Fixed point results for compact maps on closed subsets of Fréchet spaces and applications to differential and integral equations

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Abstract

In this paper, we establish fixed point results for compact maps $f : X \to \mathbb{E}$ defined on arbitrary closed subsets X of a Fréchet space $\mathbb{E}$. In particular, we obtain a continuation principle for suitable compact homotopy $h : X \times [0,1] \to \mathbb{E}$. Afterwards, those results are applied to differential equations and to Fredholm integral equations on the real line.

1 Introduction

It is well known (see [14]) that if $h : X \times [0,1] \to \mathbb{E}$ is a compact map defined on X the closure of an open set of a locally convex space $\mathbb{E}$, and if $h(x,0) \equiv \hat{x} \in \text{int}(X)$, then one of the following statements holds:

(a) $h(\cdot,1)$ has a fixed point;

(b) there exist $\lambda \in (0,1)$ and $x \in \partial X$ such that $x = h(x,\lambda)$.

In the particular case where $\mathbb{E}$ is a Banach space, this important result was widely applied, notably to nonlinear differential equations. Unfortunately, very few applications were given in the case where $\mathbb{E}$ is a locally convex space which is not normable. The problem is that in many potential applications, the appropriate set X to work with has empty interior, see for example [4].

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Furi and Pera [11] were the first to obtain a continuation principle for compact maps defined on closed convex subsets with possibly empty interior of a locally convex space. Instead of statement (b), it was required that each $(x, \lambda) \in \partial X \times [0,1)$ with $x = h(x, \lambda)$, has a neighborhood sent in X by h.

In this paper, fixed point results for compact maps $f : X \to \mathbb{E}$ are established for arbitrary closed subsets X (possibly non-convex and with empty interior). Our approach, different from Furi and Pera's one, is in the spirit of results on contractions obtained in [7, 9], where $\mathbb{E}$ is regarded as a projective limit. Of course, in allowing X to have empty interior, the statement (b) must be changed, and the point $\hat{x}$ has to be chosen in a different way. To this aim, we consider an appropriate class of compact maps and we introduce the notion of pseudo-interior of X.

In order to simplify the notations, we consider a Fréchet space $(\mathbb{E}, \{\|\cdot\|_n\}_{n\in\mathbb{N}})$. It is worthwhile to mention that our results are also true in a locally convex space $(\mathbb{E}, \{\|\cdot\|_\alpha\}_{\alpha\in\Lambda})$, where Λ is a directed set, and $\|u\|_\alpha \le c_{\alpha,\beta}\|u\|_\beta$ when $\alpha \le \beta$; see [5] for definitions.

In the last section, we present applications of our fixed point results to differential and integral equations. The first one is a result of Lee and O'Regan [12] on first order differential equations on the half line.

The second application concerns infinite systems of first order differential equations. A generalization of Peano's Theorem, and a nonlocal existence result are established.

Finally, we study Fredholm integral equations on the real line. In the very interesting papers of Anselone with Sloan [1] and with Lee [2], integral equations on the half line are treated in considering a sequence of integral equations on finite intervals. This kind of technics was also used by Lee and O'Regan [13]. Our approach is different since we always take into account the behavior of the function on the whole real line.

2 Preliminaries

2.1 Spaces

Let $\mathbb{E}$ be a Fréchet space with the topology generated by a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. For sake of simplicity, we will assume that the following condition is satisfied:

$$\|x\|_1 \le \|x\|_2 \le \cdots \quad \text{for every } x \in \mathbb{E}. \tag{$\star$}$$

To $\mathbb{E}$, we associate for every $n \in \mathbb{N}$, a normed space $\mathbb{E}_n$ as follows: For each $n \in \mathbb{N}$, we write

$$x \sim_n y \quad \text{if and only if} \quad \|x - y\|_n = 0. \tag{2.1}$$

This defines an equivalence relation on $\mathbb{E}$. We denote by $E_n = \mathbb{E}/{\sim_n}$ the quotient space, and by $\mathbb{E}_n$ the completion of E_n with respect to $\|\cdot\|_n$ (the norm on E_n induced by $\|\cdot\|_n$ and its extension to $\mathbb{E}_n$ are still denoted by $\|\cdot\|_n$). This construction defines a continuous map $\mu_n : \mathbb{E} \to \mathbb{E}_n$.

For each subset $X \subset \mathbb{E}$, and each $n \in \mathbb{N}$, we set $X_n = \mu_n(X)$, and we denote $\overline{X_n}$, and ∂X_n, respectively the closure and the boundary of X_n with respect to $\|\cdot\|_n$ in

$\mathbb{E}_n$. We denote by diam_n, the n-diameter induced by $\|\cdot\|_n$; that is, for $X \subset \mathbb{E}$,

$$\operatorname{diam}_n(X) = \sup\{\|x-y\|_n : x, y \in X\}.$$

Since the set X that we will consider can have empty interior, we introduce the notion of *pseudo-interior* of X that we define by

$$\text{pseudo-int}(X) = \{x \in X : \mu_n(x) \in \overline{X_n} \backslash \partial X_n \ \text{ for every } n \in \mathbb{N}\}.$$

Example. (1) Let $\mathbb{E}$ be the Fréchet space $C[0,\infty)$, and let $X = \{u \in C[0,\infty) : |u(t)| \leq M \text{ for every } t \in [0,\infty)\}$. Then

$$\text{pseudo-int}(X) = \{u \in C[0,\infty) : |u(t)| < M \text{ for every } t \in [0,\infty)\}.$$

(2) Let $X = [-1,1]^{\mathbb{N}}$ be in the Fréchet space $\mathbb{R}^{\mathbb{N}}$. Then

$$\text{pseudo-int}(X) = (-1,1)^{\mathbb{N}}.$$

The following result establishes that the pseudo-int(X) is independent of the choice of the family of semi-norms.

