On the asymptotic behaviour of the solutions of $-(r(t)u')' + p(t)u = 0$, where p is not of constant sign [*]

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1 Introduction

We consider a differential equation of the form

$$Lu = -(r(t)u')' + p(t)u = 0,$$

where r is a (strictly) positive continuous function on an open interval $]a, b[$ in the real numbers $\mathbb{R}$ ($a = -\infty$ and/or $b = +\infty$ are allowed) and p is a locally integrable function on $]a, b[$. In [9] and [1], we can find a complete study of the asymptotic behaviour of the solutions of $Lu = 0$, in the neighborhood of b, in the special case when $p \geq 0$. In [2], we can find a similar study for the case when $p \leq 0$. Note that in these papers, the authors assume that p is continuous, does not vanish (in the neighborhood of b) and that $b = +\infty$, but it is easily seen that most of their results remain true in the more general situation considered here. We just point out that we cannot use the "duality principle" of [1] and [2], because our hypotheses on $1/r$ and p are not symmetric. Nevertheless, when a result of [1] or [2] is proved by means of this principle, it is always possible to give a direct proof avoiding this principle.

Some results on the asymptotic behaviour of solutions are also known when p is not necessarily of constant sign (see [7], XI, 9 or [3], for example).

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In the following, we show by means of examples that if p changes of sign in every neighborhood of b, then the solutions of $Lu = 0$ are not necessarily monotone near b (contrarily to the case when p is of constant sign). Next we use results of [1], [2], [9] and [10] in order to obtain, by a very simple method, new results in the case when p is not of constant sign.

This simple method (see [7], p. 330, for example) consists in observing that if a function φ satisfies $\varphi(t) \neq 0$ on some subinterval J of $]a,b[$ and if we define $v = u/\varphi$, then

$$\mathcal{L}v = -(r\varphi^2 v')' + (\varphi L\varphi)v = \varphi Lu.$$

It follows that u is a solution on J of $Lu = 0$ if and only if v is a solution of $\mathcal{L}v = 0$.

This method has already been used in [8] and [4] (resp. [6]) in order to transform a situation where $1/r \notin L^1(a,b)$ (resp. $p \not\geq 0$) in a situation where $1/r \in L^1(a,b)$ (resp. $p \geq 0$). In the following, we take for φ a solution of $-(r\varphi')' - p^-\varphi = 0$, $-(r\varphi')' + p^+\varphi = 0$ or $-(r\varphi')' + |p|\varphi = 0$, giving respectively $\varphi L\varphi = p^+\varphi^2$, $-p^-\varphi^2$ or $-2p^-\varphi^2$. We shall also consider $\varphi(t) = \int_t^b ds/r(s)$, in order to transform a situation where $1/r \in L^1(b]$ (see the preliminaries) in a situation where $1/r \notin L^1(b]$.

2 Preliminaries

We work on an interval $]a,b[$ of $\mathbb{R}$ $(-\infty \leq a < b \leq +\infty)$ and we consider

- a continuous function $r :]a,b[\to]0,+\infty[$;
- a function $p \in L^1_{loc}(a,b)$;
- the Sturm-Liouville operator

$$L : W^{(2,r)} \to L^1_{loc}(a,b) : Lu = -(r(t)u')' + p(t)u,$$

where $W^{(2,r)}$ is the set of all $u \in \mathcal{C}(]a,b[)$ such that u has a weak derivative u' and ru' a weak derivative $(ru')'$. Note that $W^{(2,r)} \subset \mathcal{C}^1(]a,b[)$.

We define :

$$\begin{aligned}
&L^1[a) = \{u \in L^1_{loc}(a,b)\,;\, u \in L^1(a,c) \text{ for } a < c < b\} \ ;\\
&L^1(b] = \{u \in L^1_{loc}(a,b)\,;\, u \in L^1(c,b) \text{ for } a < c < b\} \ ;\\
&r_1 :]a,b[\to \mathbb{R} : r_1(t) = \int_a^t \frac{ds}{r(s)}, \text{ when } 1/r \in L^1[a) \ ;\\
&r_2 :]a,b[\to \mathbb{R} : r_2(t) = \int_t^b \frac{ds}{r(s)}, \text{ when } 1/r \in L^1(b] \ ;\\
&r_3 :]a,b[\to \mathbb{R} : r_3(t) = \int_{\overline{c}}^t \frac{ds}{r(s)}, \text{ with } \overline{c} \text{ arbitrarily fixed in }]a,b[\,.
\end{aligned}$$

When $1/r \in L^1(b]$ (resp. $1/r \notin L^1(b]$), we define :

$$\mathcal{A}_b = \{h \in L^1_{loc}(a,b)\,;\, hr_2 \text{ (resp. } hr_3) \in L^1(b]\}.$$

We obviously have : $L^1(b] \subset \mathcal{A}_b$ (resp. $\mathcal{A}_b \subset L^1(b]$). We also define :

$$\begin{aligned}
&S = \{u \in W^{(2,r)}\, ;\, u \neq 0 \text{ and } Lu = 0\}\ ;\\
&S_0 = \{u \in S\, ;\, \lim_{t\to b} u(t) = 0\}\ ;\\
&S_B = \{u \in S\, ;\, \lim_{t\to b} u(t) \text{ exists, finite, } \neq 0\}\ ;\\
&S_\infty = \{u \in S\, ;\, \lim_{t\to b} u(t) = \pm\infty\}\ ;\\
&M^+ = \{u \in S\, ;\, \exists t_u \in]a,b[\, , \forall t \in [t_u, b[\, , u(t)u'(t) > 0\}\ ;\\
&M^- = \{u \in S\, ;\, \exists t_u \in]a,b[\, , \forall t \in [t_u, b[\, , u(t)u'(t) < 0\}\ ;\\
&O = \{u \in S\, ;\, u \text{ is oscillatory near } b\}.
\end{aligned}$$

We recall that a nonzero solution u of $Lu = 0$ is said to be *oscillatory (near b)* if there exists a sequence (t_n) in $]a,b[$ such that $u(t_n) = 0$ and $t_n \to b$. It is well-known that $S = O$ or $O = \emptyset$ (see [10], for example). When $S = O$, the equation $Lu = 0$ is said to be *oscillatory (near b)*.

