A basic theory with predicates

G. Lenzi V. M. Tortorelli

Abstract

In this work we present a foundational theory with a certain degree of self-description, in which different kinds of primitive concepts allow a formalization close to common usage and open to different engraftings. We also show the relative consistency of this theory, with respect to a fragment of the Zermelo-Fraenkel set theory, by building some models.

Résumé

Dans ce travail, nous présentons une théorie auto-descriptive des fondements, où différentes sortes de concepts primitifs permettent une formalisation assez naturelle qui se prête à des extensions dans plusieurs directions. Nous en montrons enfin la consistance relative par rapport à un fragment de la théorie des ensembles de Zermelo-Fraenkel, en construisant divers types de modèles.

1 Introduction

1.1. The present work should be viewed as part of a broader activity of research. First of all we try to sketch this frame.

In the last twenty years of his life, Ennio De Giorgi was the leader of a foundational research programme at the "Scuola Normale Superiore" in Pisa. A complete bibliography can be found in [2]. This activity has been the result of a dialogue among scholars from different areas: a dialogue on the basic principles of mathematics, logic and computer science, and on their relations with the various branches of human knowledge.

The "foundational" meaning of this activity can be explained by quoting De Giorgi, [12]:

Received by the editors December 1996 – In revised form in November 2000.

Communicated by M. Boffa.

Key words and phrases : non-reductionism, open-endedness, self-description.

> *... non intendiamo rifondare la Matematica su basi più solide, ma piuttosto cercare qualche nuovo sentiero nella foresta della Matematica, della Logica e dell'Informatica senza rinunciare ad alcuna delle più geniali intuizioni degli studiosi che hanno tracciato le prime strade in questa foresta ...* [1]

We may consider the theories proposed in this research programme as foundational theories in different senses: both as a broad frame for a conceptual realm and as a way to set up a simple taxonomy of symbols, inspired by the usage of the language of mathematics.

The evolution of these researches is summed up in the title of a paper written by De Giorgi and published *post mortem* [5]: *"Overcoming set-theoretic reductionism in search of wider and deeper mutual understanding between Mathematicians and scholars of different scientific and human disciplines"*.
From the beginning, four paradigms have been outlined as informing the whole of these reflections:

i- *non-reductionism,* which proposes to take into account the idea that there are different kinds of objects, each with its peculiar intuitive meaning that we would like to preserve. A difference that is also *qualitative* and not only *quantitative.* For example, in mathematics one should consider not only numbers and sets, but also qualities, relations, tuples, operations, collections, functions, variables ....

ii- *open-endedness:* at a different level, by the same principle, a foundational theory should take into account the richness and the growth of sciences. Hence it should allow developments in several directions to frame or to code different topics in mathematics and in other disciplines.

iii- *self-description:* in particular a high degree of self-description is desirable since the theory itself is part of mathematics. The main properties, relations and operations on the objects of a foundational theory should be objects of the theory itself.

iv- *semi-formal axiomatization:* rigour and clarity are needed in view of a broader, rather than stronger, foundational framework, allowing a real exchange and a true dialogue among scholars from various branches of human knowledge. For this reason the *proposals* of these theories are written following the axiomatic method of the mathematical tradition: with *axioms expressed in the natural language* supported by a presentation with notation close to the common usage. This presentation should be suitable for *rigorous* formalizations in various formal languages.

The early activity in this area grew from a set-theoretic environment, [6]. Already at this stage the non-reductionist attitude is apparent. In fact, in this set theoretic frame there are *Urelemente* which are not mere *atoms* but correspond to various mathematical objects (numbers, tuples, operations ...). Moreover various principles of "free construction" are stated and studied, [18]. These principles allow a *wider set-formation* than the classical one, based on the idea that to have a set one must

[1]... we do not intend to give new foundations to mathematics on a more solid basis. We rather search for some new tracks in the forest of mathematics, logic and computer science, without waiving any of the most genial intuitions of the scholars that drew the early roads in this forest ...

already have all of its elements. With these principles it is possible to have sets as solutions of a large class of set-theoretic equations as, for example, $x = \{x\}$. Hence these principles extend properties that have been studied by several scholars (P. Aczel, M. Boffa, P. Finsler, P. Hájek, D. Scott, cfr. [1]) and that strongly violate the postulate of well-foundation in the Zermelian universe.

In [7], [15] the non-reductionism is achieved explicitly. Along these lines a wide and general theory for the foundations, with several kinds of objects, has been proposed in [3] to tackle the problem of self-reference. In [23] and [24] it is shown that such a theory, although relatively consistent with the classical foundational theories, is too restrictive. In fact, it is inconsistent with some natural and interesting extensions proposed in [3]. Hence such a general self-descriptive theory does not accomplish the request for open-endedness.

For this reason the evolution of these investigations led to a preference for a presentation in which only a restricted initial core is given, [8], [4]. Then, the various branches of mathematics, logic and computer sciences can be grafted onto this simple trunk: [25], [11], [20], [14], [21]. This choice is also useful in simplifying the task of finding (relative) consistency proofs, allowing a step by step "construction" of models.

In [8], [10], [4], [22] theories of this kind are called "Teorie Base" ("Basic Theories"). They adopt as primitive concepts the notions of *quality, relation, operation, natural number, collection,* and *system.* A Basic Theory has a first essentially finitary nucleus, and a second part useful in obtaining a good degree of self-description.

Subsequent simplifications of these theories opened a wider horizon. The theory called "Teoria '95", and presented in [16], marks this transition. The main difference between this theory and the previous ones is the preference given to predicates to reach self-description in the first part of the exposition. This allows one to introduce only a very weak arithmetic and to postpone stronger axioms for systems and tuples.

The last evolution is outlined in [13], [12], [9], [5]. This search of theories for the foundations of mathematics, logic and computer science, has produced a new starting point, open to the more general perspective of setting the basis of a dialogue among scholars from different fields, [17]. Hence there was a need to isolate a pre-mathematical part concerned with only a few *qualities* and *relations* of a general character. By rephrasing the introduction of [5], we can say that this frame may be suitable for a critical confrontation among the fundamental concepts of the various disciplines in Humanities and Sciences. In fact, all these disciplines are concerned with objects different in quality and with relations among these objects. All the other concepts may be grafted onto this first trunk, qualified by suitable fundamental qualities and ruled by suitable fundamental relations. Moreover, logical concepts, such as predicates, judgements and truth, have a particular importance in achieving a wide self-description without requiring any engagement with strong mathematical conceptions. In particular, in [13] a very rich structure for truth-judgments has been grafted onto the theory. This structure, dominated by the quality of being "absolutely true", fully achieves, in a specific, inner environment, the hierarchy used to escape the antinomies of self-description: a hierarchy that in classical foundational theories permeates the whole construction.

1.2. The work presented in this paper deals with a variant of these basic theories. The main difference with the presentations in [16], [12], [13], [9], [5], is that we propose axioms on predicates corresponding to the particular notion of predicate as a binary operation.
The relative consistency proofs for these kinds of theories may be done with respect to a rather weak arithmetic. On the other hand, the relative consistency proofs for strong extensions of these theories make extensive use of Set Theory ([23], [19], [20], [21]).
Since these theories are presented in a semiformal way, their formalization is the first step toward an evaluation of the strength of consistency of the theories, [22].

In this paper we give a quite natural formalization in the First Order Predicate Calculus with identity.

In the next section we present our formalization. The last paragraph is concerned with a strengthening of the theory with an internal predicate of partial truth. In section three we give a scheme to build models for the theory. These models are based on the standard concept of term-model, suitably modified to fit the purely relational presentation of the formalization. We have chosen Zermelo-Fraenkel set theory, with extensionality up to a countable set of *Urelemente*, as a frame to build these models.

ACKNOWLEDGEMENT

First of all we owe acknowledgement to Ennio De Giorgi. He died on October 25^{th}, 1996. His scientific and human heritage will always enlighten our research.

We want to remember Maurice Boffa. He died on May 29, 2001. We are deeply indebted to him for his consideration and his advice.

We due special thanks to Alessandro Berarducci for his remarks and comments, and to Marco Forti for his clarifying discussions and his patience in reviewing the paper. Moreover we are grateful to the participants to the 10th INTERNATIONAL CONGRESS OF LOGIC held in Florence in August 1995, where we presented a preliminary version of this paper.

2 The theory

Basic–like theories allow an exposition that needs only a first formal bulk to specify defined notions and terminology. After this initial task for obtaining a formal frame, it is possible to proceed with the exposition in a semiformal but fully rigorous way.

We choose a natural formalization with a finite number of symbols for constants. The symbols for constants are introduced section by section. The language has also a symbol $\|\models$ for a unary predicate, and three other symbols for predicates: $\mathbf{U}$ (binary), $\mathbf{B}$ (ternary), and $\mathbf{T}$ (quaternary). This first order language is called $\mathcal{L}$. For sake of convenience we also adopt the usual symbol for the identity $=$.
The formalization of the principal bulk of the theory needs finitely many axioms: namely the axioms from A.1 to F.2. We call this set of axioms *Basic Theory with Predicates*, briefly **BTP**.

2.1 Fundamental qualities and relations

The first symbols for constants introduced in this section are:
qqual: for the *quality of being a quality*;
qrelb: for the *quality of being a binary relation*;
qrelt: for the *quality of being a ternary relation*;
qops: for the *quality of being a simple operation*;
qopb: for the *quality of being a binary operation*;
id: for the operation of *identity*.

The symbol $\mathbf{U}$ is intended as the predicate describing the use of "unary objects". In particular we say that the object x *enjoys* the quality q when $\mathbf{U}(\mathbf{qqual}, q)$, and then write $q\,x$ for $\mathbf{U}(q, x)$. We also say that x is in the *extension* of q. Now we state with axiom A.1 that "the quality of being a quality is a quality". This is a self-referential axiom since **qqual** plays both the role of quality and the role of object enjoying the quality.

AXIOM A.1: $\mathbf{U}(\mathbf{qqual}, \mathbf{qqual})$.

AXIOM A.2: The objects **qrelb**, **qrelt**, **qops**, **qopb** are qualities.

The objects enjoying **qrelb**, **qrelt**, **qops**, **qopb** are called binary relations, ternary relations, simple operations, binary operations respectively.
The symbols $\mathbf{B}$ and $\mathbf{T}$ are intended as the predicates describing the use of "binary objects" and of "ternary objects", respectively. In particular if r is a binary relation we write $r\ xy$ for $\mathbf{B}(r, x, y)$. Similarly if s is a ternary relation we adopt the notation $s\,xyz$ for $\mathbf{T}(s, x, y, z)$. If f is a simple operation we write $f\ xy$ for $\mathbf{B}(f, x, y)$, similarly if g is a binary operation we write $g\ xyz$ for $\mathbf{T}(g, x, y, z)$. If $f\ xy$, and if $g\ abc$, we say that f is defined on x, and that g is defined on a and b (in the given order) respectively. We also say that y is a value of f, and that c is a value of g respectively.

AXIOM A.3: The operations are functional, namely: if **qops** f, $f\ xy$, $f\ xz$ then $y = z$; if **qopb** g, $g\ xyu$, $g\ xyv$ then $u = v$.

Thanks to this axiom we can adopt the usual notations for operations: if f is a simple operation we write also $f(x) = y$ or $fx = y$ for $f\,xy$, similarly if g is a binary operation we write also $g(x, y) = z$ or $gxy = z$ for $g\,xyz$. Moreover we can use $f(x)$, fx, $g(x, y)$ and $g\,xy$ as defined partial terms.

AXIOM A.4: The object **id** is a simple operation and for all objects x it holds $\mathbf{id}(x) = x$.

2.2.1 Natural Numbers

We introduce the symbols for arithmetic:
qnat: for the *quality of being a natural number*;
nadd: for the binary operation of *addition* of natural numbers;
nord: for the binary relation of *order between natural numbers*;
0: for the number *zero*;
1: for the number *one*.

AXIOM B.1: **qnat** is a quality, **nadd** a binary operation, **nord** a binary relation; further both **0** and **1** enjoy the quality **qnat**.

The objects enjoying **qnat** are called natural numbers.

AXIOM B.2: The operation **nadd** is defined exactly on all the natural numbers and all its values are natural numbers. The relation **nord** holds between natural numbers.