Proposition 2.1. *Let X be a subset of $\mathbb{E}$. Then $x \in$ pseudo-int(X) if and only if for every neighborhood of the origin U, there exists V a neighborhood of x such that*

$$V \subset X + \bigcap_{\lambda>0} \lambda U. \tag{2.2}$$

Proof. If $x \in X$ is such that for every neighborhood of 0 there exists a neighborhood satisfying (2.2), then this holds in particular with $U_n = \{y \in \mathbb{E} : \|y\|_n < 1\}$ for $n \in \mathbb{N}$. So,

$$x \in V \subset X + \bigcap_{\lambda>0} \lambda U_n,$$

and hence

$$\mu_n(x) \in \mu_n(V) \subset \mu_n\Big(X + \bigcap_{\lambda>0} \lambda U_n\Big) = X_n.$$

Since $\mu_n(V)$ is open in E_n, there exists $\delta > 0$ such that

$$\{w \in E_n : \|w - \mu_n(x)\|_n < \delta\} \subset \mu_n(V).$$

Therefore, $\mu_n(x) \in \overline{X_n} \backslash \partial X_n$.

On the other hand, let $x \in$ pseudo-int(X) and U a neighborhood of the origin. There exist $n \in \mathbb{N}$ and $\lambda > 0$ such that

$$\lambda U_n = \{y \in \mathbb{E} : \|y\|_n < \lambda\} \subset U.$$

Since $x \in$ pseudo-interior(X), there exists $\delta > 0$ such that

$$W = \{w \in E_n : \|w - \mu_n(x)\|_n < \delta\} \subset X_n.$$

Therefore,

$$x \in V = \mu_n^{-1}(W) \subset X + \bigcap_{\lambda>0} \lambda U_n \subset X + \bigcap_{\lambda>0} \lambda U.$$

∎

Now, observe that, since the condition $(\star)$ is satisfied, the semi-norm $\|\cdot\|_n$ induces a semi-norm on $\mathbb{E}_m$ for every $m \geq n$. For simplicity, this semi-norm is still denoted by $\|\cdot\|_n$. Again, the relation (2.1) defines an equivalence relation on $\mathbb{E}_m$ from which we obtain a continuous map $\mu_{n,m} : \mathbb{E}_m \to \mathbb{E}_n$ since $\mathbb{E}_m/\sim_n$ can be regarded as a subset of $\mathbb{E}_n$. Observe that $\mathbb{E}$ is the projective limit of $(\mathbb{E}_n)_{n\in\mathbb{N}}$.

The following lemma gives an important property of closed subsets of $\mathbb{E}$.

Lemma 2.2. *Assume that the condition $(\star)$ is satisfied, and let X be a closed subset of $\mathbb{E}$. Then, for every sequence $(z_n)_{n\in\mathbb{N}}$ with $z_n \in \overline{X_n}$, such that for every $n \in \mathbb{N}$, $(\mu_{n,m}(z_m))_{m\geq n}$ is a Cauchy sequence in $\overline{X_n}$, there exists $x \in X$ such that $(\mu_{n,m}(z_m))_{m\geq n}$ converges to $\mu_n(x) \in X_n$ for every $n \in \mathbb{N}$.*

The family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ obviously induces a family of semi-norms on $\mathbb{E}\times\mathbb{R}$ that we still denote by $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. Also, the continuous functions obtained from the relation (2.1) on $\mathbb{E}\times\mathbb{R}$ are still denoted $\mu_n : \mathbb{E}\times\mathbb{R} \to \mathbb{E}_n \times \mathbb{R}$. Similarly for the other notations.

In what follows, X is always a closed subset of $\mathbb{E}$, and Y is a closed subset of $\mathbb{E}$ or $\mathbb{E}\times\mathbb{R}$. By a compact map $f : Y \to \mathbb{E}$, we mean a continuous map such that $f(Y)$ is relatively compact in $\mathbb{E}$.

2.2 Multivalued maps

We recall some definitions and results concerning multivalued mappings. For more details, the reader is referred to [3] and the references therein. Let Z_1, Z_2, Z_3 be three metrizable spaces, and $I \subset \mathbb{R}$ a measurable set.