For $\alpha \in \{0, B, \infty\}$ and $\beta \in \{+, -\}$, let : $M_\alpha^\beta = S_\alpha \cap M^\beta$. The sets M_0^+ and M_∞^- being obviously void, we thus have :

$$M^+ = M_B^+ \cup M_\infty^+,\ \ M^- = M_0^- \cup M_B^-.$$

We finally denote by N_0, N_B and N_∞ the set of all $u \in S$ such that $\lim_{t\to b} r(t)u'(t)$ exists and is respectively zero, finite and nonzero, infinite. If necessary, we shall write $S(L)$, $S_0(L)$, ... , instead of S, S_0, ...

In Theorem 2.1 below, we collect the results of [9], Theorems 1, 4, Lemma 2 and of [1], Theorems 1 to 3, Lemma 1. In Theorem 2.2, we collect the results of [2], Theorems 1 and 2.

Theorem 2.1. *Assume $p \geq 0$ (almost everywhere) and that, for all $c \in]a,b[$, the measure of $\{t \in]c,b[\, ;\, p(t) \neq 0\}$ is > 0. Then :*

- *$S = M^+ \cup M^-$, $\emptyset \neq M^+ \subset N_B \cup N_\infty$ and $\emptyset \neq M^- \subset N_0 \cup N_B$;*
- *for all $u \in M^-$ and all $t \in]a,b[$, we have $u(t)u'(t) < 0$.*

Moreover, when $p \in \mathcal{A}_b$ and

- *$1/r \in L^1(b]$, $p \in L^1(b]$, we have : $S = M_B^+ \cup M^-$, $M_0^- \neq \emptyset$, $M_B^- \neq \emptyset$, $M_B^+ \subset N_B$ and $M_0^- \subset N_B$;*
- *$1/r \in L^1(b]$, $p \notin L^1(b]$, we have : $S = M_B^+ \cup M_0^-$, $M_B^+ \subset N_\infty$ and $M_0^- \subset N_B$;*
- *$1/r \notin L^1(b]$, we have : $S = M_\infty^+ \cup M_B^-$, $M_\infty^+ \subset N_B$ and $M_B^- \subset N_0$.*

Finally, when $p \notin \mathcal{A}_b$, we have : $S = M_\infty^+ \cup M_0^-$, $M_\infty^+ \subset N_\infty$ and $M_0^- \subset N_0$.

Theorem 2.2. *Assume $p \leq 0$ (almost everywhere) and that, for all $c \in]a,b[$, the measure of $\{t \in]c,b[\, ;\, p(t) \neq 0\}$ is > 0. Then, when $p \in \mathcal{A}_b$ and*

- *$1/r \in L^1(b]$, $p \in L^1(b]$, we have : $S = M_B^+ \cup M^-$, $M_B^+ \subset N_0 \cup N_B$, $M^- \subset N_B$ and M_B^-, M_0^-, $M_B^+ \cap N_0$, $M_B^+ \cap N_B \neq \emptyset$;*
- *$1/r \in L^1(b]$, $p \notin L^1(b]$, we have : $S = M^-$, $\emptyset \neq M_B^- \subset N_\infty$ and $\emptyset \neq M_0^- \subset N_B$;*
- *$1/r \notin L^1(b]$, we have : $S = M^+$, $\emptyset \neq M_B^+ \subset N_0$ and $\emptyset \neq M_\infty^+ \subset N_B$.*

When $p \notin \mathcal{A}_b$ and

- *$1/r \in L^1(b]$, we have : $S = O$ or $S = M_0^- = N_\infty$;*
- *$1/r \notin L^1(b]$, $p \in L^1(b]$, we have : $S = O$ or $S = M_\infty^+ = N_0$;*
- *$1/r \notin L^1(b]$, $p \notin L^1(b]$, we have : $S = O$.*

Six cases have been considered in Theorem 2.2. They respectively correspond to cases (C_6), (C_5), (C_3), (C_4), (C_2) and (C_1) of [2]. In the next theorem, we collect the results of [3], Propositions 2.2 and 2.3 (see also the proof of [6], Proposition 2.4), where it is not necessary to assume that $]a, b[$ is bounded. Some of these results are also contained in [7], XI, 9.

Theorem 2.3. *Without any hypothesis on the sign of p, when $p \in \mathcal{A}_b$ and*

- $1/r \in L^1(b)$, $p \in L^1(b)$, *we have :* $S = M_0^- \cup S_B$, $\emptyset \neq M_0^- \subset N_B$ *and* $\emptyset \neq S_B \subset N_0 \cup N_B$;
- $1/r \in L^1(b)$, $p \notin L^1(b)$, *we have :* $S = M_0^- \cup S_B$, $\emptyset \neq M_0^- \subset N_B$ *and* $S_B \neq \emptyset$;
- $1/r \notin L^1(b)$, *we have :* $S = S_B \cup S_\infty$, $\emptyset \neq S_B \subset N_0$ *and* $\emptyset \neq S_\infty \subset N_B$.

Assuming that $p \geq 0$ or $p \leq 0$ (almost everywhere) in $]a, b[$ and that, for all $c \in]a, b[$, the measure of $\{t \in]c, b[\,;\, p(t) \neq 0\}$ is > 0, it follows from Theorems 2.1 and 2.2 that $S = M^+ \cup M^-$ or $S = O$ (not possible if $p \geq 0$). Roughly speaking, If p does not change of sign, all the solutions of a nonoscillatory equation $Lu = 0$ are monotone near b, with a nonoscillatory derivative (direct easy proofs are given in [9] and [2]). We now show by means of three examples that the situation is different when p changes of sign in every neighborhood of b.