The usual notations are adopted. We use $x + y = z$ for $\mathbf{nadd}\,xy = z$, *i.e.* $\mathbf{T}(\mathbf{nadd}, x, y, z)$. We say that z is the sum of x and y. Similarly we use $x \leq y$ for $\mathbf{nord}\,xy$. We say that x is less than or equal to y. We write as usual $x < y$ for $x \leq y$ and $x \neq y$.

AXIOM B.3: One has $x + y = y + x$, $x + (y + z) = (x + y) + z$. If $\mathbf{qnat}\,x$, $\mathbf{qnat}\,y$ then $x \leq y$ or $y \leq x$. If $\mathbf{qnat}\,x$ then $x \leq x$. If $x \leq y \leq z$ then $x \leq z$. If $x \leq y$ and $y \leq x$ then $x = y$.

AXIOM B.4: One has $x + y = x$ if and only if $y = \mathbf{0}$. One has $\mathbf{0} < \mathbf{1}$. If $\mathbf{qnat}\,x$, and $x \neq \mathbf{0}$ then $\mathbf{1} \leq x$.

AXIOM B.5: One has $x \leq y$ if and only if $x + z = y$ for some natural number z. If $x < y$ then $x + w < y + w$.

These axioms give a very weak arithmetic. Namely the natural numbers are a discrete ordered monoid.
We also use the standard notations for numerals denoting “concrete” natural numbers. We freely speak of their family, denoted as usual by $\mathbb{N}$, that *is not* an object of the theory **BTP**.

2.2.2 Arity and fundamental relations

ar: for the operation giving *arity*;
qrel: for the quality of being a *relation*;
qop: for the quality of being an *operation*;
rfond: for the *operation generating fundamental relations.*

AXIOM B.6: The objects **qrel** and **qop** are qualities. Binary and ternary relations are relations. Simple and binary operations are operations.

AXIOM B.7: The object **ar** is a simple operation.

When $\mathbf{ar}(x) = y$ we say that the object x has arity equal to y. When the arity is a concrete positive number, 1, 2, 3 ..., we say as usual that the object is unary, binary, ternary, etc. The next axiom specifies the arities of the main kinds of objects introduced up to now. The domain and codomain of the operation **ar** are not completely specified by this axiom.

AXIOM B.8: The operation **ar** is defined on qualities, relations, operations and numbers. Each natural number has arity **0**. Each quality has arity **1**. Each binary relation and each simple operation has arity two. Each ternary relation and each binary operation has arity three.

AXIOM B.9: The object **rfond** is a simple operation defined on all the nonzero natural numbers, and $\mathbf{rfond}n$ is a relation with arity $n + 1$.

AXIOM B.10: If x is unary then $\mathbf{U}(x, y)$ if and only if $(\mathbf{rfond1})\,xy$. If x is binary then $\mathbf{B}(x, y, z)$ if and only if $(\mathbf{rfond2})\,xyz$. If x is ternary then $\mathbf{T}(x, y, z, w)$ if and only if $(\mathbf{rfond3})\,xyzw$.

Remark: In particular, according to their intended meaning, the predicates denoted by **U**, **B**, **T** correspond to the fundamental relations on unary, binary, ternary objects.

Remark: Only with the symbols **U**, **B** and **T**, we can not write axioms ruling objects with arity greater than three. For example we can not express the usual notations for ternary operations and quaternary relations, namely $g(x, y, z) = w$ and $t\,xyzw$ respectively. We complete our formalization in sections 2.4.1 and 2.4.2.

2.3 Collections

We introduce the symbols for the basic collections:
qcoll: for the *quality of being a collection*;
V: for the *the collection of all the objects*;
$\bar{\emptyset}$: for the *empty collection*;
coll: for the *collection of all the collections*;
ins: for the *collection of all the sets.*

AXIOM C.1: The object **qcoll** is a quality. If $\mathbf{qcoll}\,x$ then $\mathbf{ar}\,x = \mathbf{1}$.

If $\mathbf{qcoll}\,C$ we say that C is a collection. If C is a collection we write $x \in C$ for $\mathbf{U}(C, x)$, and we say that x is an element of C or x belongs to C.

We write $A \subseteq B$ when all the elements of the collection A are also elements of the collection B, and we say that A is included in B.

AXIOM C.2: The collections are extensional: if $A \subseteq B$ and $B \subseteq A$ then $A = B$.

AXIOM C.3: The objects **coll**, $\bar{\emptyset}$, **V** and **ins** are collections. In particular **coll** is the collection whose elements are exactly all collections, $\bar{\emptyset}$ is the collection without elements, **V** is the collection whose elements are all objects. Moreover $\bar{\emptyset} \in \mathbf{ins} \subseteq \mathbf{coll}$.

The elements of **ins** are called *sets*. In particular $\mathbf{V} \in \mathbf{coll} \in \mathbf{coll} \in \mathbf{V} \in \mathbf{V}$.

For sake of convenience we introduce the collection of natural numbers.

N is the symbol for the *collection of natural numbers*.

AXIOM C.4: **N** is a collection whose elements are exactly the natural numbers, *i.e.* $x \in \mathbf{N}$ if and only if $\mathbf{qnat}\, x$.

As already said one must distinguish between **N**, that is an object of the theory, and the family of *concrete natural numbers* $\mathbb{N}$.

2.4.1 Predicates

The main symbols of this section are:
bpred: for the *operation giving the collections of b-predicates*;
qpred: for the *quality of being a general predicate*;
qprop: for the *quality of being a general proposition*;
ide: for the *operation of diagonalization of predicates*;
et: for the *operation of conjunction of predicates*;
vel: for the *operation of disjunction of predicates*;
neg: for the *operation of negation of predicates*;
exists: for the *operation generating existential quantifications of predicates*;
univ: for the *operation generating universal quantifications of predicates.*

AXIOM D.1.1: The objects **qpred**, **qprop** are qualities, **et**, **vel**, **ide** are binary operations; **bpred**, **neg**, **exists**, **univ** are simple operations.

The objects enjoying **qpred** are called (general) predicates, the objects enjoying **qprop** are called (general) propositions.

AXIOM D.1.2: Every proposition is a predicate.

AXIOM D.2: The operation **bpred** is defined on all natural numbers. For each natural number n $\mathbf{bpred}\, n$ is a collection. If $p \in \mathbf{bpred}\,(n+1)$ then p is a predicate and also a binary operation with values in $\mathbf{bpred}\, n$. If $p \in \mathbf{bpred}\, \mathbf{0}$ then p is a proposition and also a binary operation with values in $\mathbf{bpred}\, \mathbf{0}$.

For each natural number n the elements of $\mathbf{bpred}\, n$ are called *b-predicates* of order n. If $n = \mathbf{0}$ they are called *b-propositions.*

According to axiom D.2 the b-predicates are *binary operations* with values that are b-predicates. We use the prefix “b-” to remind this fact. We choose this notion of predicate to obtain, in a finitary way, the principal manipulations on predicates without using operations like **K** (*à la* Curry), and like the operation of transposition **T** (as done, for example, in [16]). The first input of a b-predicate must be a natural number specifying in what "entry" of the predicate the second input is evaluated. At each new computation the "entries" of the predicate are "renewed". Hence the order of a b-predicate has the intuitive meaning of "number of free variables". Thus we state the following axiom:

AXIOM D.3: The b-predicate p is defined on h and x if and only if h is a positive natural number. If n is the order of p and $h > n$ then $p\,hx = p$. If $\mathbf{0} < h \leq k$ then for any two objects x and y it holds $(p\,hx)\,ky = (p\,(k+1)y)\,hx$.

If $0 < m_1 \leq m_2 \ldots \leq m_k$, we denote by $p[x_1/m_1, x_2/(m_2+1)\ldots, x_k/(m_k+k-1)]$, the predicate: $(\ldots(p\,m_1x_1)\,m_2x_2\ldots)\,m_kx_k$. If $m_1 = m_2 \ldots = m_k = 1$ we simply use the notation $p[x_1 \ldots x_k]$.

AXIOM D.4: The operations **et**, **vel**, **neg**, are defined on predicates and their values are predicates. The operations **exists**, **univ**, are defined on nonzero natural numbers, and their values are simple operations defined on b-predicates that have as values b-predicates.

We use the standard notations $p \wedge q$, $p \vee q$, $\neg p$ for $\mathbf{et}\,pq$, $\mathbf{vel}\,pq$, $\mathbf{neg}\,p$, respectively. Moreover we denote $\mathbf{exists}\,k$ and $\mathbf{forall}\,k$ by $\exists_k$, and by $\forall_k$ respectively. When p is of order one we simply write $\forall p$ and $\exists p$ instead of $\forall_1 p$, and $\exists_1 p$.

Axioms D.5, D.7, D.8 reflect the following intuitive idea: if the formula $\varphi(x_1 \ldots x_n)$ corresponds to the b-predicate p, and the formula $\psi(x_1 \ldots x_m)$ corresponds to the b-predicate q, then: the formula $\varphi(x_1 \ldots x_n) \wedge \psi(x_{n+1} \ldots x_{n+m})$ corresponds to $p \wedge q$, the formula $\exists x_k \varphi(x_1 \ldots x_n)$ corresponds to $\exists_k p$, etc.

AXIOM D.5: If p and q are propositions then $p\wedge q$, $p\vee q$, and $\neg p$ are propositions. For every $n \in \mathbf{N}$, $m \in \mathbf{N}$ one has: if $p \in \mathbf{bpred}\,n$ and $q \in \mathbf{bpred}\,m$ then $p \wedge q \in \mathbf{bpred}(n+m)$, $p \vee q \in \mathbf{bpred}(n+m)$, $\neg p \in \mathbf{bpred}\,n$. If $1 \leq k \leq n+1$ and $p \in \mathbf{bpred}(n+1)$ then $\exists_k p \in \mathbf{bpred}\,n$ and $\forall_k p \in \mathbf{bpred}\,n$. If $k > n$ and $p \in \mathbf{bpred}\,n$ then $\exists_k p = p$ and $\forall_k p = p$.

Let us now give the axiom on the binary operation **ide**. It allows the diagonalizations of a predicate by identifying an "entry" of the predicate with a previous one.

AXIOM D.6: The binary operation **ide** is defined on nonzero natural numbers and its values are operations on b-predicates with values that are b-predicates. For any $h \geq 1$, $k \geq 1$ one has $\mathbf{ide}\,hk = \mathbf{ide}\,kh$. For any b-predicate p if $h \geq 1$ then $(\mathbf{ide}hh)p = p$. For any b-predicate p of order n, if $m > n$ then for any $h \geq 1$ one has $(\mathbf{ide}\,mh)p = p$. For any b-predicate p of order $n+1$, if $1 \leq h < m \leq n+1$ then $(\mathbf{ide}\,hm)p \in \mathbf{bpred}\,n$.

We denote $\mathbf{ide}\,hk$ by i_{hk}.

The following two axioms state some commutation rules among the introduced operations.

AXIOM D.7: Let p be a b-predicate of order n, and let q be any b-predicate:

1:
$$i_{lm}(i_{hk}\,p) = \begin{cases} i_{hk}(i_{(l+1)(m+1)}\,p) & \text{for } 1 \le h < k \le l < m \le n \\ i_{hk}(i_{l(m+1)}\,p) & \text{for } 1 \le h \le l < k \le m \le n \\ i_{h(k-1)}(i_{lm}\,p) & \text{for } 1 \le h \le l < m < k \le n \end{cases}$$

2:
$$i_{hk}(\forall_m\,p) = \begin{cases} \forall_{m-1}(i_{hk}\,p) & \text{for } h < k < m \\ \forall_m(i_{h(k+1)}\,p) & \text{for } h < m \le k \\ \forall_m(i_{(h+1)(k+1)}\,p) & \text{for } 1 \le m \le h < k \end{cases}$$

$$i_{hk}(\exists_m\,p) = \begin{cases} \exists_{m-1}(i_{hk}\,p) & \text{for } h < k < m \\ \exists_m(i_{h(k+1)}\,p) & \text{for } h < m \le k \\ \exists_m(i_{(h+1)(k+1)}\,p) & \text{for } 1 \le m \le h < k \end{cases}$$

3:
$$i_{hk}(p \wedge q) = \begin{cases} (i_{hk}p) \wedge q & \text{for } 1 \le h \le k \le n \\ p \wedge (i_{(h-n)(k-n)}\,q) & \text{for } n < h \le k \end{cases}$$

$$i_{hk}(p \vee q) = \begin{cases} (i_{hk}p) \vee q & \text{for } 1 \le h \le k \le n \\ p \vee (i_{(h-n)(k-n)}\,q) & \text{for } n < h \le k \end{cases}$$

4:
$$i_{hk}(\neg p) = \neg(i_{hk}\,p).$$

We do not assume associativity or commutativity for $\vee$ and $\wedge$. Let us now give the axioms on the action of "composite" predicates.