Definition 2.3. A multivalued mapping $F : Z_1 \to Z_2$ is *upper semi-continuous* (u.s.c.) if $\{z : F(z)\cap K \neq \emptyset\}$ is closed for every closed subset K of Z_2. It is *lower semi-continuous* (l.s.c.) if $\{z : F(z)\cap U \neq \emptyset\}$ is open for every open subset U of Z_2. It is *continuous* if it is lower and upper semi-continuous. A multivalued map $F : I \to Z_2$ is *measurable* if $\{t : F(t)\cap K \neq \emptyset\}$ is measurable for every closed subset K of Z_2.

Lemma 2.4. *If $F_0 : Z_1 \to Z_2$ and $F_1 : Z_2 \to Z_3$ are two continuous multivalued mappings, then $F_1 \circ F_0 : Z_1 \to Z_3$ is continuous.*

Lemma 2.5. *Let $F : Z_1 \to Z_2$ be a continuous multivalued map with relatively compact values. Then the map $\overline{F} : Z_1 \to Z_2$ defined by $\overline{F}(z) = \overline{F(z)}$ is continuous.*

Lemma 2.6. *Let $F : I\times\mathbb{R}^m \to \mathbb{R}^n$ be a multivalued map with compact values such that F is measurable in $t \in I$, and continuous in $x \in \mathbb{R}^m$. Then for every measurable (single-valued) function $t \mapsto x(t)$, the multivalued map $t \mapsto F(t, x(t))$ is measurable.*

Definition 2.7. Let $F : I \to \mathbb{R}^n$ be a multivalued map. The *integral* of F is defined by

$$\int_I F(t)\,dt = \Big\{ z = \int_I f(t)\,dt : f \in L^1(I), f(t) \in F(t)\ \forall t \in I \Big\}.$$

Lemma 2.8. *Let $F : I \to \mathbb{R}^n$ be a multivalued map with compact values such that there exists $h \in L^1(I)$ satisfying*

$$\|F(t)\| = \sup\{\|y\| : y \in F(t)\} \leq h(t) \quad a.e.\ t \in I.$$

Then, $\int_I F(t)\,dt$ is non-empty, convex and compact.

3 Strongly admissible compact functions

In this section, we consider a particular case of the main result of this paper. The presentation is simpler and helps to a better understanding of the more general case. More precisely, we establish fixed point results for strongly admissible compact functions that we define as follows. Let X be a closed subset of $\mathbb{E}$, and Y a closed subset of $\mathbb{E}$ or $\mathbb{E} \times \mathbb{R}$.

Definition 3.1. A compact map $f : Y \to \mathbb{E}$ is called *strongly admissible* if for every $n \in \mathbb{N}$,

(1) $\|f(x) - f(y)\|_n = 0$ when $\|x - y\|_n = 0$;

(2) the function $f_n : Y_n \to \mathbb{E}_n$ defined by $f_n(\mu_n(x)) = \mu_n \circ f(x)$ admits a continuous extension $\boldsymbol{f}_n : \overline{Y_n} \to \mathbb{E}_n$.

Obviously, if f is strongly admissible then $\boldsymbol{f}_n : \overline{Y_n} \to \mathbb{E}_n$ is compact for every $n \in \mathbb{N}$.

Definition 3.2. Let $f : X \to \mathbb{E}$. We say that f is in the *class* $A^s_\partial(X)$, if f is strongly admissible, and for every $n \in \mathbb{N}$, $z \neq \boldsymbol{f}_n(z)$ for every $z \in \partial X_n$.

Remark. Let $\hat{x} \in X$ then the constant function associated to $\hat{x}$ (still denoted $\hat{x}$) is strongly admissible. Moreover, if $\hat{x} \in$ pseudo-int(X) then $\hat{x} \in A^s_\partial(X)$.

In the class $A^s_\partial(X)$, we introduce a notion of homotopy.

Definition 3.3. Let $f, g \in A^s_\partial(X)$. We say that f and g are *homotopic in* $A^s_\partial(X)$ if there exists a strongly admissible compact map $h : X \times [0,1] \to \mathbb{E}$ such that

(1) $h(\cdot, 0) = f$, $h(\cdot, 1) = g$;

(2) $h(\cdot, \lambda) \in A^s_\partial(X)$ for every $\lambda \in [0,1]$.

We write $f \approx_s g$.

Clearly, $\approx_s$ is an equivalence relation in $A^s_\partial(X)$. Now, we can establish the main fixed point result of this section.

Theorem 3.4. *Let $f \in A^s_\partial(X)$, and let $\hat{x} \in$ pseudo-int(X). If $f \approx_s \hat{x}$, then f has a fixed point.*

Proof. Let $h : X \times [0,1] \to \mathbb{E}$ be an homotopy between f and $\hat{x}$ in $A^s_\partial(X)$. Therefore, for every $n \in \mathbb{N}$,

$$\boldsymbol{h}_n : \overline{X_n} \times [0,1] \to \mathbb{E}_n,$$

is a compact homotopy between $\boldsymbol{f}_n$ and the constant function $\mu_n(\hat{x})$, without fixed point on the boundary of $\overline{X_n}$. By the Topological Transversality Theorem [6, theorem 4.4.7], $\boldsymbol{f}_n$ has a fixed point $z_n \in \overline{X_n}$.