Example 2.4. Let $]a, b[\, = \,]0, +\infty[$, $r(t) = 1$ and $p(t) = -\sin t/(\sin t + 2)$. The equation $Lu = 0$ is nonoscillatory (near $+\infty$) since $u(t) = \sin t + 2$ is a solution. The derivative $u'(t) = \cos t$ is oscillatory since $u'(t_n) = 0$ with $t_n = 2n\pi + \pi/2 \to +\infty$. We now show that every solution w of this equation has also an oscillatory derivative. There exists α, $\beta \in \mathbb{R}$ such that

$$w(t) = \alpha u(t) + \beta u(t) \int_0^t \frac{ds}{u^2}, \quad w'(t) = u'(t) \left[\alpha + \beta \int_0^t \frac{ds}{u^2}\right] + \frac{\beta}{u(t)}.$$

We may assume $\beta \neq 0$. Letting $t'_n = 2n\pi + \pi$, we have

$$w'(t_n) = \frac{\beta}{3}, \quad w'(t'_n) = -\left[\alpha + \beta \int_0^{t'_n} \frac{ds}{u^2}\right] + \frac{\beta}{2} \to (-\beta)(+\infty).$$

This shows that the derivative of each solution is not only oscillatory, but it changes of sign in every neighborhood of $+\infty$. Consequently, no solution is monotone near $+\infty$.

Example 2.5. Let $]a, b[\, = \,]1, +\infty[$, $r(t) = 1/t$ and $p(t) = -(t \sin t + \cos t + 2)/t^2(\sin t + 2t)$. The equation $Lu = 0$ is nonoscillatory (near $+\infty$) since $u(t) = \sin t + 2t$ is a solution. The derivative $u'(t) = \cos t + 2$ is nonoscillatory. Each solution w has also a nonoscillatory derivative. Indeed, writing

$$w(t) = \alpha u(t) + \beta u(t) \int_1^t \frac{ds}{ru^2} \quad \text{(we may assume } \beta \neq 0\text{)},$$

we have :

$$w'(t) = u'(t) \left[\alpha + \beta \int_1^t \frac{ds}{ru^2}\right] + \frac{\beta}{r(t)u(t)} \to \beta(+\infty), \quad \text{when } t \to +\infty.$$

It follows that each solution of $Lu = 0$ is monotone near $+\infty$.

Example 2.6. Let $]a, b[=]1, +\infty[$, $r(t) = 1$ and $p(t) = -\sin t/(\sin t + t)$. The equation $Lu = 0$ is nonoscillatory (near $+\infty$) since $u(t) = \sin t + t$ is a solution. The derivative $u'(t) = \cos t + 1$ is oscillatory but does not change of sign (u is increasing). If $\alpha \geq 0$ and $\beta > 0$, the solution

$$w(t) = \alpha u(t) + \beta u(t) \int_1^t \frac{ds}{u^2}$$

is increasing and has a nonoscillatory derivative since

$$w'(t) = u'(t) \left[\alpha + \beta \int_1^t \frac{ds}{u^2}\right] + \frac{\beta}{u(t)} > 0.$$

But if α and β are chosen such that $\alpha + \beta K < 0$ and $\beta > 0$, where $K = \int_1^\infty ds/u^2 < +\infty$, the solution w has an oscillatory derivative changing of sign in every neighborhood of $+\infty$ (hence w is not monotone near $+\infty$). It suffices indeed to define $t_n = 2n\pi - \pi$, $t_n' = 2n\pi + \pi/2$ and to observe that :

$$w'(t_n) = \frac{\beta}{2n\pi - \pi} > 0,$$

$$w'(t_n') \leq \alpha + \beta K + \frac{\beta}{1 + 2n\pi + \pi/2} < 0, \quad \text{for } n \text{ large enough.}$$

We recall that if the equation $Lu = 0$ is nonoscillatory (near b), there always exists a solution u such that $\int_c^b ds/ru^2 = +\infty$, where $c \in]a, b[$ exceeds the largest zero of u ; such a solution is uniquely determined up to a constant factor and is called a *principal solution* (see [7], p. 355, Theorem 6.4, where the continuity of $q = -p$ is not necessary). We define :

$$\begin{aligned}
P &= \{u \in S\,;\, u \text{ is a principal solution}\}\ ;\\
\tilde{M}^+ &= \{u \in S\,;\, \exists t_u \in]a, b[\,, \forall t \in [t_u, b[\,,\ u(t)u'(t) \geq 0\}\ ;\\
\tilde{M}^- &= \{u \in S\,;\, \exists t_u \in]a, b[\,, \forall t \in [t_u, b[\,,\ u(t)u'(t) \leq 0\}.
\end{aligned}$$

Since each solution u is nonoscillatory, we may choose t_u such that $u(t) \neq 0$ for all $t \in [t_u, b[$. The example 2.5 is a special case of (i) in the next result.

Proposition 2.7. *If the equation $Lu = 0$ is nonoscillatory, we then have :*

(i) $P \cap \tilde{M}^+ \neq \emptyset \Rightarrow S = \tilde{M}^+$ *(and $P \cap M^+ \neq \emptyset \Rightarrow S = M^+$) ;*

(ii) $\tilde{M}^- \neq \emptyset \Rightarrow P \subset \tilde{M}^-$ *(and $M^- \neq \emptyset \Rightarrow P \subset M^-$).*

Proof. (i) Let $u \in P \cap \tilde{M}^+$. We may assume that $u(t) > 0$ and $u'(t) \geq 0$ for all $t \in [t_u, b[$. Every solution w of $Lu = 0$, linearly independent of u, can be written (for $t \geq t_u$)

$$w(t) = u(t) \left[\alpha + \beta \int_{t_u}^t \frac{ds}{ru^2}\right] \quad (\beta \neq 0)$$

and then

$$w'(t) = u'(t) \left[\alpha + \beta \int_{t_u}^t \frac{ds}{ru^2}\right] + \frac{\beta}{r(t)u(t)}.$$

Since $\int_{t_u}^b ds/ru^2 = +\infty$, it follows that $w(t) \neq 0$ and $w'(t) \neq 0$ for t near b, with the same sign as β, and we conclude that $w \in \tilde{M}^+$.