AXIOM D.8: Let p be a b-predicate of order n, and let q be any b-predicate:

1:
$$(i_{hk}\,p)\,mx = \begin{cases} i_{(h-1)(k-1)}(p\ mx) & \text{for } 1 \le m < h \\ i_{h(k-1)}(p\ mx) & \text{for } 1 \le h < m < k \\ i_{hk}(p\ (m{+}1)\,x) & \text{for } 1 \le k \le m \ne h \end{cases}$$

$$(i_{hk}\,p)\,hx = ((p\ hx)\ (k{-}1)\,x) = p[x/h, x/k] \quad \text{for } 1 \le h < k.$$

2:
$$(\forall_h p)\,kx = \begin{cases} \forall_h(p\ (k{+}1)\,x) & \text{for } 1 \le h \le k \\ \forall_{h-1}(p\,kx) & \text{for } k < h \end{cases}$$

$$(\exists_h p)\,kx = \begin{cases} \exists_h(p\ (k{+}1)\,x) & \text{for } 1 \le h \le k \\ \exists_{h-1}(p\,kx) & \text{for } k < h \end{cases}$$

3:
$$(p \wedge q)\,hx = \begin{cases} (p\,hx) \wedge q & \text{for } h \le n \\ p \wedge (q\ (h{-}n)\,x) & \text{for } h > n \end{cases}$$

$$(p \vee q)\,hx = \begin{cases} (p\,hx) \vee q & \text{for } h \le n \\ p \vee (q\ (h{-}n)\,x) & \text{for } h > n \end{cases}$$

4:
$$(\neg p)\,kx = \neg(p\,kx).$$

2.4.2 Basic predicates

Let us now introduce *basic* predicates related to the "action" of the objects of the theory itself. The fundamental object introduced in this section is:

gbp: for the *operation generating basic predicates.*

AXIOM D.9: The object **gbp** is a simple operation associating to any object a predicate. If $\mathbf{ar}\, x = n > \mathbf{0}$ then $\mathbf{gbp}x$ is a b-predicate of order n.

We adopt the notation "$xy_1 \dots y_k$" for the predicate $(\mathbf{gbp}\, x)[y_1 \dots y_k]$. We use simply the notation "x" for $\mathbf{gbp}\, x$. If x is a collection we use the notation "$y \in x$" for "$x\, y$". We also use the notation "$x = y$" for $(\mathbf{gbp}\,\mathbf{id})[x, y]$. If $0 < m_1 \leq m_2 \ldots \leq m_k$, the predicates "$x$"$[y_1/m_1, y_2/(m_2 + 1) \ldots, y_k/(m_k + k - 1)]$, are called *basic*.

Since $\mathbf{rfond}\, h$ is the relation of arity $h + 1$ describing the objects of arity h, we give the following axiom:

AXIOM D.10: If $\mathbf{ar}\, x = h > 0$ then "$(\mathbf{rfond}\, h)\, x$" = "x".

Now we can find a b-proposition corresponding exactly to each sentence of the language $\mathcal{L}\backslash\{|\!\models\}$. This can be done by replacing **U**, **B**, **T** with "**rfond1**", "**rfond2**", "**rfond3**" respectively.

2.5 The notion of truth

Now we consider the symbol $|\!\models$ as a unary predicate. The intended meaning of $|\!\models x$ is: x is a proposition and x is true. We have made this choice simply to get a finite and meaningful formalization of the theory.

AXIOM E.1: If $|\!\models x$ then x is a proposition.

AXIOM E.2: If p is a b-proposition then $|\!\models p$ if and only if not $|\!\models \neg p$.
If p and q are b-propositions then:
$|\!\models p \wedge q$ if and only if $|\!\models p$ and $|\!\models q$, and $|\!\models p \vee q$ if and only if $|\!\models p$ or $|\!\models q$.
If p is a b-predicate of order one then:
$|\!\models \forall p$ if and only if for every x it holds $|\!\models px$, and $|\!\models \exists p$ if and only if for some x it holds $|\!\models px$.

AXIOM E.3: If $\mathbf{ar}x = 1$ then $\mathbf{U}(x, y)$ if and only if $|\!\models$"$x\, y$". If $\mathbf{ar}x = 2$ then $\mathbf{B}(x, y, z)$ if and only if $|\!\models$"$x\, yz$". If $\mathbf{ar}x = 3$ then $\mathbf{T}(x, y, z, w)$ if and only if $|\!\models$"$x\, yzw$".

We can now generalize the notation $x\, y_1 \dots y_k$ for $|\!\models$"$x\, y_1 \dots y_k$", also for values of $\mathbf{ar}x = k$ greater than three. Thanks to Axiom D.10, if $\mathbf{ar}x = k$ then $x\, y_1 \dots y_k$ if and only if $(\mathbf{rfond}k)\, xy_1 \dots y_k$. When $x\, y_1 \dots y_k$ holds we say that the object x acts on the objects y_1 ... y_k.

Remark: in this way we can express in our formalization, with a finite number of axioms and symbols, the action of objects with arity greater than three. We remark that, for each k, the notation $|\!\models$"$x\, y_1 \dots y_k$" corresponds to a Σ_1 formula of the first order language $\mathcal{L}$ in the free variables $x, y_1, \dots, y_k$.

Remark: at this stage the axioms E.1, E.2, E.3 have no counterparts in terms of the internal predicates of the theory.
In fact the truth predicate $|\!\models$ can *not* have an internal counterpart. More precisely we can adapt the traditional *"liar"* argument to obtain the following theorem, cfr.[16]:

Theorem 1: *Let p be a b-predicate of order one. There is a b-proposition q such that $|\!\models q$ if and only if $|\!\not\models p[q]$. In particular there is no b-predicate p such that: $|\!\models p[x]$ if and only if $|\!\models x$, for any object x.*

Proof: Put $\bar\theta = (\exists_1 i_{24} i_{15}((\neg p) \wedge \text{“}\mathbf{rfond}3\text{”}))\,[\mathbf{1}/2]$, and $q = \bar\theta[\bar\theta] = \exists i_{12}\left((\neg p) \wedge \text{“}\bar\theta\ \mathbf{1}\bar\theta\text{”}\right)$. Then

$$
\begin{aligned}
&|\!\models q \text{ if and only if for some } x,\ |\!\models \left(i_{12}\left((\neg p) \wedge \text{“}\bar\theta\,\mathbf{1}\,\bar\theta\text{”}\right)\right)[x]\\
&\text{if and only if for some } x,\ |\!\models \left((\neg p) \wedge \text{“}\bar\theta\,\mathbf{1}\,\bar\theta\text{”}\right)[x/1, x/2]\\
&\text{if and only if for some } x,\ |\!\models \neg p[x] \text{ and } |\!\models \text{“}\bar\theta\ \ \mathbf{1}\,\bar\theta\,x\text{”}\\
&\text{if and only if for some } x,\ |\!\models \neg p[x] \text{ and } \mathbf{T}(\bar\theta, \mathbf{1}, \bar\theta, x)\\
&\text{if and only if for some } x, \text{ not } |\!\models p[x], \text{ and } \bar\theta[\bar\theta] = x\\
&\text{if and only if not } |\!\models p[\bar\theta[\bar\theta]]\\
&\text{if and only if}\quad \text{not } |\!\models p[q].
\end{aligned}
$$
■

Different formalizations of the theory can be completely described in terms of the predicates of the theory itself: *e.g.* taking an infinite language having a sequence of symbols for predicates $\mathbf{P}_n$ (each of them having arity $n+1$ and corresponding to $\mathbf{rfond}n$). This choice leads to an infinite axiomatization.

2.6 Verifiability and falsifiability

In this section we partially reflect into the theory the notions of being a true proposition and of being a false proposition. The constant symbols of this section are:

qfals: for the *quality of being falsifiable*;

qver: for the *quality of being verifiable*.

AXIOM F.1: The objects **qver**, **qfals** are qualities of b-propositions.

AXIOM F.2: If $\mathbf{qver}\,x$ then $|\!\models x$. If $\mathbf{qfals}\,x$ then $|\!\models \neg x$.

We call *judgement* any proposition of the form “$\mathbf{qver}\,x$” or “$\mathbf{qfals}\,x$”. A proposition x is called *judgeable* if either $\mathbf{qver}\,x$ holds or $\mathbf{qfals}\,x$ holds.

Remark: The following natural properties of **qver** and **qfals** hold thanks to axioms E.2 and F.2:

- $\mathbf{qver}\,x$ and $\mathbf{qfals}\,x$ can not both hold;
- if p and $\neg p$ are both judgeable, then $\mathbf{qver}\,\neg p$ if and only if $\mathbf{qfals}\,p$;
- if p, q and $p \wedge q$ are all judgeable, then $\mathbf{qver}\ p \wedge q$ if and only if $\mathbf{qver}\,p$ and $\mathbf{qver}\,q$;
- if p, q and $p \vee q$ are all judgeable, then $\mathbf{qver}\ p \vee q$ if and only if $\mathbf{qver}\,p$ or $\mathbf{qver}\,q$;

- if $\forall_1 p$ is judgeable and for every x the proposition $p[x]$ is judgeable, then $\mathbf{qver}\,\forall_1 p$ if and only if for all x $\mathbf{qver}\,p[x]$;
- if $\exists_1 p$ is judgeable and for every x the proposition $p[x]$ is judgeable, then $\mathbf{qver}\,\exists_1 p$ if and only if for some x $\mathbf{qver}\,p[x]$.

The following axiom schemata provide judgeable propositions. Let n be a concrete natural number:

AXIOM F.3.n: $\mathbf{qver}$“$x\,y_1 \ldots y_n$” if and only if $\|\!\models$“$x\,y_1 \ldots y_n$”.

AXIOM F.4.n: $\mathbf{qfals}$“$x\,y_1 \ldots y_n$” if and only if $\|\!\models\neg$“$x\,y_1 \ldots y_n$” .

These axioms can be synthetized by saying that all basic propositions are judgeable.

As already noticed, the axioms from A.1 to F.4 do not guarantee any “closure” property of judgeable propositions. The main closure properties can be stated as follows.

AXIOM F.5: If p is judgeable then $\neg p$ is judgeable as well. If p, q are judgeable then both $p \wedge q$ and $p \vee q$ are judgeable.

AXIOM F.6: Let p be a predicate of order one. If for every x it holds $\mathbf{qver}\,p[x]$ or for some x it holds $\mathbf{qfals}\,p[x]$, then $\forall p$ is judgeable. If for every x it holds $\mathbf{qfals}\,p[x]$ or for some x it holds $\mathbf{qver}\,p[x]$, then $\exists p$ is judgeable.

Taken together, these axioms would allow a good reflection of truth into the theory. But (un)fortunately they are inconsistent.

Theorem 2: *The axioms from A.1 to F.6 are inconsistent.*

Proof : We prove more: the axioms from A.1 to F.5 are inconsistent with a weaker form of F.6, namely:

(*) If for every x the proposition $p[x]$ is judgeable then both $\forall\ p$ and $\exists\ p$ are judgeable.

The argument is the following. By theorem 1 there is some proposition y such that $\|\!\models$“$\mathbf{qver}\,y$” if and only if $\|\!\not\models y$. Hence either $\|\!\models$“$\mathbf{qver}\ y$” (*i.e.* $\mathbf{qver}\ y$) holds and $\|\!\not\models y$, or $\mathbf{qver}\ y$ does not hold and $\|\!\models y$. The first possibility is excluded by F.2, and so is the second one, when y is judgeable. In fact in this case by E.2 and F.2 we have $\mathbf{qver}\ y$. Following the proof of theorem 1, with $p =$ “$\mathbf{qver}$”, we can build y as the b-proposition $q = \exists i_{12}\Big((\neg\text{“}\mathbf{qver}\text{”}) \wedge \text{“}\bar\theta\ \mathbf{1}\bar\theta\text{”}\Big)$, which is judgeable by axioms (*), F.3, F.4, F.5.