Obviously $\mu_{n,m}(z_m) = \boldsymbol{f}_n(\mu_{n,m}(z_m))$ for every $m \geq n$. By compactness, the sequence $(\mu_{1,m}(z_m))_{m\geq 1}$ has a subsequence $(\mu_{1,m}(z_m))_{m\in N_1}$ converging to $y_1 \in \overline{X_1}$. It follows from the continuity of $\boldsymbol{f}_1$ that $y_1 = \boldsymbol{f}_1(y_1)$.

Again, the sequence $(\mu_{2,m}(z_m))_{m\in N_1}$ has a subsequence $(\mu_{2,m}(z_m))_{m\in N_2}$ converging to $y_2 \in \overline{X_2}$, with $y_2 = \boldsymbol{f}_2(y_2)$. By uniqueness of the limit, $\mu_{1,2}(y_2) = y_1$.

In repeating this argument, we obtain for every $n \in \mathbb{N}$, $y_n \in \overline{X_n}$ such that $y_n = \boldsymbol{f}_n(y_n)$; and $\mu_{n,m}(y_m) = y_n$ for every $m \geq n$. It follows from Lemma 2.2 the existence of $y \in X$ such that $y = f(y)$. ∎

Corollary 3.5. *Let $f : X \to \mathbb{E}$ be a strongly admissible compact map. If $0 \in$ pseudo-int(X), then one of the following statements holds:*

(a) f has a fixed point;

(b) there exist $n \in \mathbb{N}$, $\lambda \in (0,1]$, and $z \in \partial X_n$ such that $z = \lambda \boldsymbol{f}_n(z)$.

Proof. Since $\overline{\text{co}}(\{0\} \cup \overline{f(X)})$ is compact, $h : X \times [0,1] \to \mathbb{E}$ defined by $h(x,\lambda) = \lambda f(x)$ is compact and obviously strongly admissible. The conclusion follows directly from Definition 3.3 and Theorem 3.4. ∎

4 Admissible compact functions

As we have seen in the previous section, strongly admissible compact functions must satisfy a very restrictive condition, namely:

$$\|f(x) - f(y)\|_n = 0 \quad \text{whenever} \quad \|x - y\|_n = 0.$$

In this section, compact functions which may not satisfy this condition are considered. In this case, $f_n(\mu_n(x)) = \mu_n \circ f(x)$ is not well defined, and hence we can not proceed as in the previous section. We define for every $n \in \mathbb{N}$, the multivalued map $S_n : Y \to Y$ by

$$S_n(x) = \{y \in Y : \|x - y\|_n = 0\}.$$

Definition 4.1. A compact map $f : Y \to \mathbb{E}$ is called *admissible* if for every $n \in \mathbb{N}$,

(1) the multivalued map $F_n : Y_n \to \mathbb{E}_n$ defined by

$$F_n(\mu_n(x)) = \overline{\text{co}}\Big(\mu_n \circ f \circ S_n(x)\Big)$$

admits an upper semi-continuous extension $\boldsymbol{F}_n : \overline{Y_n} \to \mathbb{E}_n$ with convex, compact values;

(2) for every $\varepsilon > 0$, there exists $m \geq n$ such that for every $x \in Y$,

$$\text{diam}_n\Big(f(S_m(x))\Big) < \varepsilon.$$

Definition 4.2. Let $f : X \to \mathbb{E}$ be a compact map. We say that f is in the *class* $A_\partial(X)$, if f is admissible, and for every $n \in \mathbb{N}$, $z \notin \boldsymbol{F}_n(z)$ for every $z \in \partial X_n$.

Clearly, a strongly admissible function is admissible, and $A^s_\partial(X) \subset A_\partial(X)$. As before, we introduce a notion of homotopy in the class $A_\partial(X)$.

Definition 4.3. Let $f, g \in A_\partial(X)$. We say that f and g are *homotopic in* $A_\partial(X)$ if there exists an admissible compact map $h : X \times [0,1] \to \mathbb{E}$ such that

(1) $h(\cdot, 0) = f$, $h(\cdot, 1) = g$;

(2) $h(\cdot, \lambda) \in A_\partial(X)$ for every $\lambda \in [0,1]$.

We write $f \approx g$.

Now, we can establish our main fixed point theorem.

Theorem 4.4. *Let $f \in A_\partial(X)$, and let $\hat{x} \in$ pseudo-int(X). If $f \approx \hat{x}$, then f has a fixed point.*

Proof. In using the Topological Transversality Theorem for upper semi-continuous compact map with convex values [6, section 5.11], we deduce as in the previous section that $\boldsymbol{F}_n$ has a fixed point $z_n \in \overline{X_n}$.

In using Lemma 2.2, the compactness and upper semi-continuity of $\boldsymbol{F}_n$, and in arguing as in the proof of Theorem 3.4, we obtain the existence of $y \in X$ such that $\mu_n(y) \in \boldsymbol{F}_n(\mu_n(y))$ for every $n \in \mathbb{N}$.