(ii) Let $u \in \tilde{M}^-$. If $u \in P$, then $P \subset \tilde{M}^-$. If $u \notin P$, then $v(t) = u(t) \int_t^b ds/ru^2$ is in P ([7], p. 355, Corollary 6.3) and it is clear that $v \in \tilde{M}^-$. ∎

3 The asymptotic behaviour of the solutions

In order to study the asymptotic behaviour of the solutions of $Lu = 0$ in some situations where $p \notin \mathcal{A}_b$ and is not of constant sign, we consider the following conditions :

$(\mathcal{P}+)$ $\forall c \in]a, b[$, the measure of $\{t \in]c, b[\,;\, p(t) > 0\}$ is > 0 ;

$(\mathcal{P}-)$ $\forall c \in]a, b[$, the measure of $\{t \in]c, b[\,;\, p(t) < 0\}$ is > 0.

Theorem 3.1. *Assume that $p^- \in \mathcal{A}_b$ and $p^+ \notin \mathcal{A}_b$. Then, if*

(1) $1/r \in L^1(b]$, $p^- \in L^1(b]$, *we have :* $S = M_\infty^+ \cup M_0^-$, $\emptyset \neq M_\infty^+ \subset N_\infty$ *and* $\emptyset \neq M_0^- \subset N_0$ *;*

(2) $1/r \in L^1(b]$, $p^- \notin L^1(b]$, *we have :* $S = M_0^- \cup S_\infty$, $M_0^- \neq \emptyset$ *and* $S_\infty \neq \emptyset$ *;*

(3) $1/r \notin L^1(b]$, *we have :* $S = M_\infty^+ \cup S_0$, $\emptyset \neq M_\infty^+ \subset N_\infty$ *and* $\emptyset \neq S_0 \subset N_0$.

Proof. We may assume that condition $(\mathcal{P}-)$ is true. Indeed, if not, then $p^- = 0$ and $p = p^+$ almost everywhere in the neighborhood of b and the result follows from Theorem 2.1. Suppose now there exists a solution $\varphi \in \mathcal{C}(]a, b])$ of the equation

$$\Lambda\varphi = -(r\varphi')' - p^-\varphi = 0,$$

such that $\varphi(b) > 0$. Without lost of generality (since we are only interested by the behaviour of the solutions of $Lu = 0$ in the neighborhood of b), we may then assume that $\varphi(t) > 0$ for all $t \in]a, b]$. Letting $v = u/\varphi$, we immediately verify that

$$\mathcal{L}v = -(r\varphi^2 v')' + (p^+\varphi^2)v = \varphi Lu,$$

hence that u is a solution of $Lu = 0$ if and only if v is a solution of $\mathcal{L}v = 0$. It is easy to see that $1/r \in L^1(b]$ if and only if $1/r\varphi^2 \in L^1(b]$ and that r and $r\varphi^2$ define the same set $\mathcal{A}_b$.

(1) By Theorem 2.2, we may choose $\varphi = \varphi_1 \in M_B^+(\Lambda) \cap N_B(\Lambda)$ or $\varphi = \varphi_2 \in M_B^-(\Lambda) \cap N_B(\Lambda)$. We denote by $\mathcal{L}_k$ the operator $\mathcal{L}$ corresponding to $\varphi = \varphi_k$ ($k = 1, 2$). By Theorem 2.1, we know that for $k = 1, 2$, we have :

$$S(\mathcal{L}_k) = M_\infty^+(\mathcal{L}_k) \cup M_0^-(\mathcal{L}_k),$$

$$\emptyset \neq M_\infty^+(\mathcal{L}_k) \subset N_\infty(\mathcal{L}_k),\ \emptyset \neq M_0^-(\mathcal{L}_k) \subset N_0(\mathcal{L}_k).$$

Given functions u and v such that $v = u/\varphi_k$, the following equivalences hold

$$u \in S \Leftrightarrow v \in S(\mathcal{L}_k),\ \ u \in S_\infty \Leftrightarrow v \in M_\infty^+(\mathcal{L}_k),\ \ u \in S_0 \Leftrightarrow v \in M_0^-(\mathcal{L}_k),$$

and it follows obviously that $S = S_\infty \cup S_0$, $S_\infty \neq \emptyset$ and $S_0 \neq \emptyset$.

Given $u \in S_\infty$, we have $v = u/\varphi_1 \in M_\infty^+(\mathcal{L}_1) \cap N_\infty(\mathcal{L}_1)$ and, since $u' = v'\varphi_1 + v\varphi_1'$ with $\varphi_1 \in M_B^+(\Lambda) \cap N_B(\Lambda)$, it follows that $u \in M_\infty^+ \cap N_\infty$. In the same way, if $u \in S_0$, then $v = u/\varphi_2 \in M_0^-(\mathcal{L}_2) \cap N_0(\mathcal{L}_2)$ with $\varphi_2 \in M_B^-(\Lambda) \cap N_B(\Lambda)$ and, consequently, $u \in M_0^- \cap N_0$. The result (1) follows.

(2) The proof is similar but, here, using Theorem 2.2, we can only choose $\varphi \in M_B^-(\Lambda) \cap N_\infty(\Lambda)$ and, by Theorem 2.1, we have $S(\mathcal{L}) = M_\infty^+(\mathcal{L}) \cup M_0^-(\mathcal{L})$.

(3) Here, we can only choose $\varphi \in M_B^+(\Lambda) \cap N_0(\Lambda)$ and we have again $S(\mathcal{L}) = M_\infty^+(\mathcal{L}) \cup M_0^-(\mathcal{L})$. ∎

Corollary 3.2. *If $p^- \in \mathcal{A}_b$, then the equation $Lu = 0$ is nonoscillatory.*

Proof. If $p \in \mathcal{A}_b$, it follows from Theorem 2.3 that the equation $Lu = 0$ is not oscillatory. If $p^+ \notin \mathcal{A}_b$ and $p^- \in \mathcal{A}_b$, this follows from Theorem 3.1. ∎

Remark 3.3. More generally, if the equation $-(r\varphi')' - p^-\varphi = 0$ is nonoscillatory, then $Lu = 0$ is also nonoscillatory. That follows easily from the beginning of the proof of Theorem 3.1 (since $p^+\varphi^2 \geq 0$), but it is also a consequence of the Sturm's comparison theorem ([10], Theorem 1.1).