In fact by (*), D.7, D.8 it is enough to prove that for every x the proposition $(\neg$“$\mathbf{qver}\ x$”$) \wedge$ “$\bar\theta[\bar\theta] = x$” is judgeable. By E.6 it is enough to prove that for every x both $\neg$“$\mathbf{qver}\ x$” and “$\bar\theta[\bar\theta] = x$” are judgeable. Thanks to F.3 and F.4. it is enough to prove:

$$\forall x\Big(\Big((\|\!\models\neg\text{“}\mathbf{qver}\ x\text{”}) \vee (\|\!\not\models\neg\text{“}\mathbf{qver}\ x\text{”})\Big) \wedge \Big((\|\!\models\text{“}\bar\theta[\bar\theta] = x\text{”}) \vee (\|\!\not\models\text{“}\bar\theta[\bar\theta] = x\text{”})\Big)\Big).$$

This is a theorem of first order logic. ∎

However, both the group of axioms from A.1 to F.5 and the group of axioms from A.1 to F.3 together with F.5, F.6, are relatively consistent with respect to Zermelo-Fraenkel set theory. This will be shown in the next sections.

3 Some models

In this section we prove the relative consistency with respect to **ZFU** (Zermelo-Fraenkel set theory with extensionality up to a countable set of *Urelemente*) of two extensions of **BTP**, namely: **BTP** and axioms F.3, F.4, F.5, **BTP** and axioms F.3, F.5, F.6.

The relative consistency proofs are done by giving a scheme to build suitable first order structures, over the first order language $\mathcal{L}$ introduced in section 2. In this way we stress quite neatly a division between the main ideas underlying these constructions and the rather long and tedious coding needed to complete the construction of the models. We propose structures of three different kinds.

The structures $\mathcal{U}$ of the first type are first order structures on the language $\mathcal{L}$ and verify **BTP**, F.5, F.6. The scheme to build them is divided in two parts. We begin with the construction of models of the axioms from A.1 to D.10. Such a structure is a sort of term model. Then we encode formulae by elements of the term model and we define the interpretation of $||\!\!=$ in $\mathcal{U}$. We conjecture that similar constructions can be done in suitable fragments of Recursive Arithmetic.

The models of the second kind, denoted by $\mathcal{U}^\infty$, verify **BTP**, F.3, F.4, F.5. Since they satisfy the axiom schemata F.3, F.4, all basic propositions are judgeable. These models are given by an iterative and increasing process based on a structure of the first kind.

The models of the third kind, denoted by $\mathcal{U}^\tau$, satisfy **BTP**, F.3, F.5, F.6. The construction of this kind of models recalls the one presented in [26]. To get the desired properties we use a countable transfinite iteration based on a structure of the first kind.

• We consider first order structures based on a countable set of *Urelemente* U including $\mathbb{N}$, and such that $U \setminus \mathbb{N}$ is infinite.

3.1 The general scheme

In this section we prove the consistency of **BTP**, F.5, F.6 with respect to **ZFU**. Thanks to the notations introduced in section 2, we assume that all the axioms are formulated in the formal first order language $\mathcal{L}$ (which has 33 symbols of constants, and the four symbols of predicates $\mathbf{U}$, $\mathbf{B}$, $\mathbf{T}$ and $||\!\!=$).

3.1.1 Extension functions and basic structures

Definition 1: Fixed two distinct elements, ar and $rfond$, of U, we consider:

- $Ar : U \to \mathbb{N}$ such that: $Ar(ar) = 2$, $Ar(rfond) = 2$ and, for $n \in \mathbb{N}$, $Ar(n) = 0$, $Ar^{-1}(\{n\})$ is infinite ;
- $Rfond : \mathbb{N} \setminus \{0\} \to U \setminus \{ar, rfond\}$ an injective function such that: $Ar(Rfond(n)) = n + 1$ for $n \in \mathbb{N}$;
- $Ext : U \setminus Ar^{-1}(\{0\}) \to V_\omega(U)$, being $V_\omega(U)$ the level ω of the Von Neumann hierarchy over U, such that:

$Ext(x) \subseteq U^{Ar(x)}$ for $x \in U \setminus Ar^{-1}(\{0\})$, and $Ext(ar) = Ar$;
$Ext(rfond) = Rfond$ and $Ext(Rfond(h)) = \bigcup_{Ar(x)=h} (\{x\} \times Ext(x))$.

We call $(Ar, ar, Rfond, rfond)$ a *fundamental structure* and $((Ar, ar, Rfond, rfond), Ext)$ a *coherent basis*, and we also say that Ext is an *extension function* coherent with the given fundamental structure.

Definition 2: We call a first order structure $\mathcal{U} = (U, \mathcal{I})$, on $\mathcal{L}$, a *basic structure* if for some coherent basis $((Ar, ar, Rfond, rfond), Ext)$ one has:

A) the interpretation $\mathcal{I}$ is injective, $\mathcal{I}(\mathbf{ar}) = ar$, $\mathcal{I}(\mathbf{0}) = 0$, $\mathcal{I}(\mathbf{1}) = 1$, $\mathcal{I}(\mathbf{rfond}) = rfond$;
the interpretation of each symbol of constant is an element of $Ar^{-1}(n)$, where n is the arity assigned to this symbol in the corresponding axiom of **BTP**;

B) $\mathcal{I}(\mathbf{U}) = Ext(Rfond(1))$, $\mathcal{I}(\mathbf{B}) = Ext(Rfond(2))$, $\mathcal{I}(\mathbf{T}) = Ext(Rfond(3))$.

Remark: Fixed an interpretation of constants satisfying the condition A, a coherent extension function Ext determines uniquely, up to the interpretation of $\|\models$, a basic structure. We denote such a basic structure by $\mathcal{U}_{Ext}$.

Notation: If $\mathcal{I}$ is an interpretation satisfying condition A we denote the $\mathcal{I}$-interpretation of a symbol for a constant by its name in *italic* characters (instead of **boldface**). We adopt this convention also for defined terms. This is coherent with the notation adopted in the above definitions.

Consider a fundamental structure $(Ar, ar, Rfond, rfond)$ and an interpretation $\mathcal{I}$ of the constants of $\mathcal{L}$ satisfying condition A. Our purpose is to extend these data, $\mathcal{I}$, $(Ar, ar, Rfond, rfond)$, to a basic structure that satisfies the desired axioms.

3.1.2 First conditions on the extension function

We introduce distinguished sets $Qual$, $Coll$, Rel_h and Op_h $(h \in \mathbb{N})$, $Pred$, $Prop$, such that:

- qualities are collected in the set $Qual \subset Ar^{-1}(1)$ that contains the $\mathcal{I}$-interpretations of all the symbols of constants for qualities: $qqual$, $qrelb$, $qrelt$, $qops$, $qopb$, $qcoll$, $qnat$, $qrel$, qop, $qpred$, $qprop$, $qfals$, $qver$;

- collections are collected in the infinite set $Coll \subset (Ar^{-1}(1) \setminus Qual)$, containing V, $\mathcal{I}(\bar{\emptyset})$, $coll$, ins, N;

- relations of arity equal to h are collected in the set $Rel_h \subset Ar^{-1}(h)$ such that $nord \in Rel_2$, $Rfond(h) \in Rel_{h+1}$;

- operations of arity equal to $h+1$ are collected in the set $Op_h \subseteq (Ar^{-1}(h+1) \setminus Rel_{h+1})$. The set Op_1 is infinite. Moreover: $id \in Op_1$, $nadd \in Op_2$, $ar \in Op_1$, $bpred \in Op_1$, $ide \in Op_2$, $et \in Op_2$, $vel \in Op_2$, $neg \in Op_1$, $exists \in Op_1$, $univ \in Op_1$, $rfond \in Op_1$, $gbp \in Op_1$;

- predicates and propositions are collected in the sets $Pred$ and $Prop$ such that: $Prop \subseteq Pred \subseteq Op_2$, $Pred \setminus Prop$ and $Prop$ are countable.

We put $Rel = \bigcup_h Rel_h$, $Op = \bigcup_h Op_h$.

The elements of the above sets are the "qualified" elements of the model. Some among the remaining elements of the universe U can be "activated" to get models of extensions of the theory.

Remark: The conditions on the powers of $Coll$, Op_1, $Pred \setminus Prop$ and $Prop$ will be used in section 3.1.3.

Now we give a first group of conditions that the extension function Ext must satisfy in order to allow the construction of our models. These conditions will be kept fixed throughout this section. For sake of simplicity the extensions of several objects are only partially specified. In particular, the extensions of $qver$, $qfals$ are *completely unspecified.*

1) $Ext(ar) = Ar$, $Ext(rfond) = Rfond$, $Ext(x) \subseteq U^{Ar(x)}$ for any $x \in U \setminus Ar^{-1}(\{0\})$.
2) $Ext(qqual) = Qual$, $Ext(qrelb) = Rel_2$, $Ext(qrelt) = Rel_3$, $Ext(qops) = Op_1$, $Ext(qopb) = Op_2$,
   $Ext(qnat) = \mathbb{N}$, $Ext(qcoll) = Coll$, $Ext(qrel) = Rel$, $Ext(qop) = Op$.
   $Ext(V) = U$, $Ext(\mathcal{I}(\bar{\emptyset})) = \emptyset$, $Ext(coll) = Coll$, $Ext(N) = \mathbb{N}$.
   $\{\mathcal{I}(\bar{\emptyset}), N\} \subseteq Ext(ins) \subseteq Coll$.
   $Ext(nord) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : m \le n\}$.
   $Ext(id) = \{(x, x) : x \in U\}$, $Ext(nadd) = \{(m, n, s) \in \mathbb{N}^3 : m + n = s\}$.
   $Ext(qprop) = Prop$, $Ext(qpred) = Pred$.
3) For the extensions of operations different from id, $nadd$, ar, $rfond$, we only assume that they are univalent graphs so as to satisfy axiom A.3.
   For the extensions of collections different from $V, \mathcal{I}(\bar{\emptyset})$, $coll$, ins, N we only assume that they are extensional so as to satisfy axiom C.2.
4) $Ext(Rfond(h)) = \bigcup_{Ar(x)=h} \{x\} \times Ext(x)$.

Remark: The condition 4) gives inductively the extension of each $Rfond(h)$ for $h \ge 1$. It is well posed *no matter how* we choose the extensions of the objects different from $Rfond(h)$. In fact, the extension of $Rfond(h)$, which has arity $h+1$, depends only on the extensions $Ext(x)$ for $Ar(x) = h$.

It needs only a straightforward check to prove the following proposition:

Proposition: *If Ext verifies 1), 2), 3), 4) then:* $\mathcal{U}_{Ext} \models A.1 - A.4,\ B.1 - B.10,\ C.1 - C.4,\ D.1$.

3.1.3 Coding functions and predicative extensions

Let us now specify the extension function so as to get models satisfying also the axioms from D.2 to D.10 and E.1, E.2, E.3. We begin by defining a first order language $\mathcal{L}$ associated to (Ar, U).

Language of the arity. The first order language relative to (Ar, U), has a symbol for constant $\mathbf{c}_x$ for each $x \in U$, a symbol for predicate $\mathbf{p}_x$ with arity equal to $Ar(x)$

for each $x \in U$ such that $Ar(x) \neq 0$, and it does not have the symbol = for equality. Moreover we distinguish, *à la* Takeuti, a sequence of symbols for *free variables*, say $\{\xi_n\}$, and a sequence of symbols for *bounded variables*, say $\{\eta_n\}$.

We denote the set of the well formed formulae of $^*\mathcal{L}$ by $\mathcal{F}^*\mathcal{L}$.

Normal Form. A $^*\mathcal{L}$-formula φ is in normal form if:

(a) ξ_n is the free variable whose first occurrence in φ is the n^{th} first occurrence of a free variable in φ;

(b) the n^{th} quantifier appearing in φ bounds η_n;

(c) the formula φ has no subformula of the kind $\mathbf{p}_{Rfond(h)}(\mathbf{c}_x, t_1, \dots t_h)$, with $Ar(x) = h > 0$.

