad n \geq n_0, (x, \theta) \in \mathcal{E}.$$

By uniform boundedness principle it follows that there exists $L_1 \geq 1$ such that

$$||\Phi(\theta, t_n)|| \leq L_1, \qquad n \geq n_0, \theta \in \Theta$$

and then we have

$$||\Phi(\theta, t_n)|| \leq L := \max\{L_1, Me^{\omega t_{n_0}}\}, \qquad n \in \mathbf{N}, \theta \in \Theta.$$

Without lost of generality, we may suppose that N is left continuous - if not we can consider the function $\tilde{N}(t) = \lim_{s \nearrow t} N(s)$ and the proof is unchanged.

Let $(O_N, |\cdot|_N)$ be the Orlicz sequence space associated to N and Y_N the Young function associated to N.

Let $x \in X \setminus \{0\}$ and $\beta(x) = \min\{\alpha(x), 1/KL||x||\}$. If $\tilde{x} = \beta(x)x$ and $\theta \in \Theta$, then the sequence

$$\varphi_{\tilde{x},\theta} : \mathbf{N} \to \mathbf{R}_+, \quad \varphi_{\tilde{x},\theta}(n) = ||\Phi(\theta, t_n)\tilde{x}||$$

verifies the inequality

$$Y_N(\varphi_{\tilde{x},\theta}(n)) = Y_N(\beta(x)||\Phi(\theta, t_n)x||) \leq$$

$$\leq \beta(x)||\Phi(\theta, t_n)x||\, N(\beta(x)||\Phi(\theta, t_n)x||) \leq \frac{1}{K} N(\alpha(x)||\Phi(\theta, t_n)x||),\, n \in \mathbf{N}$$

and hence $M_N(\varphi_{\tilde{x},\theta}) \leq 1$. It follows that $\varphi_{\tilde{x},\theta} \in O_N$ and $|\varphi_{\tilde{x},\theta}|_N \leq 1$. Because $\varphi_{\tilde{x},\theta} = \beta(x)\varphi_{x,\theta}$ and O_N is a linear space, we obtain that $\varphi_{x,\theta} \in O_N$ and

$$|\varphi_{x,\theta}|_N \leq K(x) := \max\{\frac{1}{\alpha(x)}, KL||x||\}, \quad (x, \theta) \in \mathcal{E}.$$

By Theorem 4.1. we obtain that π is uniformly exponentially stable. ∎

Theorem 3.3. *The linear skew-product semiflow* $\pi = (\Phi, \sigma)$ *on* $\mathcal{E} = X \times \Theta$ *is uniformly exponentially stable if and only if there is a Banach function space* $B \in \mathcal{B}(\mathbf{R}_+)$ *with the following properties:*

(i) for every $(x, \theta) \in \mathcal{E}$ *the function*

$$\Psi_{x,\theta} : \mathbf{R}_+ \to \mathbf{R}_+, \quad \Psi_{x,\theta}(t) = ||\Phi(\theta, t)x||$$

belongs to B;
(ii) there exists $K : X \to (0, \infty)$ *such that*

$$|\Psi_{x,\theta}|_B \leq K(x), \qquad (x, \theta) \in \mathcal{E}.$$

Proof: Necessity. It is a simple exercise for $B = L^1(\mathbf{R}_+, \mathbf{C})$.
Sufficiency. Let S_B be the Banach function space associated to B via Remark 2.5. Since $B \in \mathcal{B}(\mathbf{R}_+)$ there exists a strictly increasing sequence (t_n) of positive real numbers with $t_n \to \infty, \delta := \sup_n (t_{n+1} - t_n) < \infty$ and $\inf_n |\chi_{[t_n, t_{n+1})}|_B > 0$. For every $(x, \theta) \in \mathcal{E}$ the function

$$\varphi_{x,\theta} : \mathbf{N} \to \mathbf{R}_+, \quad \varphi_{x,\theta}(n) = ||\Phi(\theta, t_{n+1})x||$$

satisfies

$$\varphi_{x,\theta}(n) \leq ||\Phi(\sigma(\theta, t), t_{n+1} - t)||\, ||\Phi(\theta, t)x|| \leq$$

$$\leq Me^{\omega\delta}||\Phi(\theta, t)x|| = ||\Phi(\theta, t)\tilde{x}||, \quad n \in \mathbf{N}, (x, \theta) \in \mathcal{E}, t \in [t_n, t_{n+1}),$$

where $\tilde{x} = Me^{\omega\delta}x$ and M, ω are given by Proposition 2.1. It follows that

$$\sum_{n=0}^{\infty} \varphi_{x,\theta}(n)\chi_{[t_n, t_{n+1})} \leq \Psi_{\tilde{x},\theta}$$

and hence $\varphi_{x,\theta} \in S_B$ and

$$|\varphi_{x,\theta}|_{S_B} \leq |\Psi_{\tilde{x},\theta}|_B \leq K(\tilde{x}) = K(Me^{\omega\delta}x), \quad (x, \theta) \in \mathcal{E}.$$

Then by Theorem 3.1. we conclude that π is uniformly exponentially stable. ∎

Corollary 3.2. *The linear skew-product semiflow* $\pi = (\Phi, \sigma)$ *on* $\mathcal{E} = X \times \Theta$ *is uniformly exponentially stable if and only if there are* $p \in [1, \infty)$ *and* $K : X \to (0, \infty)$ *such that*

$$\int_0^\infty ||\Phi(\theta, t)x||^p dt \leq K(x), \qquad (x, \theta) \in \mathcal{E}.$$