Theorem 3.4. *Assume that $p^+ \in \mathcal{A}_b$ and $p^- \notin \mathcal{A}_b$. Then, if*
(1) *$1/r \in L^1(b]$, $p^+ \in L^1(b]$, we have : $S = O$ or $S = M_0^- = N_\infty$;*
(2) *$1/r \in L^1(b]$, $p^+ \notin L^1(b]$, we have : $S = O$ or $S = S_0$;*
(3) *$1/r \notin L^1(b]$, $p^- \in L^1(b]$, we have : $S = O$ or $S = S_\infty$;*
(4) *$1/r \notin L^1(b]$, $p^- \notin L^1(b]$, we have : $S = O$.*

Proof. We may assume that condition $(\mathcal{P}+)$ is true. Indeed, if not, then $p^+ = 0$ and $p = -p^-$ almost everywhere in the neighborhood of b and the result follows from Theorem 2.2. Let $\varphi \in \mathcal{C}(]a, b])$ be a solution of the equation

$$\Lambda\varphi = -(r\varphi')' + p^+\varphi = 0,$$

such that $\varphi(b) > 0$. We may assume that $\varphi(t) > 0$ for all $t \in]a, b]$. Letting $v = u/\varphi$, we have

$$\mathcal{L}v = -(r\varphi^2 v')' - (p^-\varphi^2)v = \varphi Lu$$

and, consequently, u is a solution of $Lu = 0$ if and only if v is a solution of $\mathcal{L}v = 0$.

(1) By Theorem 2.1, we may choose $\varphi \in M_B^-(\Lambda) \cap [N_0(\Lambda) \cup N_B(\Lambda)]$. By Theorem 2.2, we know that

$$S(\mathcal{L}) = O(\mathcal{L}) \text{ or } S(\mathcal{L}) = M_0^-(\mathcal{L}) \subset N_\infty(\mathcal{L}).$$

The result (1) follows.

(2) We can only choose $\varphi \in M_B^+(\Lambda) \cap N_\infty(\Lambda)$ and we have :

$$S(\mathcal{L}) = O(\mathcal{L}) \text{ or } S(\mathcal{L}) = M_0^-(\mathcal{L}) \subset N_\infty(\mathcal{L}).$$

(3) We can only choose $\varphi \in M_B^-(\Lambda) \cap N_0(\Lambda)$ and we have :

$$S(\mathcal{L}) = O(\mathcal{L}) \text{ or } S(\mathcal{L}) = M_\infty^+(\mathcal{L}) \subset N_0(\mathcal{L}).$$

(4) We can choose $\varphi \in M_B^-(\Lambda)$ and we have $S(\mathcal{L}) = O(\mathcal{L})$. ∎

Remark 3.5. The result (4) of Theorem 3.4 can be stated as follows :

$$1/r \notin L^1(b], p^+ \in \mathcal{A}_b \text{ and } p^- \notin L^1(b] \Rightarrow Lu = 0 \text{ is oscillatory.}$$

On the other hand, the proof of Theorem 2.2 uses the next Leighton's result :

$$1/r \notin L^1(b] \text{ and } \lim_{t\to b}\int_c^t p(s)\,ds = -\infty \Rightarrow Lu = 0 \text{ is oscillatory}$$

(c is any fixed member of]a,b[). This result, without any hypothesis on the sign of p, can be found in [10], Theorem 2.24 (with stronger hypotheses than here, but the proof continues to operate). Since, we obviously have

$$p^+ \in L^1(b], p^- \notin L^1(b] \Rightarrow \lim_{t\to b} \int_c^t p(s)\,ds = -\infty \ \ (\Rightarrow p^- \notin L^1(b]),$$

the result (4) of Theorem 3.4 can be improved as follows :

$$1/r \notin L^1(b], p^+ \in L^1(b] \text{ and } p^- \notin L^1(b] \Rightarrow Lu = 0 \text{ is oscillatory.}$$

Examples 3.6. In the result (4) of Theorem 3.4, improved as above, the condition $p^+ \in L^1(b]$ cannot be omitted. There exist, indeed, examples such that

$$1/r \notin L^1(b], p^+ \notin L^1(b], p^- \notin L^1(b] \text{ and } O = \emptyset.$$

We may take $]a, b[=]0, 1[$, $r(t) = (1-t)^2$ and

$$p(t) = \frac{-\frac{1}{(1-t)^2} \sin \frac{1}{1-t}}{2 + \sin \frac{1}{1-t}}.$$

The equation $-(ru')' + p(t)u$ is not oscillatory near 1 since $u(t) = 2 + \sin(1/(1-t))$ is a solution. We may also consider example 2.4.

In view of the next Theorem 3.7 we assume, this time, that $p \notin \mathcal{A}_b$ and that condition $(\mathcal{P}-)$ is satisfied. Considering the equation

$$\Lambda\varphi = -(r\varphi')' + |p|\varphi = 0,$$

it follows from Theorem 2.1 that

$$S(\Lambda) = M_\infty^+(\Lambda) \cup M_0^-(\Lambda),$$

$$\emptyset \neq M_\infty^+(\Lambda) \subset N_\infty(\Lambda), \ \emptyset \neq M_0^-(\Lambda) \subset N_0(\Lambda).$$

We choose $0 \leq \varphi \in M_0^-(\Lambda) \cap N_0(\Lambda)$ and, by Theorem 2.1 again, we know that $\varphi(t)$, $-\varphi'(t) > 0$ for all $t \in]a, b[$. We also consider the equation

$$\mathcal{L}v = -(r\varphi^2 v')' - (2p^- \varphi^2)v = 0.$$

As before, u is a solution of $Lu = 0$ if and only if $v = u/\varphi$ is a solution of $\mathcal{L}v = 0$ but, here, since $\varphi(b) = 0$, the asymptotic behaviours of u and v cannot be easily compared, except the fact that u is oscillatory if and only if v is oscillatory.