Reduction steps. Given any $^*\mathcal{L}$-formula ψ, we define:

- $S_a(\psi)$ as the $^*\mathcal{L}$-formula obtained by replacing the m free variables of ψ with $\xi_1 \dots \xi_m$ in such a way to satisfy point (a) above;

- $S_b(\psi)$ as the $^*\mathcal{L}$-formula obtained by replacing in the formula ψ each free occurrence of a variable in the scope of the h^{th} quantifier with η_h in such a way to satisfy point (b) above;

- $S_c(\psi)$ as the $^*\mathcal{L}$-formula obtained by replacing in ψ every subformula of the kind $\mathbf{p}_{Rfond(h)}(\mathbf{c}_x, t_1, \dots t_h)$, with $Ar(x) = h > 0$, by the formula $\mathbf{p}_x(t_1, \dots t_h)$.

Remark: Starting from any formula ψ and applying S_a, S_b and a finite number of steps S_c one gets a formula in normal form. The normal form of a formula is uniquely determined. We call $\mathcal{N}^*\mathcal{L}$ the set of all the formulae in normal form.

Coding–function. We call *coding–function* a bijection G between $\mathcal{N}^*\mathcal{L}$ and *Pred*, mapping the formulae without free variables onto *Prop*. If φ has normal form ψ we denote by $\overline{\varphi}$ the predicate $G(\psi)$.

Since both *Coll* and Op_1 are infinite we can choose the following distinguished elements of U in such a way that:

- $bpred_n \in Coll \setminus \{V,\ \bar{\emptyset},\ coll,\ ins,\ N\}$, $n \in \mathbb{N}$, $bpred_n \neq bpred_m$ if $n \neq m$, $Coll \setminus \{bpred_n :\ n \in \mathbb{N}\}$ is infinite;

- $univ_n$, $exists_n \in Op_1 \setminus \{id,\ ar,\ bpred,\ neg,\ exists,\ univ,\ rfond,\ gbp\}$, $n \geq 1$, $univ_n \neq univ_m$ and $exists_n \neq exists_m$ if $n \neq m$.
Moreover $univ_n \neq exists_m$ for all m, n;

- ide_{mn}, $n, m \geq 1$, are elements of Op_1 different from each $univ_n$ and $exists_n$ for $n \geq 1$, and from $id,\ ar,\ bpred,\ neg,\ exists,\ univ,\ rfond,\ gbp$.
Moreover $ide_{mn} = ide_{nm}$, ide_{mn} are all different for $m \leq n$, and $Op_1 \setminus \{ide_{mn} : m, n \in \mathbb{N}\} \cup \{univ_n :\ n \in \mathbb{N}\} \cup \{exists_n :\ n \in \mathbb{N}\}$ is infinite.

Now we give a second group of conditions that the extension function *Ext* must satisfy in order to allow the construction of models for the axioms from A.1 to D.10. Also these conditions will be kept fixed throughout the section.

5) $Ext(qpred) = Pred = G\,(\mathcal{N}^*\mathcal{L})$, $Ext(qprop) = Prop = \{G(\varphi) :\ \varphi$ has no free variables$\}$;

- $Ext(bpred) = \{(n, bpred_n) : n \in \mathbb{N}\}$, $Ext(bpred_n) = \{G(\varphi) : \varphi$ has exactly n free variables$\}$;
- $Ext(neg) = \{(G(\varphi), G(\neg\varphi)) : \varphi \in \mathcal{NL}\}$;
- $Ext(et) = \{(G(\varphi), G(\psi), G(\varphi \wedge \psi')) : \varphi, \psi \in \mathcal{NL}\}$, $Ext(vel) = \{(G(\varphi), G(\psi), G(\varphi \vee \psi')) : \varphi, \psi \in \mathcal{NL}\}$,
where ψ' is obtained from ψ by replacing each occurrence of ξ_i with ξ_{n+i} and each occurrence of η_j with η_{m+j}, where n and m are respectively the number of free variables and the number of bounded variables in φ;

- $Ext(univ) = \{(n, univ_n) : n \in \mathbb{N}\}$, $Ext(exists) = \{(n, exists_n) : n \in \mathbb{N}\}$.
Let $\forall_n \varphi$ and $\exists_n \varphi$ be the normal forms of the formulae "obtained by quantifying" the n^{th} free variable of φ with $\forall$ and with $\exists$ respectively:
$Ext(univ_n)$ is the graph of the correspondence that associates to $G(\varphi)$ the predicate $G(\varphi)$ itself if φ has less than n free variables, and the predicate $G(\forall_n \varphi)$ if φ has at least n free variables,
$Ext(exists_n)$ is the graph of the correspondence that associates to $G(\varphi)$ the predicate $G(\varphi)$ itself if φ has less than n free variables, and the predicate $G(\exists_n \varphi)$ if φ has at least n free variables.

- $Ext(ide) = \{(n, m, ide_{nm}) : m, n \in \mathbb{N}\}$, $Ext(ide_{mn}) = \{(G(\varphi), G(\varphi'')) : \varphi \in \mathcal{NL}\}$, where φ'' is φ if either $m = n$ or $n > m \geq 1$ and φ has less than n free variables, else, for $n > m \geq 1$, φ'' is the normal form of the formula obtained from φ by replacing each occurrence of ξ_n with ξ_m;

6) For each $\varphi \in \mathcal{NL}$ let $Ext(G(\varphi))$ be the graph of the correspondence that associates to (n, x), $n \in \mathbb{N}$ and $x \in U$, the predicate $G(\varphi''')$, where φ''' is φ itself if this formula has less than n free variables, else the normal form of the formula obtained from φ by replacing each occurrence of ξ_n with $\mathbf{c}_x$;

7) $Ext(gbp)$ is the graph of the correspondence that associates to $x \in U$ the predicate $G(\mathbf{p}_x(\xi_1, \ldots, \xi_n))$ if $Ar(x) = n > 0$.

Predicative extension. Any extension function Ext fullfilling conditions 1, 2, 3, 4, 5, 6 and 7 is called a *predicative extension.*

Proposition: *If Ext is a predicative extension then:* $\mathcal{U}_{Ext} \models A.1 - A.4,\ B.1 - B.10,\ C.1 - C.4,\ D.1 - D.10.$

Sketch of the proof:
By condition 5 the axioms D.2, D.4, D.5, D.6, D.7 are satisfied.
By condition 6 the axioms D.3, D.8 are satisfied.
By conditions 6 and 7 the axioms D.9, D.10 are satisfied. Notice that condition (c) in the definition of normal form has been chosen so as to satisfy axiom D.10. ■

Notation:
- We write $\neg\overline{\varphi}$ for $\overline{\neg\varphi}$. When φ and ψ are sentences we write: $\overline{\varphi} \wedge \overline{\psi}$, $\overline{\varphi} \vee \overline{\psi}$ for $\overline{\varphi \wedge \psi}$, $\overline{\varphi \vee \psi}$, respectively. We write $\forall_n \overline{\varphi}$, $\exists_n \overline{\varphi}$ for $\overline{\forall_n \varphi}$, $\overline{\exists_n \varphi}$, respectively.

- If $Ar(x) = n > 0$ we denote $\overline{\mathbf{p}_x(\xi_1, \ldots \xi_n)}$ also by "x".
Similarly "$x\ x_1 \ldots x_k$" stands for $\overline{\mathbf{p}_x(\mathbf{c}_{x_1}, \ldots, \mathbf{c}_{x_k}, \xi_{k+1}, \ldots \xi_n)}$.

Notice that: "$Rfond(n+1)x$" $= \overline{\mathbf{p}_{Rfond(n+1)}(\mathbf{c}_x, \xi_2, \ldots, \xi_{n+1})} = \overline{\mathbf{p}_x(\xi_1, \ldots, \xi_n)} =$ "x".

3.1.4 Truth structures

With the previous constructions the coding is over. In this section we give the interpretation of $\|\models$.

Predicative structures. Given a predicative extension function Ext we call *predicative structure* the first order ${}^*\!\mathcal{L}$-structure ${}^*\mathcal{U}_{Ext} = (U, J)$ such that $J(\mathbf{c}_x) = x,\ J(\mathbf{p}_x) = Ext(x)$.

Definition 3: A basic structure $\mathcal{U}$ relative to a predicative extension Ext is called *truth structure* if it satisfies conditions A and B in definition 2 and:

C) $$\mathcal{I}(\|\models) = \{\bar{\varphi} : \bar{\varphi} \in Prop\ ,\ \ {}^*\mathcal{U}_{Ext} \models \varphi\}\,.$$

Proposition: *If* $\mathcal{U}$ *is a truth structure, then:* $\mathcal{U} \models \textsf{BTP} \setminus \{F.1,\ F.2\}$.

Proof : By definition 3 the following key identity holds for the elements $x \in U$ such that $Ar(x) = n \neq 0$:

$$\{(x_1, \ldots x_n) \in U^n : \mathcal{U} \models \|\models\text{“}xx_1 \ldots x_n\text{”}\} = \\ \{(x_1, \ldots x_n) \in U^n :\ {}^*\mathcal{U}_{Ext} \models \mathbf{p}_x(x_1 \ldots x_n)\} = Ext(x)$$

The proof of axioms E.1, E.2, E.3 needs only a “word-by-word translation” according to the above identity. Axiom E.1 is trivial. We only check that $\mathcal{U}$ satisfies the first clause of axiom E.2. The remaining proofs are similar and we omit them.

Thanks to the above identity we have to prove that for each $x \in Prop$, $x = \overline{\varphi} \in \mathcal{I}(\|\models)$ if and only if $\overline{\neg\varphi} \notin \mathcal{I}(\|\models)$. This follows since ${}^*\mathcal{U}_{Ext} \models \varphi$ if and only if ${}^*\mathcal{U}_{Ext} \not\models \neg\varphi$.■

In order to get a trivial model of the axioms F.1, F.2, F.5, F.6, one may simply assume that there are neither verifiable nor falsifiable propositions, and put: $Ext(qver) = Ext(qfals) = \emptyset$.

Proposition: *If* $\mathcal{U}$ *is a truth structure and* $Ext(qver) = Ext(qfals) = \emptyset$ *, then:* $\mathcal{U} \models \textsf{BTP} \cup \{F.5, F.6\}$.

We say that $x \in U$ is *judgeable* in a basic structure if $x \in Ext(qver) \cup Ext(qfals)$. We say that a ${}^*\!\mathcal{L}$-sentence φ is *judgeable* in a predicative structure ${}^*\mathcal{U}_{Ext}$ if $\overline{\varphi}$ is judgeable in $\mathcal{U}_{Ext}$, *i.e.* ${}^*\mathcal{U}_{Ext} \models \mathbf{p}_{qver}(\mathbf{c}_{\overline{\varphi}}) \vee \mathbf{p}_{qfals}(\mathbf{c}_{\overline{\varphi}})$.

3.2 Getting judgements

In this section we modify the model defined in the previous sections by suitably enlarging the extensions of $qver$ and $qfals$. This cannot be done in the straightforward and naïve way. In fact let us consider $\mathcal{U}$ and ${}^*\mathcal{U}$, the truth structure and the predicative structure relative to the same predicative extension function, without judgeable propositions. If one adds to the extension of $qver$ the propositions that are true in ${}^*\mathcal{U}$, and to the extension of $qfals$ those that are false in ${}^*\mathcal{U}$, one gets a new predicative extension function, hence a new truth structure $\tilde{\mathcal{U}}$ satisfying all the axioms from A.1 to E.3. This structure $\tilde{\mathcal{U}}$ verifies also F.1, F.5, F.6 (and the natural conditions on **qver** and **qfals** listed after F.2) in a non-trivial way. But then $\tilde{\mathcal{U}}$ does not satisfy F.2. In fact ${}^*\mathcal{U} \models \mathbf{p}_{qqual}(\mathbf{c}_{qqual})$, hence $\tilde{\mathcal{U}} \models \mathbf{qver}$“$qqual\ qqual$”, *i.e.* ${}^*\tilde{\mathcal{U}} \models \mathbf{p}_{qver}(\mathbf{c}_{\text{“}qqual\ qqual\text{”}})$, and ${}^*\tilde{\mathcal{U}} \not\models \neg\mathbf{p}_{qver}(\mathbf{c}_{\text{“}qqual\ qqual\text{”}})$. The latter is in turn equivalent to $\tilde{\mathcal{U}} \not\models \|\!\models(\neg\text{“}qver\ \text{“}qqual\,qqual\text{” ”})$. On the other hand $\mathcal{U} \models \forall x \neg \mathbf{qver} x$, in particular $\mathcal{U} \models \neg\mathbf{qver}\text{“}qqual\ qqual\text{”}$, *i.e.* ${}^*\mathcal{U} \models \neg\mathbf{p}_{qver}(\mathbf{c}_{\text{“}qqual\ qqual\text{”}})$. Hence $\tilde{\mathcal{U}} \models \mathbf{qver}(\neg\text{“}qver\ \text{“}qqual\,qqual\text{” ”})$, and $\tilde{\mathcal{U}} \not\models \|\!\models(\neg\text{“}qver\ \text{“}qqual\,qqual\text{” ”})$, therefore F.2 does not hold in $\tilde{\mathcal{U}}$.