We have to show that $y = f(y)$. If this is false, there exists $n \in \mathbb{N}$ and $\alpha > 0$ such that $\|y - f(y)\|_n = \alpha$. Let $\varepsilon < \alpha/2$. By Definition 4.1(2), there exists $m \geq n$ such that $\operatorname{diam}_n\big(f(S_m(y))\big) < \varepsilon$. We have

$$\operatorname{diam}_n\Big(f(S_m(y))\Big) = \operatorname{diam}_n\Big(\operatorname{co}\Big(f(S_m(y))\Big)\Big).$$

On the other hand, since $\mu_m(y) \in \boldsymbol{F}_m(\mu_m(y))$, we can take $w \in \operatorname{co}\big(f(S_m(y))\big)$ such that $\|y - w\|_m < \varepsilon$. Thus,

$$\alpha = \|y - f(y)\|_n \leq \|y - w\|_n + \|w - f(y)\|_n < \|y - w\|_m + \varepsilon < 2\varepsilon < \alpha;$$

a contradiction. ■

Remark. It can be seen in the last proof that condition (2) of Definition 4.1 is not needed for $h(\cdot, \lambda)$, $\lambda \in (0,1)$.

Corollary 4.5. *Assume that $0 \in$ pseudo-int(X). If $f : X \to \mathbb{E}$ is an admissible compact map, then one of the following statements holds:*

(a) f has a fixed point;

(b) there exist $n \in \mathbb{N}$, $\lambda \in (0,1]$, and $z \in \partial X_n$ such that $z \in \lambda \boldsymbol{F}_n(z)$.

5 Applications

5.1 First order differential equations

We consider the problem

$$\begin{aligned} &x'(t) = g(t, x(t)), \quad t \in I, \\ &x(0) = 0, \end{aligned} \tag{5.1}$$

with I a real interval which will be precised later.

5.1.1 Finite systems on $[0, \infty)$

We start with the following known result on finite systems of first order differential equations on the half line, see [12]. For sake of simplicity, we assume that g is continuous; we could have treated the Carathéodory case. Also, the following theorem can be generalized to differential equations in a Banach space, in the K-Carathéodory context, see [10].

Theorem 5.1. *Let $g : [0, \infty) \times \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function with $N \in \mathbb{N}$. Assume there exist $\theta \in L^1_{loc}[0, \infty)$ and $\psi : [0, \infty) \to (0, \infty)$ a continuous function such that $\|g(t, x)\| \leq \theta(t)\psi(\|x\|)$ for all $t \in [0, \infty)$, and $x \in \mathbb{R}^N$. Let*

$$T = \sup \left\{ t \geq 0 : \int_0^\infty \frac{dz}{\psi(z)} > \|\theta\|_{L^1[0,t]} \right\}.$$

Then the problem (5.1) *has a solution on* $[0, T)$.

Proof. For $t < T$, set $M(t) > 0$ such that

$$\int_0^{M(t)} \frac{dz}{\psi(z)} > \|\theta\|_{L^1[0,t]}.$$

Take $\mathbb{E} = C([0, T), \mathbb{R}^N)$, $X = \{x \in \mathbb{E} : \|x(t)\| \leq M(t)$ for every $t \in [0, T)\}$, and define $f : X \to \mathbb{E}$ by

$$f(x)(t) = \int_0^t g(s, x(s))\, ds.$$

It is easy to show that f is a strongly admissible compact map. By standard arguments and the choice of $M(t)$, we deduce that $\lambda f \in A^s_\partial(X)$ for every $\lambda \in [0, 1]$, see [12]. Since $0 \in$ pseudo-int(X), the existence of a solutions follows from Corollary 3.5. ■

5.1.2 Infinite systems of differential equations

On $\mathbb{R}^{\mathbb{N}}$, let us define the family of semi-norms:

$$|||(x_1, x_2, \dots)|||_n = \left(|x_1|^2 + \cdots + |x_n|^2 \right)^{1/2}.$$

The first result of this paragraph is a generalization of Peano's Theorem to infinite systems of first order differential equations.

Theorem 5.2. *Let $r > 0$ and $g : [0,T] \times [-r,r]^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ a continuous function such that*

(1) $\sup\{M_n : n \in \mathbb{N}\} < \infty$, *where* $M_n = \max\Big\{|||g(t,x)|||_n : (t,x) \in [0,T] \times [-r,r]^{\mathbb{N}}\Big\}$;

(2) for every $n \in \mathbb{N}$, there exists a sequence $(k_m^n)_{m\geq n}$ converging to 0 such that for every $m \geq n$,
$$|||g(t,x) - g(t,y)|||_n \leq k_m^n$$
for all $t \in [0,T]$, $x, y \in [-r,r]^{\mathbb{N}}$ such that $|||x-y|||_m = 0$.

Then there exists $\tau \in (0,T]$ such that the system (5.1) *has a solution on $[0,\tau]$.*

Proof. Take $\tau \in (0,T]$ such that
$$\tau \sup_{n\in\mathbb{N}} M_n < r.$$
Consider the Fréchet space $\mathbb{E} = C([0,\tau],\mathbb{R}^{\mathbb{N}})$ endowed with the family of semi-norms:
$$\|u\|_n = \max_{t\in[0,\tau]} |||u(t)|||_n.$$
Set $X = \{u \in \mathbb{E} : |u_n(t)| \leq r \text{ for every } t \in [0,\tau],\ n \in \mathbb{N}\}$. Define $f : X \to \mathbb{E}$ by
$$f(u)(t) = \int_0^t g(s,u(s))\,ds.$$
It is easy to show that the function f is continuous and compact.