We recall that $r_3(t)$ is defined by means of a fixed member $\overline{c}$ of $]a, b[$ (see the introduction). We now define :

$$R_3(t) = \int_{\overline{c}}^t \frac{ds}{r(s)\varphi(s)^2}.$$

Since $(\varphi^2)' = 2\varphi\varphi' < 0$, we have :

$$\varphi(t)^2 R_3(t) = \int_{\overline{c}}^t \frac{\varphi(t)^2\,ds}{r(s)\varphi(s)^2} \leq \int_{\overline{c}}^t \frac{\varphi(s)^2\,ds}{r(s)\varphi(s)^2} = r_3(t).$$

On the other hand, $\overline{\varphi}(t) = \varphi(t)R_3(t)$ is a solution of $\Lambda\varphi = 0$, linearly independent of φ. Hence $\overline{\varphi} \in M^+_\infty(\Lambda)$, which shows that $\lim_{t\to b} R_3(t) = +\infty$, i.e. $1/r\varphi^2 \notin L^1(b]$. The function $p^-\varphi^2$ is in the space $\mathcal{A}_b$ corresponding to $r\varphi^2$ if and only if $p^-\varphi^2 R_3 = p^-\varphi\overline{\varphi} \in L^1(b]$ or, equivalently, if $p^-\varphi\psi \in L^1(b]$, where ψ is any solution of $\Lambda\varphi = 0$, linearly independent of φ. Note also that $|p|\varphi \in L^1(b]$ is always true, hence $p^-\varphi^2 \in L^1(b]$, since $\varphi \in N_0(\Lambda)$ and

$$\int_{\overline{c}}^{t} |p|\varphi\, ds = \int_{\overline{c}}^{t} (r\varphi')'ds = r(t)\varphi'(t) - r(\overline{c})\varphi'(\overline{c}) \ \ (\overline{c} < t < b).$$

Now it follows from Theorem 2.2 that :

- $p^-\varphi\psi \in L^1(b] \Rightarrow S(\mathcal{L}) = M^+(\mathcal{L})$, $\emptyset \neq M^+_B(\mathcal{L}) \subset N_0(\mathcal{L})$ and $\emptyset \neq M^+_\infty(\mathcal{L}) \subset N_B(\mathcal{L})$;
- $p^-\varphi\psi \notin L^1(b] \Rightarrow S(\mathcal{L}) = O(\mathcal{L})$ or $S(\mathcal{L}) = M^+_\infty(\mathcal{L}) = N_0(\mathcal{L})$.

Unfortunately we can only deduce from this that :

$$p^-\varphi\psi \in L^1(b] \Rightarrow O(L) = \emptyset \text{ and } S_0(L) \cap N_0(L) \neq \emptyset.$$

The relations $p^-\varphi\overline{\varphi} = p^-\varphi^2 R_3 \leq p^- r_3$ show that the condition $p^- \in \mathcal{A}_b$ implies $p^-\varphi\psi \in L^1(b]$, when $1/r \notin L^1(b]$. This is also true when $1/r \in L^1(b]$ because, in this case, we have $p^-\varphi\overline{\varphi} \leq \lambda p^- r_2$ (in the neighborhood of b). Indeed, φ is a linear combination of $\overline{\varphi}$ and

$$\widetilde{\varphi}(t) = \overline{\varphi}(t)\int_t^b \frac{ds}{r(s)\overline{\varphi}(s)^2}$$

and it follows that $\varphi = \lambda\widetilde{\varphi}$ for some $\lambda > 0$, hence that

$$\varphi(t)\overline{\varphi}(t) = \lambda\overline{\varphi}(t)^2 \int_t^b \frac{ds}{r(s)\overline{\varphi}(s)^2} \leq \lambda \int_t^b \frac{\overline{\varphi}(s)^2\, ds}{r(s)\overline{\varphi}(s)^2} = \lambda r_2(t).$$

We have thus obtained the next result (in which we may now forget the condition $(\mathcal{P}-)$).

Theorem 3.7. *Assume that* $p \notin \mathcal{A}_b$ *and let* φ, ψ *be two solutions of the equation* $-(r\varphi')' + |p|\varphi = 0$ *such that* $\lim_{t\to b}\varphi(t) = 0$ *and* $\lim_{t\to b}\psi(t) = \pm\infty$. *Then :*

$$(p^- \in \mathcal{A}_b \Rightarrow)\ p^-\varphi\psi \in L^1(b] \Rightarrow O = \emptyset \ \textit{and}\ S_0 \cap N_0 \neq \emptyset.$$

Example 3.8. When $p \notin \mathcal{A}_b$, the condition $p^-\varphi\psi \in L^1(b]$ may be true even if $p^- \notin \mathcal{A}_b$. To show it, we modify the example 4.1 of [5]. Let $]a, b[=]0, 1[$, $r(t) = 1 - t$ and

$$p(t) = -\frac{t^2 - 2t + 2}{t^2(1-t)} < 0 \ \text{ if } \ t \in [1 - \alpha_n, 1 - \beta_n] \ \ (n = 1, 2, \ldots),$$
$$p(t) = \frac{t^2 - 2t + 2}{t^2(1-t)} > 0 \ \text{ in the other cases } \ (0 < t < 1),$$

where the sequences $\alpha_n = \exp\left(\frac{1}{2n^2} - n\right)$ and $\beta_n = \exp\left(-\frac{1}{2n^2} - n\right)$ have the following properties :

$$0 < \alpha_{n+1} < \beta_n < \alpha_n < 1 \ (n = 1, 2, \ldots), \ \lim \alpha_n = 0.$$