The basic idea in this section is that of *level of judgement*. First we add to the extension of *qver* only those basic propositions that are not judgements and are true in ${}^*\mathcal{U}$ and the sentences that are logically true. Similarly we add to the extension of $qfals$ the basic propositions that are not judgements and are false in ${}^*\mathcal{U}$, the judgements on objects that are not propositions, and the sentences that are logically refutable. Moreover, to get also F.5, we add all the boolean combinations of these particular propositions. Then we add just the boolean combinations of judgements involving them, and so on. Thus we build a sequence of predicative extensions based on this hierarchy.
At each step the extensions of both *qver* and $qfals$ increase. Moreover at each step the only changing extensions are those of *qver*, $qfals$, and $Rfond(h)$, $h \geq 1$. Consequently the extension of each $Rfond(h)$ changes by increasing, thanks to the inductive definition of extension, given with condition 4 in section 3.1.2. In fact the extension of $Rfond(1)$ changes depending on the new extensions of *qver* and $qfals$. Similarly for each $h > 0$ the extension of $Rfond(h+1)$ changes according to the new extension of $Rfond(h)$.
Since the scheme for extensions given with condition 4 in section 3.1.2 is continuous for increasing union, we consider the limit extension function simply by taking the pointwise increasing union. This limit truth structure satisfies **BTP** and F.5 in a non-trivial way, and also the axiom scheme F.3. In fact all the basic propositions which are true in the model are verifiable.

This limit structure does not satisfy the axiom scheme F.4. In fact there are a lot of basic predicates that are not judgeable. They are exactly the judgements on predicates excluded from the hierarchy of the level of judgement. To get rid of this we need to modify suitably the coding-function so as to get all the judgements to be “well founded”: *e.g.* we have to avoid self-references of the kind “*qver x*” $= x$. This allows to extend inductively the previous hierarchy to all the judgements.

We start with a basic structure. We define inductively the partition $\mathcal{S} = \{Q\} \cup \{S_n\}_{n\in\mathbb{N}}$ of $\mathcal{N}{}^*\!\mathcal{L}$, as follows. Fix a countable partition $\mathcal{P} = \{\bar{Q}\} \cup \{\bar{S}_n\}_{n\in\mathbb{N}}$ of *Prop* such that each component of the partition is countable. Put $\bar{S}_\infty = \bigcup_{n\in\mathbb{N}} \bar{S}_n$.

Definition 1: Let $\varphi \in \mathcal{N}{}^*\!\mathcal{L}$.
- $\varphi \in Q$ if and only if neither $\models_{{}^*\mathcal{L}} \varphi$ nor $\models_{{}^*\mathcal{L}} \neg\varphi$, *i.e.* φ is neither logically valid nor logically refutable, and either φ begins with a quantifier or one of its boolean components begins with a quantifier and is in turn neither logically valid nor logically refutable;
- $\varphi \in S_0$ if and only if φ is a sentence, or a boolean combination of sentences, of the following kinds:
either $\models_{{}^*\mathcal{L}} \varphi$ or $\models_{{}^*\mathcal{L}} \neg\varphi$, *i.e.* φ is logically valid or logically refutable,
or φ is $\mathbf{p}_x(\mathbf{c}_{x_1}\ldots\mathbf{c}_{x_m})$, with $x \neq qver,\ qfals$,
or φ is $\mathbf{p}_{qver}(\mathbf{c}_x)$, $\mathbf{p}_{qfals}(\mathbf{c}_x)$ with $x \notin Prop$,

or φ is $\mathbf{p}_{qver}(\mathbf{c}_x)$, $\mathbf{p}_{qfals}(\mathbf{c}_x)$ with $x \in \bar{Q}$;
- $\varphi \in S_{n+1}$ if and only if $\varphi \notin S_0 \cup \cdots \cup S_n$ and φ is either $\mathbf{p}_{qver}(\mathbf{c}_x)$, $\mathbf{p}_{qfals}(\mathbf{c}_x)$, for some $x \in \bar{S}_n$, or a boolean combination of this kind of sentences and of sentences belonging to $S_0 \cup \cdots \cup S_n$.

We say that a $^*\!\mathcal{L}$-sentence φ has *level of judgement* equal to n if its normal form is in S_n. We define the function γ giving the level of judgement by: $\gamma(\varphi) = n$ if the normal form of φ is in S_n. Put $S_\infty = \bigcup_{n\in\mathbb{N}} S_n$.

Since $\mathcal{P}$ is a partition of *Prop*, $\mathcal{S}$ is a partition of $\mathcal{N}^*\!\mathcal{L}$. In fact, if $\varphi \in \mathcal{N}^*\!\mathcal{L}$ is of the kind $\mathbf{p}_x(x_1, \ldots, x_n)$, $Ar(x) = n > 0$, then φ belongs to some S_h. If $\varphi \in \mathcal{N}^*\!\mathcal{L}$ begins with a quantifier, then φ belongs to $S_0 \cup Q$. Since every $\varphi \in \mathcal{N}^*\!\mathcal{L}$ is a boolean combination of atoms with normal form of the previous kinds, φ belongs to $S_h \cup Q$ for some h.

In this section we consider only coding functions inducing bijections between Q and $\bar{Q}$, and between S_n and $\bar{S}_n$, for all $n \in \mathbb{N}$.
For each $x = \overline{\varphi} \in \bar{S}_\infty$ we put $\gamma(x) = \gamma(\varphi)$.

A model for the axioms from A.1 to F.5.

We start from a predicative extension function Ext such that $Ext(qver) = Ext(qfals) = \emptyset$. Notice that:

- For each $n \in \mathbb{N}$ the normal forms of boolean combinations of elements of $S_0 \cup S_n$ are in turn elements of $S_0 \cup S_n$. Similarly for $S_0 \cup Q$. Moreover judgements transform S_n and Q into S_{n+1} and S_0, respectively, *i.e.* if $\varphi \in S_n$ then both $\mathbf{p}_{qver}(\mathbf{c}_{\bar{\varphi}})$ and $\mathbf{p}_{qfals}(\mathbf{c}_{\bar{\varphi}})$ belong to S_{n+1}, and similarly for $\varphi \in Q$.

It is useful to fix the notation for extension functions obtained by changing the values of a previous one only on $qver$ and $qfals$:

Given an extension function Ext on U, if A, B are disjoint subsets of *Prop*, we define $Ext(A, B)$ as follows:

$$Ext(A,B)(x) = Ext(x)\ ,\quad \text{if } x \neq qver\ ,\quad qfals\ ,\quad Rfond(h)\quad (h \geq 1);$$
$$Ext(A,B)(qver) = A\ ,\quad Ext(A,B)(qfals) = B;$$
$$Ext(A,B)(Rfond(h)) = \bigcup_{Ar(x)=h} \Big(\{x\} \times Ext(A,B)(x)\Big).$$

- If $A \subseteq A'$, and $B \subseteq B'$, then $Ext(A, B)(x) \subseteq Ext(A', B')(x)$ for all x. In particular equality holds unless x is $qver$, $qfals$, or $Rfond(h)$.
- If Ext is a predicative extension function then also $Ext(A, B)$ is a predicative extension function based on the same coding-function.

We denote by $^*\!\mathcal{U}(A, B)$ and by $\mathcal{U}(A, B)$, the predicative structure, and the truth structure, respectively, relative to the predicative extension function $Ext(A, B)$.

Definition 2: $Ext^0 = Ext,\ A^0 = Ext(qver) = \emptyset,\ B^0 = Ext(qfals) = \emptyset$,
$^*\mathcal{U}^n = {}^*\mathcal{U}_{Ext^n},\ \mathcal{U}^n = \mathcal{U}_{Ext^n};$
$A^{n+1} = \left\{\bar{\varphi}:\ \gamma(\varphi) \leq n \text{ and } ^*\mathcal{U}^n \models \varphi\right\}$,
$B^{n+1} = \left\{\bar{\varphi}:\ \gamma(\varphi) \leq n \text{ and } ^*\mathcal{U}^n \models \neg\varphi\right\}$,
$Ext^{n+1} = Ext^n\left(A^{n+1}, B^{n+1}\right)$;
$Ext^\infty = Ext\Big(\bigcup_{n\in\mathbb{N}} A^n, \bigcup_{n\in\mathbb{N}} B^n\Big)$,
$^*\mathcal{U}^\infty = {}^*\mathcal{U}_{Ext^\infty}$, $\mathcal{U}^\infty = \mathcal{U}_{Ext^\infty}$.

We notice that the propositions judgeable in $\mathcal{U}^{n+1}$ are exactly the propositions which have level of judgement at most n. Hence the propositions judgeable in $\mathcal{U}^\infty$ are exactly those not belonging to $\bar{Q}$. In particular *all the basic propositions* are judgeable in $\mathcal{U}^\infty$.

Theorem 1: *$\mathcal{U}^n \models$ **BTP** $\cup \{F.5\}$ for all $n \in \mathbb{N}$. $\mathcal{U}^\infty \models$ **BTP** $\cup \{F.3, F.4, F.5\}$.*

Proof : All $\mathcal{U}^n$, $n \in \mathbb{N}$, as well as $\mathcal{U}^\infty$, are truth structures. Hence all of them verify the axioms from A.1 to E.3. They verify F.1 since A^n and B^n are disjoint subsets of $Prop$.
All $\mathcal{U}^n$, $n \in \mathbb{N}$, verify F.5, since $\bar{S}_0 \cup \bar{S}_n$ is closed under boolean combinations. On the other hand a finite family of propositions judgeable in $\mathcal{U}^\infty$ is contained in $A^m \cup B^m$, for a suitable $m \in \mathbb{N}$. Hence $\mathcal{U}^\infty \models\ F.5$ since $Ext^\infty(qver)$ and $Ext^\infty(qfals)$ are the unions of all the A^n and of all the B^n, respectively.
Moreover $\mathcal{U}^n$, $n \in \mathbb{N}$, and $\mathcal{U}^\infty$ verify the natural properties concerning **qver** and **qfals** listed after F.2.
In order to prove that $\mathcal{U}^n \models\ F.2$, and $\mathcal{U}^\infty \models\ F.2,\ F.3,\ F.4$, we make the following claims, to be proved later on.

Claim 1: If φ is judgeable in $^*\mathcal{U}^{n+1}$ then: $^*\mathcal{U}^{n+1} \models \varphi$ if and only if $^*\mathcal{U}^n \models \varphi$;
Claim 2: If φ is judgeable in $^*\mathcal{U}^{n+1}$ then: $^*\mathcal{U}^\infty \models \varphi$ if and only if $^*\mathcal{U}^n \models \varphi$.

Assuming the claims, we prove that F.2 is true in $\mathcal{U}^n$ and in $\mathcal{U}^\infty$. For $n = 0$ this is trivial, since the premises of the axiom are false in $\mathcal{U}^0$. Assume φ judgeable in $^*\mathcal{U}^{n+1}$. Then:

$\mathcal{U}^{n+1} \models$ **qver** $\overline{\varphi}$ if and only if $^*\mathcal{U}^n \models \varphi$ if and only if (by claim 1) $^*\mathcal{U}^{n+1} \models \varphi$ if and only if $\mathcal{U}^{n+1} \models\ \mid\models\overline{\varphi}$.

Similarly for **qfals**. Therefore $\mathcal{U}^{n+1} \models\ F.2$.
The propositions judgeable in $\mathcal{U}^\infty$ are judgeable in some $\mathcal{U}^{n+1}$. Hence:

$\mathcal{U}^\infty \models$ **qver** $\overline{\varphi}$ if and only if $^*\mathcal{U}^n \models \varphi$ if and only if (by claim 2) $^*\mathcal{U}^\infty \models \varphi$ if and only if $\mathcal{U}^\infty \models\ \mid\models\overline{\varphi}$.

Similarly for **qfals**. Therefore $\mathcal{U}^\infty \models\ F.2$.