We have that for every $n \in \mathbb{N}$, $X_n = \overline{X_n}$, and the function $S_n^* : X_n \to X$ defined by $S_n^*(u) = \{u\} \times \Gamma_n$, where
$$\Gamma_n = \Big\{(v_{n+1}, v_{n+2}, \dots) \in \prod_{i=n+1}^{\infty} C[0,\tau] : |v_m(t)| \leq r \text{ for all } t \in [0,\tau] \text{ and } m > n\Big\},$$
is continuous, since it is the product of a continuous function with a constant multi-valued map. It follows from Lemma 2.4 that $f \circ S_n^*$ is continuous, and hence $\overline{f \circ S_n^*}$ is continuous by Lemma 2.5.

To deduce that f is admissible, we want to show that for every $n \in \mathbb{N}$, $F_n = \boldsymbol{F}_n = \overline{f \circ S_n^*}$. To this end, observe that for $u \in X_n$,
$$\begin{aligned}\overline{f \circ S_n^*}(u) &= \mathrm{cl}\Big(\Big\{w \in C([0,\tau],\mathbb{R}^n) : w(t) = \int_0^t (g_1,\dots,g_n)(s,v(s))\,ds \\ &\qquad\qquad \text{with } v \in X \text{ and } \|u-v\|_n = 0\Big\}\Big) \\ &= \Big\{w \in C([0,\tau],\mathbb{R}^n) : w(t) \in \int_0^t G_n(s,u(s))\,ds\Big\},\end{aligned}$$
where $G_n : [0,\tau] \times [-r,r]^n \to \mathbb{R}^n$ is defined by
$$G_n(t,(x_1,\dots,x_n)) = \Big\{(g_1(t,y),\dots,g_n(t,y)) : \\ y = (x_1,\dots,x_n,y_{n+1},\dots) \in [-r,r]^{\mathbb{N}}\Big\}.$$

From the continuity of g, we deduce that the multivalued map G_n has compact values, $t \mapsto G_n(t,x)$ is measurable, and $x \mapsto G_n(t,x)$ is continuous. It follows from Lemmas 2.6 and 2.8 that $\overline{f \circ S_n^*}$ has convex, compact values. Thus $\boldsymbol{F}_n = \overline{f \circ S_n^*}$.

It follows directly from assumption (2) that for every $n \in \mathbb{N}$, and every $m \geq n$,

$$\operatorname{diam}_n\Big(f(S_m(u))\Big) \leq k_m^n T \quad \text{for every } u \in X.$$

Hence, f is admissible.

The choice of τ with standard arguments permit to conclude that statement (b) of Corollary 4.5 does not hold. Therefore, the problem (5.1) has a solution on $[0,\tau]$. ∎

Now, we present a generalization of Wintner's Theorem to infinite systems of differential equations, see [15] or [12] for finite systems.

Theorem 5.3. *Let $g : [0,T] \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be a continuous function such that*

(1) there exists $\hat{n} \in \mathbb{N}$ such that for every $m \geq \hat{n}$ there exist $M_m > 0$ and $\psi_m : [0,\infty) \to (0,\infty)$ continuous such that

$$\int_0^{M_m} \frac{ds}{\psi_m(s)} > T \quad \textit{and} \quad |||g(t,x)|||_m \leq \psi_m(|||x|||_m) \textit{ for all } t \in [0,T], x \in \mathbb{R}^{\mathbb{N}};$$

(2) for every $n \geq \hat{n}$, there exists a sequence $(k_m^n)_{m \geq n}$ converging to 0 such that for every $m \geq n$,

$$|||g(t,x) - g(t,y)|||_n \leq k_m^n$$

for all $t \in [0,T]$, $x, y \in \mathbb{R}^{\hat{n}-1} \times \prod_{i=\hat{n}}^{\infty}[-M_i, M_i]$ such that $|||x-y|||_m = 0$.

Then the system (5.1) has a solution on $[0,T]$.

Proof. Without lost of generality, we may assume that $\hat{n} = 1$. Take $\mathbb{E} = C([0,T], \mathbb{R}^{\mathbb{N}})$, $X = \{u \in \mathbb{E} : |u_n(t)| \leq M_n \text{ for every } t \in [0,T],\ n \in \mathbb{N}\}$, and define $f : X \to \mathbb{E}$ by

$$f(u)(t) = \int_0^t g(s, u(s))\, ds.$$

In arguing as in the proof of the previous theorem, we deduce that f is admissible.

Assumption (1) with standard arguments (see for example [8]) permit to conclude that statement (b) of Corollary 4.5 does not hold. Therefore, the problem (5.1) has a solution on $[0,T]$. ∎

5.2 Integral equations

We consider the integral equation

$$x(t) = \int_{\mathbb{R}} g(t,s,u(s))\, ds, \quad t \in \mathbb{R}, \tag{5.2}$$

where $g : \mathbb{R}^{N+2} \to \mathbb{R}^N$, $N \in \mathbb{N}$.