Since $1/r \in L^1[0)$ and $1/r \notin L^1(1]$, we have

$$\mathcal{A}_1 = \{h \in L^1_{loc}(0,1)\,;\, hr_1 \in L^1(1]\},$$

with

$$r_1(t) = \int_0^1 \frac{ds}{1-s} = -\ln(1-t) \;\; (0 \le t < 1).$$

We observe that $p \notin \mathcal{A}_1$ and it is easy to see that the functions

$$\varphi(t) = \frac{(1-t)(t+2)}{t}, \;\; \psi(t) = \frac{t^2}{1-t},$$

are solutions of the equation $\Lambda\varphi = -(r\varphi')' + |p|\varphi = 0$, with the required properties. Since $\varphi(t)\psi(t) = t(t+2)$, it follows that, for $h \in L^1_{loc}(0,1)$, the condition $h\varphi\psi \in L^1(1]$ will mean that $h \in L^1(1]$ and the condition $h \in \mathcal{A}_1$ will mean that $-h(t)\ln(1-t) \in L^1(1]$. Given the function

$$\begin{aligned} &q(t) = -\frac{1}{1-t} \;\text{ if }\; t \in [1-\alpha_n, 1-\beta_n] \;\; (n = 1, 2, ...), \\ &q(t) = 0 \;\text{ in the other cases }\; (0 < t < 1), \end{aligned}$$

we have :

$$\begin{aligned} &\int_{1-\alpha_n}^{1-\beta_n} q(t)\,dt = [-\ln(1-t)]_{1-\alpha_n}^{1-\beta_n} = \frac{1}{n^2}\;; \\ &\int_{1-\alpha_n}^{1-\beta_n} -q(t)\ln(1-t)\,dt = [\frac{1}{2}\ln^2(1-t)]_{1-\alpha_n}^{1-\beta_n} = \frac{1}{n}\;; \\ &\int_0^1 q(t)\,dt = \sum_{n=1}^{+\infty} \frac{1}{n^2} < +\infty\;; \\ &\int_0^1 -q(t)\ln(1-t)\,dt = \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty. \end{aligned}$$

Since $p^-(t) = q(t)(t^2 - 2t + 2)/t^2$, we conclude that $p^- \in L^1(1]$ and that $-p^-(t)\ln(1-t) \notin L^1(1]$. Note, finally, that Theorem 3.7 says that the equation $-(ru')' + pu = 0$ is not oscillatory near 1 and that it has nonzero solutions u such that $\lim_{t\to 1} u(t) = \lim_{t\to 1} r(t)u'(t) = 0$. The next theorem shows there also exist solutions u such that $\lim_{t\to 1} u(t) = \pm\infty$.

Theorem 3.9. *Assume that $p \notin \mathcal{A}_b$ and let φ, ψ be two solutions of the equation $\Lambda\varphi = -(r\varphi')' + |p|\varphi = 0$ such that $\lim_{t\to b}\varphi(t) = 0$ and $\lim_{t\to b}\psi(t) = \pm\infty$. Then :*

$$p^-\varphi\psi \in L^1(b] \text{ and } p^-\psi^2 \notin L^1(b] \Rightarrow S_\infty \neq \emptyset.$$

The conditions $p \notin \mathcal{A}_b$ and $p^-\psi^2 \notin L^1(b]$ are true, in particular, if $p^- \notin \mathcal{A}_b$.

Proof. We may assume that the condition $(\mathcal{P}-)$ is satisfied. By Theorem 2.1, we know that $\varphi \in M_0^-(\Lambda) \cap N_0(\Lambda)$ and $\psi \in M_\infty^+(\Lambda) \cap N_\infty(\Lambda)$. We may assume without lost of generality that $\psi(t) \neq 0$ for all $t \in]a, b[$. The function u is a solution of $Lu = 0$ if and only if $v = u/\psi$ is a solution of

$$\mathcal{L}v = -(r\psi^2 v')' - (2p^-\psi^2)v = 0.$$

It follows from [9], Theorem 2, that $1/r\psi^2 \in L^1(b]$ and we may define

$$R_2(t) = \int_t^b \frac{ds}{r(s)\psi(s)^2}.$$

Since $\overline{\psi}(t) = \psi(t)R_2(t)$ is a solution of $\Lambda\varphi = 0$, linearly independent of ψ, there exists real numbers α, β such that $\varphi = \alpha\psi + \beta\overline{\psi} = \psi(\alpha + \beta R_2)$, and this implies that $\alpha = 0$. It follows that $p^-\psi^2$ is in the space $\mathcal{A}_b$ corresponding to $r\psi^2$ if and only if $p^-\psi^2 R_2 = p^-\psi\overline{\psi}$ is in $L^1(b]$, hence if and only if $p^-\varphi\psi \in L^1(b]$. It remains to use the hypotheses and Theorem 2.2 to obtain $S(\mathcal{L}) = M^-(\mathcal{L})$, with $M_B^-(\mathcal{L}), M_0^-(\mathcal{L}) \neq \emptyset$, in order to conclude that $S_\infty(L) \neq \emptyset$.

Finally, if $1/r \in L^1(b]$, we obviously have $\lim_{t\to b}(\psi^2(t)/r_2(t)) = +\infty$ and, if $1/r \notin L^1(b]$, we obtain by the L'Hospital's rule $\lim_{t\to b}(\psi^2(t)/r_3(t)) = \lim_{t\to b} 2\psi(t)r(t)\psi'(t) = +\infty$. This shows that if $p^-\psi^2 \in L^1(b]$, then $p^- \in \mathcal{A}_b$. The proof is complete. ∎

4 Oscillation and nonoscillation criteria

Many results on oscillation and nonoscillation of solutions of second order differential equations are very often given (maybe only for convenience) in the situation when $b = +\infty$ and $r(t) = 1$ or, equivalently, by a classical change of variable (see the proof of Corollary 4.3 (1)), when $b \leq +\infty$ and $1/r \notin L^1(b]$. In this section, we show how to apply theses results to the case $1/r \in L^1(b]$ and to the case when p is not of constant sign.

Given $q \in L^1_{loc}(0, +\infty)$ such that $q(t) \leq 0$ almost everywhere, we consider the differential equation

$$lv = -v''(s) + q(s)v(s) = 0 \;\; (0 < s < +\infty)$$

and we define :

$$g(s) = s\int_s^{+\infty} |q(\sigma)|\, d\sigma \;\; (= +\infty \text{ if } q \notin L^1(+\infty]),$$

$$g_\star = \liminf_{s\to+\infty} g(s), \;\; g^\star = \limsup_{s\to+\infty} g(s).$$

The next result, due to Hille, can be found in [10], Theorem 2.1.