It remains to prove that the schemes F.3, F.4 are true in $\mathcal{U}^\infty$. In order to prove F.3 take x with $Ar(x) = m > 0$ and different from $Rfond(m-1)$, and assume $\mathcal{U}^\infty \models (\mid\models$"$x\ x_1 \ldots x_m$"$)$. We have to show that $\mathcal{U}^\infty \models$ **qver** "$x\ x_1 \ldots x_m$". In fact "$x\ x_1 \ldots x_m$" is a basic proposition, therefore it is judgeable in $\mathcal{U}^\infty$. But $\mathcal{U}^\infty \models F.2$, hence if $\mathcal{U}^\infty \models (\mid\models y)$, and y is judgeable then $\mathcal{U}^\infty \models$ **qver** y.

The proof of the axiom scheme F.4 is quite similar. In fact "$x\ x_1 \dots x_m$" is judgeable in $\mathcal{U}^\infty$. Hence also $\neg$"$x\ x_1 \dots x_m$" is judgeable in $\mathcal{U}^\infty$, *i.e.* it belongs to $Ext^\infty(qver) \cup Ext^\infty(qfals)$.
If $\mathcal{U}^\infty \models \neg(\Vdash$"$x\ x_1 \dots x_m$"$)$, then also $\mathcal{U}^\infty \models (\Vdash\neg$"$x\ x_1 \dots x_m$"$)$, and then the thesis follows since $\mathcal{U}^\infty$ verifies F.3 and F.2. ■

Proof of claim 1: Assume first that φ has level of judgement equal to 0. We distinguish five cases.
If φ is either logically true or logically false, then the thesis is straightforward.
If φ is a sentence of the kind $\mathbf{p}_x(\mathbf{c}_{x_1}, \dots \mathbf{c}_{x_n})$, with x different from $qver$, $qfals$ and $Rfond(n-1)$, then the thesis follows since the extension of x does not change when passing from ${}^*\mathcal{U}^n$ to ${}^*\mathcal{U}^{n+1}$.
If φ is of the kind $\mathbf{p}_{qver}(\mathbf{c}_x)$, $\mathbf{p}_{qfals}(\mathbf{c}_x)$, with $x \notin Prop$, then the thesis follows since the extensions of both $qver$ and $qfals$ are included in $Prop$.
If φ is of the kind $\mathbf{p}_{qver}(\mathbf{c}_x)$, $\mathbf{p}_{qfals}(\mathbf{c}_x)$, with $x \in \bar{Q}$ then the thesis follows because φ is false in every ${}^*\mathcal{U}^n$. In fact $\bar{Q}$ is disjoint from each $A^n \cup B^n$.
Finally, if φ is a boolean combination of sentences of the previous kinds, the statement follows by induction on the boolean complexity.

In order to prove the claim in general we proceed by induction on $n \in \mathbb{N}$.
- If $n = 0$ then by definition φ has level of judgement equal to 0. The thesis has been proved just above.
- In order to prove the inductive step we proceed by induction on the boolean complexity of the decomposition of φ. Clearly, it suffices to prove the case of an atomic formula or of a formula beginning with a quantifier. Now by hypothesis we consider only judgeable formulae, so that the only case to prove is when φ has the form $\mathbf{p}_{qver}(\mathbf{c}_{\bar{\theta}})$ or $\mathbf{p}_{qfals}(\mathbf{c}_{\bar{\theta}})$, $\bar{\theta} \notin \bar{Q}$. In fact every judgeable sentence beginning with a quantifier has level 0 as well as all the atomic sentences which are not judgements of propositions and all the judgements on elements of $\bar{Q}$.

Hence assume that $\varphi \in S_1 \cup \dots \cup S_{n+1}$ has the form $\mathbf{p}_{qver}(\mathbf{c}_{\bar{\theta}})$ $(\bar{\theta} \notin \bar{Q})$. Then by definition θ has normal form in $S_0 \cup \dots \cup S_n$: *i.e.* θ is judgeable in ${}^*\mathcal{U}^{n+1}$. Therefore one has ${}^*\mathcal{U}^{n+1} \models \varphi$ if and only if $\bar{\theta} \in Ext^{n+1}(qver) = A^{n+1}$, by definition of predicative structure. This is equivalent to ${}^*\mathcal{U}^n \models \theta$, by definition of A^{n+1}. Then by inductive hypothesis one gets ${}^*\mathcal{U}^n \models \theta$ if and only if ${}^*\mathcal{U}^{n+1} \models \theta$. Since $\gamma(\theta) \leq n < n+1$ this happens if and only if $\bar{\theta} \in Ext^{n+2}(qver)$. This is equivalent to ${}^*\mathcal{U}^{n+2} \models \varphi$.
If $\varphi = \mathbf{p}_{qfals}(\mathbf{c}_{\bar{\theta}})$, one argues similarly.

Proof of claim 2: Arguing as in the above proof, we limit ourselves to prove the thesis for judgements φ on propositions $\bar{\theta}$ which are judgeable in ${}^*\mathcal{U}^{n+1}$. The proof is a straightforward consequence of the fact that $Ext^\infty(qver)$, and $Ext^\infty(qfals)$, are the increasing unions of A^n, and of B^n, respectively, taking into account that $A^n = A^{n+1} \cap \{p : \gamma(p) \leq n-1\}$, $B^n = B^{n+1} \cap \{p : \gamma(p) \leq n-1\}$. ■

3.3 A model of axiom F.6

In this section we propose a model of **BTP** together with axioms F.3, F.5, F.6 (clearly, one cannot satisfy also F.4, by theorem 2 of section 2.6). The clauses in the following definitions are chosen so as to get an increasing sequence of extension functions.

Definition 1: Let E be a predicative extension function on U. We define $\mathcal{E}(E)$ as the collection of the extension functions W, such that:

$W(qver) \supseteq E(qver), \quad W(qfals) \supseteq E(qfals);$

$W(qver) \cap W(qfals) = \emptyset, \quad W(qver) \cup W(qfals) \subseteq Prop;$

$W = E(W(qver), W(qfals)).$

- Since E is a predicative extension function defined by a coding-function G, then each $W \in \mathcal{E}(E)$ is a predicative extension based on the same G.

In the following we start from a predicative extension Ext such that $Ext(qver) = Ext(qfals) = \emptyset$.

Definition 2: $E^0 = Ext$

$E^{\alpha+1}(qver) = \{\bar{\varphi} : {}^*\mathcal{U}_W \models \varphi, \text{ for all } W \in \mathcal{E}(E^\alpha)\},$

$E^{\alpha+1}(qfals) = \{p : \neg p \in E^{\alpha+1}(qver)\};$

$E^{\alpha+1} = E^\alpha\,(E^{\alpha+1}(qver), E^{\alpha+1}(qfals))$, for all ordinals α;

$E^\lambda(x) = \bigcup_{\gamma<\lambda} E^\gamma(x)$, for $x \in U \setminus Ar^{-1}(\{0\})$, for each limit ordinal λ.

Lemma 1. For each ordinal β one has:
1.1: $E^\beta(qver) \cap E^\beta(qfals) = \emptyset$, $E^\beta(qver) \cup E^\beta(qfals) \subseteq Prop$. In particular the definition is well posed.

1.2: $E^\beta = E^0\left(E^\beta(qver), E^\beta(qfals)\right)$.

1.3: $E^\beta(qver) \subseteq E^{\beta+1}(qver)$, and $E^\beta(qfals) \subseteq E^{\beta+1}(qfals)$. Hence the functions $\alpha \mapsto E^\alpha(qver)$, $\alpha \mapsto E^\alpha(qfals)$ are monotone non decreasing functions with respect to set inclusion.

1.4: For all $\alpha \leq \beta$ $E^\beta \in \mathcal{E}(E^\alpha)$. In particular $E^\beta = E^\alpha(E^\beta(qver), E^\beta(qfals))$.

Proof : **1.1** is straightforward by ordinal induction.
1.2. By ordinal induction. For $\beta = 0$ it is the definition of $E(A, B)$.
Assume $E^\beta = E^0\left(E^\beta(qver), E^\beta(qfals)\right)$.
Since $E^{\beta+1} = E^\beta\left(E^{\beta+1}(qver), E^{\beta+1}(qfals)\right)$ by definition, and $(E(A, B))(C, D)$ $= E(C, D)$ in general for any extension function E, the desired equality follows.
If β is a limit ordinal the equality $E^\beta(x) = E^0(E^\beta(qver), E^\beta(qfals))(x)$ is straightforward for $x = qver, qfals$ by definition of local modification.
If $x \neq qver, qfals$ then $E^\beta(x) = \bigcup_{\gamma<\beta} E^\gamma(x) = \bigcup_{\gamma<\beta} E^0\,(E^\gamma(qver), E^\gamma(qfals))(x)$, by the inductive hypothesis.
When $x \neq qver, qfals, Rfond(h), h \in \mathbb{N}$, each term of the union $\bigcup_{\gamma<\beta} E^0\,(E^\gamma(qver), E^\gamma(qfals))(x)$ is equal to $E^0(x)$, as observed after the definition of local modification in section 3.2. In particular $E^\beta(x) = E^0(E^\beta(qver), E^\beta(qfals))(x)$ $= E^0(x)$.

If x is $Rfond(h)$, $h \in \mathbb{N}$ then:

$$E^{\beta}(Rfond(h)) = \bigcup_{\gamma<\beta} E^{\gamma}(Rfond(h)) = \bigcup_{\gamma<\beta} \bigcup_{Ar(x)=h} \Big(\{x\} \times E^{\gamma}(x)\Big) =$$

$$\bigcup_{Ar(x)=h} \Big(\{x\} \times \bigcup_{\gamma<\beta} E^{\gamma}(x)\Big).$$

Using the inductive hypothesis we get the thesis.
1.3. It is sufficient to prove $E^{\beta}(qver) \subseteq E^{\beta+1}(qver)$. By ordinal induction. The basis is true: $E^0(qver) = \emptyset$.
For $\beta = \alpha + 1$, if $x \in E^{\beta}(qver)$, then ${}^{*}\mathcal{U}_W \models \mathbf{p}_{qver}(\mathbf{c}_x)$ for all $W \in \mathcal{E}(E^{\alpha})$. By the inductive hypothesis and point 1.1, one has: $\mathcal{E}(E^{\alpha+1}) \subseteq \mathcal{E}(E^{\alpha})$. Thus by definition $x \in E^{\beta+1}(qver)$.
For β limit ordinal one has $\mathcal{E}(E^{\beta}) \subseteq \mathcal{E}(E^{\gamma})$, for all $\gamma < \beta$. If $\bar{\varphi} = x \in E^{\beta}(qver)$ then for some $\gamma < \beta$ we have $x \in E^{\gamma}(qver)$, in particular ${}^{*}\mathcal{U}_W \models \varphi$ for all $W \in \mathcal{E}(E^{\beta})$. That is $x \in E^{\beta+1}(qver)$.
1.4 follows from the previous points for each fixed $\alpha \leq \beta$. ■

Definition 3: For each ordinal β, we put: ${}^{*}\mathcal{U}^{\beta} = {}^{*}\mathcal{U}_{E^{\beta}}$ (the predicative structure relative to E^{β}) and $\mathcal{U}^{\beta} = \mathcal{U}_{E^{\beta}}$ (the truth structure relative to E^{β}).

Proposition 1: *For each ordinal β one has $\mathcal{U}^{\beta} \models$ **BTP** $\cup \{F.5\}$. If β is limit then also $\mathcal{U}^{\beta} \models F.3$.*

Proof : For any ordinal β the extension function E^{β} is predicative by 1.2 above. Hence the truth structure $\mathcal{U}^{\beta}$ satisfies **BTP** $\setminus \{F.1,\ F.2\}$. Moreover $qver$ and $qfals$ have disjoint extensions in $\mathcal{U}^{\beta}$ by 1.1. It follows that the structure $\mathcal{U}^{\beta}$ verifies all the properties of **qver** and **qfals** listed after axiom F.2.
Moreover, by definition 2, the set $E^{\beta}(qver) \cup E^{\beta}(qfals)$ is closed under boolean combinations. Hence $\mathcal{U}^{\beta} \models F.5$.
We have to prove F.2. If $\alpha \leq \beta$ then $E^{\beta} \in \mathcal{E}(E^{\alpha})$ by 1.4. If $\beta = \gamma + 1$ and $\mathcal{U}^{\beta} \models$ **qver** x, *i.e.* $x = \bar{\varphi} \in E^{\beta}(qver)$, then ${}^{*}\mathcal{U}^{\beta} \models \varphi$, *i.e.* $\mathcal{U}^{\beta} \models (\Vdash x)$, since $E^{\beta} \in \mathcal{E}(E^{\gamma})$. Similarly if β is limit and $\mathcal{U}^{\beta} \models$ **qver** x, *i.e.* $x = \bar{\varphi} \in E^{\gamma+1}(qver)$ for some $\gamma < \beta$, then $\mathcal{U}^{\beta} \models \Vdash x$ since $E^{\beta} \in \mathcal{E}(E^{\gamma})$. Thus we have proved F.2.