Denote $\mathbb{E} = C(\mathbb{R}, \mathbb{R}^N)$ the Fréchet space endowed with the family of semi-norms

$$\|u\|_n = \max_{t\in[-n,n]} \|u(t)\|.$$

Let X be a closed subset of $\mathbb{E}$ which will be determined later. Define $f : X \to \mathbb{E}$ by

$$f(u)(t) = \int_{\mathbb{R}} g(t, s, u(s))\, ds.$$

We assume that the following conditions are satisfied:

(H1) g is continuous;

(H2) for every $t \in \mathbb{R}$, there exists $h_t \in L^1(\mathbb{R})$ such that

$$\|g(t, s, u(s))\| \le h_t(s) \quad \text{for all } u \in X \text{ and all } s \in \mathbb{R};$$

(H3) for every $t \in \mathbb{R}$,

$$\sup_{u\in X} \left\| \int_{\mathbb{R}} g(t, s, u(s)) - g(r, s, u(s))\, ds \right\| \to 0 \quad \text{as } r \to t.$$

Lemma 5.4. *Under (H1) – (H3), f is continuous and compact. Moreover, for every $\varepsilon > 0$ and every $n \in \mathbb{N}$, there exists $m \ge n$ such that for every $u \in X$, $diam_n\big(f(S_m(u))\big) < \varepsilon$.*

Proof. It follows from (H1) – (H3) that f is well defined.

Let $n \in \mathbb{N}$, and $\varepsilon > 0$. Assumption (H3) implies that for every $t \in [-n, n]$, there exists $\delta_t > 0$ such that for every $r \in (t - \delta_t, t + \delta_t)$,

$$\left\| \int_{\mathbb{R}} g(t, s, u(s)) - g(r, s, u(s))\, ds \right\| < \varepsilon, \quad \text{for every } u \in X. \tag{5.3}$$

The open cover $\{(t-\delta_t, t+\delta_t)\}_{\{t\in[-n,n]\}}$ has a finite subcover $\{(t_i-\delta_i, t_i+\delta_i)\}_{\{i=1,\ldots,l\}}$.

Now, take (u_k) a sequence in X converging to u_0. We have to show that $\|f(u_k)-f(u_0)\|_n \to 0$. From (H1) and (H2), we have that for every $i \in \{1, \ldots, l\}$, there exists $K_i \in \mathbb{N}$ such that for every $k \ge K_i$,

$$\|f(u_k)(t_i) - f(u_0)(t_i)\| < \varepsilon.$$

This inequality combined with (5.3) implies that for every $t \in [-n, n]$, and every $k \ge K = \max\{K_1, \ldots, K_l\}$,

$$\begin{aligned} \|f(u_k)(t) - f(u_0)(t)\| &\le \|f(u_k)(t) - f(u_k)(t_i)\| + \|f(u_k)(t_i) - f(u_0)(t_i)\| \\ &\quad + \|f(u_0)(t_i) - f(u_0)(t)\| \\ &< 3\varepsilon, \end{aligned}$$

with $t \in (t_i - \delta_i, t_i + \delta_i)$. Hence f is continuous.

On the other hand, let h_i be the function given by (H2) associated to t_i, $i \in \{1,\dots,l\}$. Again, in using (5.3), we deduce that for every $u \in X$ and every $t \in [-n,n]$,

$$\begin{aligned}\|f(u)(t)\| &\leq \|f(u)(t) - f(u)(t_i)\| + \|f(u)(t_i)\| \\ &\leq \varepsilon + \max\Big\{\|h_1\|_{L^1},\dots,\|h_l\|_{L^1}\Big\}.\end{aligned}$$

So that $f(X)|_{[-n,n]}$ is bounded in $C([-n,n],\mathbb{R}^N)$. It is equicontinuous by (H3). The compacity of f follows from Arzela-Ascoli's Theorem.

Finally, for every $i \in \{1,\dots,l\}$, there exits $r_i > 0$ such that

$$\|h_i\|_{L^1(\mathbb{R}\setminus[-r_i,r_i])} < \varepsilon.$$

Take $m \in \mathbb{N}$ such that $m \geq \max\{r_1,\dots,r_l\}$. It follows that for every $t \in [-n,n]$, and every $u,v \in X$ such that $\|u-v\|_m = 0$,

$$\begin{aligned}\|f(u)(t) - f(v)(t)\| &\leq \|f(u)(t) - f(u)(t_i)\| + \|f(u)(t_i) - f(v)(t_i)\| \\ &\quad + \|f(v)(t_i) - f(v)(t)\| \\ &< 2\varepsilon + \left\| \int_{\mathbb{R}\setminus[-m,m]} g(t_i,s,u(s)) - g(t_i,s,v(s))\, ds \right\| \\ &\leq 2\varepsilon + 2\|h_i\|_{L^1(\mathbb{R}\setminus[-m,m])} \\ &< 4\varepsilon,\end{aligned}$$

with $t \in (t_i - \delta_i, t_i + \delta_i)$. Thus,

$$\sup_{u\in X} \operatorname{diam}_n\Big(f(S_m(u))\Big) < 4\varepsilon.$$

■

Proposition 5.5. *Let $M : \mathbb{R} \to (0,\infty)$ be a continuous function, and g a function satisfying (H1)–(H3) with $X = \{u \in C(\mathbb{R},\mathbb{R}^N) : \|u(t)\| \leq M(t)\ \forall t \in \mathbb{R}\}$. Then f is admissible.*