Theorem 4.1. (Hille) *If* $g^\star < 1/4$*, then the equation* $lv = 0$ *is nonoscillatory. If* $g_\star > 1/4$ *or* $g^\star > 1$*, then* $lv = 0$ *is oscillatory.*

An equation $-(r(t)u')' + p(t)u = 0$ is said to be *strongly oscillatory* (resp. *strongly nonoscillatory*) if, for all $\lambda > 0$, the equation $-(r(t)u')' + \lambda p(t)u = 0$ is oscillatory (resp. nonoscillatory). The next result, due to Nehari, can be found in [10], Theorem 2.9.

Theorem 4.2. (Nehari) *The above equation* $lv = 0$ *is strongly oscillatory if and only if* $g^\star = +\infty$*. It is strongly nonoscillatory if and only if* $\lim_{s\to+\infty} g(s) = 0$*.*

Corollary 4.3. (1) *If $1/r \notin L^1(b]$, $p \in L^1_{loc}(a,b)$ and $p \leq 0$ almost everywhere, then the conclusions of Theorems 4.1 and 4.2 are true for the equation $Lu = -(r(t)u')' + p(t)u = 0$, if we define g, $g_\star$ and $g^\star$ by*

$$g(t) = r_3(t)\int_t^b |p(\tau)|\,d\tau,\ \ g_\star = \liminf_{t\to b} g(t),\ \ g^\star = \limsup_{t\to b} g(t).$$

(2) *The same is true when $1/r \in L^1(b]$, if the function g is defined by*

$$g(t) = \frac{1}{r_2(t)}\int_t^b |p(\tau)| r_2(\tau)^2\,d\tau.$$

Proof. (1) The usual change of variable $s = r_3(t) = \int_{\overline{c}}^t ds/r(s)$ reduces the equation

$$-(r(t)u'(t))' + p(t)u(t) = 0\ \ (\overline{c} \leq t < b)$$

to the equation

$$-v''(s) + p(r_3^{-1}(s))r(r_3^{-1}(s))v(s) = 0\ \ (0 \leq s < +\infty).$$

The function g associated with this last equation is

$$g(s) = s\int_s^{+\infty} |p(r_3^{-1}(\sigma))| r(r_3^{-1}(\sigma))\,d\sigma$$

and, consequently :

$$\begin{aligned} g(t) = g(s) = g(r_3(t)) &= r_3(t)\int_{r_3(t)}^{+\infty} |p(r_3^{-1}(\sigma))| r(r_3^{-1}(\sigma))\,d\sigma \\ &= r_3(t)\int_t^b |p(\tau)|\,d\tau. \end{aligned}$$

(2) If $v = u/r_2$, we see that

$$\mathcal{L}v = -(rr_2^2 v')' + (pr_2^2)v = r_2 Lu$$

and, consequently, the equation $Lu = 0$ is (strongly) oscillatory if and only if the equation $\mathcal{L}v = 0$ is (strongly) oscillatory. Defining $R_3(t) = \int_{\overline{c}}^t ds/r(s)r_2(s)^2$, we have :

$$\begin{aligned} R_3(t) = -\int_{r_2(\overline{c})}^{r_2(t)} \frac{d\sigma}{\sigma^2} &= \frac{1}{r_2(t)} - \frac{1}{r_2(\overline{c})} \\ &= \frac{r_2(\overline{c}) - r_2(t)}{r_2(\overline{c})r_2(t)} = \frac{r_3(t)}{r_3(b)r_2(t)}. \end{aligned}$$

Hence $\lim_{t\to b} R_3(t) = +\infty$, i.e. $1/rr_2^2 \notin L^1(b]$. By (1), the function g associated with the equation $\mathcal{L}v = 0$ is $g(t) = R_3(t)\int_t^b |p| r_2^2\,d\tau$. Since $\lim_{t\to b}(r_3(t)/r_3(b)) = 1$, we may replace g by the one defined in (2). ∎

Remark 4.4. The proof of (2) shows that the next result, where p is not necessarily of constant sign, can be deduced from Leighton's criterion given in Remark 3.5 :

$$1/r \in L^1(b] \text{ and } \lim_{t\to b}\int_c^t p(s)r_2(s)^2\,ds = -\infty \Rightarrow Lu = 0 \text{ is oscillatory.}$$

This is a special case of a result due to Moore (see [10], p. 74).

Theorem 4.5. *Without any hypothesis on the integrability of* $1/r$ *or the sign of* p*, we assume that* $p \notin \mathcal{A}_b$*, we choose any solution* φ *of the equation* $-(r\varphi')' + |p|\varphi = 0$ *such that* $lim_{t\to b}\varphi(t) = 0$ *and we consider the function*

$$g(t) = \int_{\bar c}^t \frac{d\tau}{r(\tau)\varphi(\tau)^2}\int_t^b 2p^-(\tau)\varphi(\tau)^2\,d\tau.$$

The conclusions of Theorem 4.1 are then true for the equation $Lu = 0$*.*

Proof. This follows easily from the proof of Theorem 3.7 and from Corollary 4.3 (1). ∎

Remark 4.6. The condition $p^-\varphi\psi \in L^1(b]$ of Theorem 3.7 implies that $\lim_{t\to b} g(t) = 0$. Indeed, using the notations of the proof of Theorem 3.7, we obtain successively for $\bar c \le t < b$:

$$\begin{aligned}0 \le g(t) = R_3(t)\int_t^b 2p^-(\tau)\varphi(\tau)^2\,d\tau &\le 2\int_t^b p^-(\tau)\varphi(\tau)^2 R_3(\tau)\,d\tau\\ &= 2\int_t^b p^-\varphi\overline{\varphi}\,d\tau \to 0 \text{ (when } t\to b).\end{aligned}$$

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