Now we suppose λ to be a limit ordinal. We have to prove that $\mathcal{U}^{\lambda}$ verifies F.3. It is sufficient to prove that for any basic proposition $p =$ "$x\ y_1 \ldots y_h$", if $\mathcal{U}^{\lambda} \models \Vdash p$, then $\mathcal{U}^{\lambda} \models$ **qver** p.
If $\mathcal{U}^{\lambda} \models (\Vdash$"$x\ y_1 \ldots y_h$") and $x \neq qver, qfals, Rfond(h)$, then: $Ar(x) = h \neq 0$ and ${}^{*}\mathcal{U}^{\lambda} \models \mathbf{p}_x(\mathbf{c}_{y_1} \ldots \mathbf{c}_{y_h})$, *i.e.* $(y_1, \ldots y_h) \in E^{\lambda}(x) = E^0(x) = W(x)$, for any $W \in \mathcal{E}(E^0)$. Moreover if $W \in \mathcal{E}(E^{\alpha})$ then $W \in \mathcal{E}(E^0)$ since $E^{\alpha} \in \mathcal{E}(E^0)$. It follows that ${}^{*}\mathcal{U}_W \models \mathbf{p}_x(\mathbf{c}_{y_1} \ldots \mathbf{c}_{y_h})$ for all $\gamma < \lambda$ and all $W \in \mathcal{E}(E^{\gamma})$, *i.e.* "$x\ y_1 \ldots y_h$" $\in E^{\gamma+1}(qver)$ $\subseteq E^{\lambda}(qver)$. Thus $\mathcal{U}^{\lambda} \models$ **qver**"$x\ y_1 \ldots y_h$".
If $\mathcal{U}^{\lambda} \models (\Vdash$"$qver\ y$"), *i.e* ${}^{*}\mathcal{U}^{\lambda} \models \mathbf{p}_{qver}(\mathbf{c}_y)$, *i.e.* $y \in E^{\lambda}(qver)$ then $y \in E^{\gamma+1}(qver)$ for some $\gamma < \lambda$. Hence $y \in W(qver) \supseteq E^{\gamma+1}(qver)$ for all $W \in \mathcal{E}(E^{\gamma+1})$. This means that ${}^{*}\mathcal{U}_W \models \mathbf{p}_{qver}(\mathbf{c}_y)$, for all $W \in \mathcal{E}(E^{\gamma+1})$. Hence "$qver\ y$" $\in E^{\gamma+2}(qver) \subseteq E^{\lambda}(qver)$, since λ is a limit ordinal. Thus $\mathcal{U}^{\lambda} \models$ **qver** "$qver\,y$".
The case $\mathcal{U}^{\lambda} \models (\Vdash$"$qfals\ y$") is analogous.
Finally if $x = Rfond(h)$, $h > 0$, the thesis follows by a simple induction on h, since $\mathcal{U}^{\lambda} \models D.10$. ■

Lemma 2: *There is a countable limit ordinal τ such that $E^\tau = E^{\tau+1}$*

Proof : The functions $\alpha \mapsto E^\alpha(qver)$, $\alpha \mapsto E^\alpha(qfals)$ are monotone increasing with respect to set inclusion. Then, since U is countable, there is a countable ordinal α such that $E^{\alpha+1}(qver) = E^\alpha(qver)$ and $E^{\alpha+1}(qfals) = E^\alpha(qfals)$. By definition $E^{\alpha+1} = E^\alpha$. Put τ equal to the first limit greater than α. (Notice that in fact the first such α is limit). ∎

Proposition 2: *If τ is limit and $E^\tau = E^{\tau+1}$ then $\mathcal{U}^\tau \models$ **BTP** $\cup \{F.3,\ F.5,\ F.6\}$.*

Proof : We have to show that $\mathcal{U}^\tau \models F.6$. Let $\bar{\varphi} = p$ be a b-predicate of order 1, where φ is a $\mathcal{N}^*\mathcal{L}$-formula with only ξ_1 free. If for all $u \in U$: $p[u] \in E^\tau(qver) = E^{\tau+1}(qver)$, then for all $W \in \mathcal{E}(E^\tau)$, and for all $u \in U$: ${}^*\mathcal{U}_W \models \varphi[u/\xi_1]$, that is for all $W \in \mathcal{E}(E^\tau)$ it holds ${}^*\mathcal{U}_W \models \forall\eta_1\varphi[\eta_1/\xi_1]$. Hence $\forall_1 p \in E^{\tau+1}(qver) = E^\tau(qver)$. If for some $u \in U$: $p[u] \in E^\tau(qfals) = E^{\tau+1}(qfals)$, then for all $W \in \mathcal{E}(E^\tau)$: ${}^*\mathcal{U}_W \models \neg\varphi[u/\xi_1]$, that is for all $W \in \mathcal{E}(E^\tau)$ it holds ${}^*\mathcal{U}_W \models \exists\eta_1\neg\varphi[\eta_1/\xi_1]$, *i.e.* ${}^*\mathcal{U}_W \models \neg\forall\eta_1\varphi[\eta_1/\xi_1]$. Hence $\neg\forall_1 p \in E^{\tau+1}(qver) = E^\tau(qver)$, that is $\forall_1 p \in E^\tau(qfals)$. The same argument works for the existential quantifier. ∎

References

[1] P.Aczel – *Non-Well-Founded Sets*, CSLI Lecture Notes N.14, Stanford 1988.

[2] L.Ambrosio, G.Dal Maso, M.Forti, M.Miranda, S.Spagnolo – Necrologio di Ennio De Giorgi, *Boll. Un. Mat. It.*, (B) Febbraio (1999), 1-31.

[3] M.Clavelli, E.De Giorgi, M.Forti, V.M. Tortorelli – A selfreference oriented theory for the Foundations of Mathematics, in *Analyse Mathématique et applications – Contributions en l'honneur de Jacques-Louis Lions*, Gauthier-Villars, Paris 1988, pp. 67-115.

[4] E.De Giorgi – *Fundamental Principles of Mathematics*, relation held at the Plenary Session of the 'Accademia Pontificia delle Scienze', 25-29 October 1994.

[5] E.De Giorgi – Dal superamento del riduzionismo insiemistico alla ricerca di una più ampia e profonda comprensione tra matematici e studiosi di altre discipline scientifiche e umanistiche, *Rend. Mat. Acc. Lincei* (9) 9 (1998), 71-80.

[6] E.De Giorgi, M.Forti – *Premessa a nuove teorie assiomatiche dei Fondamenti della Matematica*, Quaderni del Dipartimento di Matematica dell'Università di Pisa (54), Pisa 1984.

[7] E.De Giorgi, M.Forti – Una teoria quadro per i fondamenti della Matematica, *Atti Acc. Naz. Lincei Cl. Sci. Fis. Mat. Nat. Rend. Lincei Mat. Appl.* (8) 79 (1985), 55-67.

[8] E.De Giorgi, M.Forti – *"5×7": A Basic Theory for the Foundations of Mathematics*, Preprint di Matematica della Scuola Normale Superiore di Pisa (74), Pisa 1990.

[9] E.De Giorgi, M.Forti – Dalla matematica e dalla logica alla sapienza, in *Pensiero Scientifico, Fondamenti ed Epistemologia (Ancona 1996)*, A.Repola Boatto eds., Quaderni 'Innovazione Scuola' (29), Ancona 1997, 17-36.

[10] E.De Giorgi, M.Forti, G.Lenzi – Una proposta di teorie base dei fondamenti della matematica, *Rend. Mat. Acc. Lincei* (9) 5-1 (1994), 11-22.

[11] E.De Giorgi, M.Forti, G.Lenzi – Introduzione delle variabili nel quadro delle teorie base dei Fondamenti della Matematica, *Rend. Mat. Acc. Lincei* (9) 5-2 (1994), 117-128.

[12] E.De Giorgi, M.Forti, G.Lenzi – Verso i sistemi assiomatici del 2000 in Matematica, Logica e Informatica, *Nuova Civiltà delle Macchine* 57-60 (1997), 248-259.

[13] E.De Giorgi, M.Forti, G.Lenzi – Verità e giudizi in una nuova prospettiva assiomatica, in *Il fare della Scienza: i fondamenti e le palafitte*, F.Barone, G.Basti, A.Testi eds., 'Con-tratto, rivista di filosofia tomista e contemporanea', 1996, 233-252.

[14] E.De Giorgi, M.Forti, G.Lenzi, V.M. Tortorelli – Calcolo dei predicati e concetti metateorici in una teoria base dei fondamenti della matematica, *Rend. Mat. Acc. Lincei* (9) 6-2 (1995), 79-92.

[15] E.De Giorgi, M.Forti, V.M. Tortorelli – Sul problema dell'autoriferimento, *Atti Acc. Naz. Lincei Cl. Sc. Fis. Mat. Nat. Rend. Lincei Mat. App.* (8) 80 (1986), 363-372.

[16] E.De Giorgi, G.Lenzi – La Teoria '95: una proposta di una teoria aperta e non riduzionista dei Fondamenti della Matematica, *Rend. Acc. Naz. Sci.* XL *Mem. Mat. Appl.* (5) 20 (1996), 7-34.

[17] M.Forti, L.Galleni – An axiomatization of biological concepts within the foundational theory of Ennio De Giorgi, *Rivista di Biologia - Biology Forum*, 92 (1999), 77-104.

[18] M.Forti, F.Honsell – Set theory with Free Construction Principles, *Ann. Scuola Norm. Sup. Pisa, Cl. Sci.* (4) 10 (1983), 493-522.

[19] M.Forti, F.Honsell – Models of self-descriptive set theories, in *Partial Differential Equations and the Calculus of Variations – Essays in Honor of Ennio De Giorgi*, F. Colombini et al. eds., Boston 1989, pp. 473-518.

[20] M.Forti, F.Honsell, M.Lenisa – Operations, Collections and Sets within a general axiomatic framework, in *Logic and Foundations of Mathematics*, Florence, August 1995 (A.Cantini, E.Casari and P.Minari, eds.). *Synthese Library* **280**, Kluwer, Dordrecht 1999, 1-24.

[21] M.Forti, F.Honsell, M.Lenisa – An axiomatization of partial n-place operations, *Math. Struct. Computer Sci.* 7 (1997), 283-302.

[22] M.Grassi, G.Lenzi, V. M.Tortorelli – Formalizations and Models of a Basic Theory for the foundations of Mathematics, *Rend. Acc. Naz. Sci.* XL *Mem. Mat. Appl.* 1130 XIX-1 (1995), 129-157.

[23] G.Lenzi – *Modelli di Teorie dei Fondamenti della Matematica con proprietà di Autoriferimento*, tesi di laurea, Pisa 1988.

[24] G.Lenzi – Estensioni contraddittorie della teoria Ampia, *Atti Acc. Naz. Lincei Cl. Sc. Fis. Mat. Nat. Rend. Lincei Mat. App.* (8) 83 (1989), 13-28.

[25] G.Lenzi, V. M.Tortorelli – *Introducing predicates into a basic theory for the foundations of Mathematics*, Preprint di Matematica della Scuola Normale Superiore di Pisa (51), Luglio 1989.

[26] D.Scott – Combinators and Classes, in λ-*Calculus and Computer Science*, C.Böhm ed., Springer Lecture Notes In Computer Science **37**, Berlin 1975, 1-26.

[27] J.R.Shoenfield–*Mathematical Logic*, Reading 1967.

[28] D.Van Dalen – *Logic and Structure*, Berlin 1983.

Giacomo Lenzi: Via Pesenti 11, 56124 Pisa;

Vincenzo Maria Tortorelli: Dipartimento di Matematica Università di Pisa,
Via Buonarroti 2, 56127 Pisa.