Proof: Necessity. It is trivial.
Sufficiency. It results by Theorem 3.3. for $B = L^p(\mathbf{R}_+, \mathbf{C})$. ∎

Theorem 3.4. *The linear skew-product semiflow $\pi = (\Phi, \sigma)$ on $\mathcal{E} = X \times \Theta$ is uniformly exponentially stable if and only if there exist a nondecreasing function $N : \mathbf{R}_+ \to \mathbf{R}_+$ and a constant $K > 0$ with the following properties:*

(i) $N(0) = 0$ and $N(t) > 0$, for all $t > 0$;
(ii) for every $x \in X$ there exists $\alpha(x) > 0$ such that

$$\int_0^\infty N(\alpha(x)||\Phi(\theta, t)x||)dt \leq K, \qquad \theta \in \Theta.$$

Proof: Necessity. It results immediately for $N(t) = t$.
Sufficiency. Let M, ω given by Proposition 2.1. If $(x, \theta) \in \mathcal{E}$ and $\beta(x) = \alpha(x)/Me^\omega$ then:

$$\sum_{n=0}^\infty N(\beta(x)||\Phi(\theta, n+1)x||) \leq \sum_{n=0}^\infty \int_n^{n+1} N(\alpha(x)||\Phi(\theta, t)x||)dt \leq K.$$

Then by Theorem 3.2. it results that π is uniformly exponentially stable. ∎

Remark 3.1. Theorem 3.2., Corollary 3.2. and Theorem 3.3. are generalizations for the case of linear skew-product semiflows of well-known results due to Zabczyk ([17]), Datko ([7]) and Neerven ([14]) for C_0 -semigroups of linear operators. Theorem 3.4. is a variant of Rolewicz's theorem (see [15]) for linear skew-product semiflows.

References

[1] C. Chicone, Y. Latushkin - *Evolution Semigroups in Dynamical Systems and Differential Equations*, Math. Surveys and Monographs, vol. 70, Amer. Math. Soc., 1999

[2] S. N. Chow, H. Leiva - *Dynamical spectrum for time-dependent linear systems in Banach spaces*, Japan J. Indust. Appl. Math. 11 (1994), 379-415

[3] S. N. Chow, H. Leiva - *Existence and Roughness of the exponential dichotomy for linear skew-product semiflow in Banach space*, J. Differential Equations, 102 (1995), 429-477

[4] S. N. Chow, H. Leiva - *Two definitions of exponential dichotomy for skew-product semiflow in Banach spaces*, Proc. Amer. Math. Soc., vol. 124, no. 4 (1996), 1071-1081

[5] S. N. Chow, H. Leiva - *Dynamical spectrum for skew-product flow in Banach spaces*, Boundary Problems for Functional Differential Equations, World Sci. Publ., Singapore (1995), 85-105

[6] S. N. Chow, H. Leiva - *Unbounded Perturbation of the Exponential Dichotomy for Evolution Equations*, J. Differential Equations, vol. 129 (1996), 509-531

[7] S. N. Chow, Y. Yi - *Center manifolds and stability for skew-product flows*, J. Dynam. Differential Equations, 6 (1994)

[8] R. Datko - *Uniform asymptotic stability of evolutionary processes in Banach spaces*, SIAM J. Math. Anal. 3 (1972), 428-445

[9] J. L. Daleckii, M. Krein - *Stability of Solutions of Differential Equations in Banach space*, Trans. Math. Monographs, vol. 43, Amer. Math. Soc., Providence, R.I., 1974

[10] Y. Latushkin, A. M. Stepin - *Linear skew-product flows and semigroups of weighted composition operators*, Lecture Notes in Math., vol. 1486 (98–111), Springer-Verlag, New-York, 1991

[11] Y. Latushkin, S. Montgomery-Smith, T. Randolph - *Evolutionary semigroups and dichotomy of linear skew-product flows on locally compact spaces with Banach fibres*, J. Differential Equations, 125 (1996), 75-116

[12] P. Meyer-Nieberg - *Banach Lattices*, Springer Verlag, Berlin, Heidelberg, New York, 1991

[13] M. Megan, A.L. Sasu, B. Sasu - *On uniform exponential stability of periodic evolution operators in Banach spaces*, Acta Math. Univ. Comenianae, vol. 69 (2000), no. 1, 97-106

[14] J. M. A. M. van Neerven - *The Asymptotic Behaviour of Semigroups of Linear Operators*, Operator Theory Adv. Appl. 88, Birkhäuser, Bassel, 1996

[15] S. Rolewicz - *On uniform N - equistability*, J. Math. Anal. Appl. 115 (1986), 434-441

[16] R. J. Sacker, G. R. Sell - *Dichotomies for linear evolutionary equations in Banach spaces*, J. Differential Equations, 113 (1994), 17-67

[17] Z. Zabczyk - *Remarks on the control of discrete-time distributed parameter systems*, SIAM J. Control Optim. 12 (1974), 721-735

Department of Mathematics, University of the West, Timişoara,
Bul. V.Pârvan 4, 1900 - Timişoara, România
e-mail addresses: megan@hilbert.math.uvt.ro, sasu@hilbert.math.uvt.ro,
lbsasu@yahoo.com