Proof. For every $n \in \mathbb{N}$,

$$X_n = \overline{X_n} = \{u \in C([-n,n],\mathbb{R}^N) : \|u(t)\| \leq M(t)\ \forall t \in [-n,n]\,\},$$

and the function $S_n^* : X_n \to X$ defined by

$$S_n^*(u) = \{v \in X : v \text{ is a continuous extension of } u\}$$

is continuous. Therefore $f \circ S_n^*$ is continuous and compact by Lemmas 2.4 and 5.4, and hence $\overline{f \circ S_n^*}$ is continuous by Lemma 2.5.

On the other hand, for $u \in X_n$,

$$\overline{f \circ S_n^*}(u)(t) = \int_{\mathbb{R}} G_n(t,s,u(s))\, ds,$$

where $G_n : [-n,n] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is given by

$$G_n(t,s,x) = \begin{cases} g(t,s,x), & \text{if } |s| \le n, \\ \{g(t,s,y) : \|y\| \le M(s)\}, & \text{if } |s| > n. \end{cases}$$

From (H1) and (H2), we deduce that G_n has compact values, $s \mapsto G_n(t,s,x)$ is measurable, and $x \mapsto G_n(t,s,x)$ is continuous. So, it follows from Lemmas 2.6 and 2.8 that for every $u \in X_n$ and every $t \in [-n,n]$, $\overline{f \circ S_n^*(u)(t)}$ is convex, and hence $\overline{f \circ S_n^*}$ has convex, compact values. So, $F_n = \boldsymbol{F}_n = \overline{f \circ S_n^*}$.

If follows from Lemma 5.4 that f satisfies condition (2) of Definition 4.1. Thus, f is admissible. ■

Theorem 5.6. *Let $g : \mathbb{R}^{N+2} \to \mathbb{R}^N$ be a continuous function. Assume that there exist $h, l \in C(\mathbb{R}^2, [0,\infty))$ such that for every $t \in \mathbb{R}$, $h(t,\cdot) \in L^1(\mathbb{R})$, $t \mapsto \|h(t,\cdot)\|_{L^1}$ is continuous, $l(t,s) = 0$ for $|s| \ge |t|$ and*

$$\|g(t,s,u)\| \le h(t,s) + l(t,s)\|u\|.$$

Then the integral equation (5.2) *has a solution.*

Proof. Let $M \in C(\mathbb{R},(0,\infty))$ be a function which will be determined later, and $X = \{u \in C(\mathbb{R},\mathbb{R}^N) : \|u(t)\| \le M(t)$ for every $t \in \mathbb{R}\}$. It is easy to verify that (H1)–(H3) are satisfied.

Let $n \in \mathbb{N}$, and $T \in [0,n]$. Assume that for some $u \in X_n$ and some $\lambda \in (0,1]$, $u \in \lambda \boldsymbol{F}_n(u)$. Then, for all $t \in [-T,T]$,

$$\begin{aligned} \|u(t)\| &\le \int_{-|t|}^{|t|} h(t,s) + l(t,s)\|u(s)\|\,ds + \int_{[-|t|,|t|]^c} h(t,s)\,ds \\ &\le \int_{-|t|}^{|t|} m_T(s) + k_T\|u(s)\|\,ds + a_T, \end{aligned}$$

where

$$\begin{aligned} m_T(s) &= \max\Big\{h(t,s) : t \in [-T,-|s|] \cup [|s|,T]\Big\}, \\ k_T &= \max\Big\{l(t,s) : (t,s) \in [-T,T] \times \mathbb{R}\Big\}, \end{aligned}$$

and

$$a_T = \sup\Big\{\|h(t,\cdot)\|_{L^1([-|t|,|t|]^c)} : t \in [-T,T]\Big\}.$$

So, for all $t \in [0,T]$,

$$z(t) \le a_T + 2\int_0^t m_T(s) + k_T z(s)\,ds,$$

with $z(t) = \max\big\{\|u(t)\|, \|u(-t)\|\big\}$. By Gronwall's inequality, we deduce that for every $t \in [0,T]$,

$$z(t) \le a_T e^{2k_T t} + 2\int_0^t m_T(s) e^{2k_T(t-s)}\,ds.$$

Fix $M \in C(\mathbb{R},(0,\infty))$ an even function such that for $t \ge 0$

$$M(t) > a_t e^{2k_t t} + 2\int_0^t m_t(s) e^{2k_t(t-s)}\,ds.$$

The conclusion follows from Corollary 4.5, Proposition 5.5, and the choice of M. ■

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