m-systems of polar spaces and SPG reguli

D. Luyckx *

Abstract

It will be shown that every m-system of $W_{2n+1}(q)$, $Q^-(2n+1,q)$ or $H(2n,q^2)$ is an SPG regulus and hence gives rise to a semipartial geometry. We also briefly investigate the semipartial geometries, associated with the known m-systems of these polar spaces.

1 Introduction

A *partial m-system* $\mathcal{M}$ of a polar space $\mathcal{P}$ is a set of m-dimensional subspaces $\pi_1, \ldots, \pi_t$ of $\mathcal{P}$ such that each generator of $\mathcal{P}$ containing an element $\pi_i \in \mathcal{M}$ has an empty intersection with $(\pi_1 \cup \ldots \cup \pi_t) \setminus \pi_i$. Partial m-systems of polar spaces were introduced by Shult and Thas in [5]. They show that there exists an upper bound, which is independent of m, on the number of elements of a partial m-system and they call a partial m-system which meets this upper bound an *m-system.* We mention the size of an m-system $\mathcal{M}$ for the finite classical polar spaces:

$$\text{if} \quad \mathcal{P} = W_{2n+1}(q), \quad \text{then} \quad |\mathcal{M}| = q^{n+1} + 1, \tag{1}$$
$$\text{if} \quad \mathcal{P} = Q(2n,q), \quad \text{then} \quad |\mathcal{M}| = q^{n} + 1, \tag{2}$$
$$\text{if} \quad \mathcal{P} = Q^+(2n+1,q), \quad \text{then} \quad |\mathcal{M}| = q^{n} + 1, \tag{3}$$
$$\text{if} \quad \mathcal{P} = Q^-(2n+1,q), \quad \text{then} \quad |\mathcal{M}| = q^{n+1} + 1, \tag{4}$$
$$\text{if} \quad \mathcal{P} = H(2n,q^2), \quad \text{then} \quad |\mathcal{M}| = q^{2n+1} + 1, \tag{5}$$
$$\text{if} \quad \mathcal{P} = H(2n+1,q^2), \quad \text{then} \quad |\mathcal{M}| = q^{2n+1} + 1. \tag{6}$$

*The author is Research Assistant of the Fund for Scientific Research – Flanders (Belgium)(F.W.O.)

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The union of the elements of an m-system $\mathcal{M}$ will be denoted by $\tilde{\mathcal{M}}$. It can be shown that m-systems of certain polar spaces have two intersection numbers with hyperplanes.

Theorem 1.1 (Shult and Thas [5]). *Every m-system of a polar space $\mathcal{P} \in \{W_{2n+1}(q), Q^-(2n+1,q), H(2n,q^2)\}$ has two intersection numbers with respect to hyperplanes, namely:*

(a) If $\mathcal{P} = W_{2n+1}(q)$ and $H = p^\perp$, where $p^\perp$ denotes the unique image of p with respect to the symplectic polarity defining $W_{2n+1}(q)$, with p a point of $\tilde{\mathcal{M}}$, respectively p a point of $W_{2n+1}(q) \setminus \tilde{\mathcal{M}}$, then

$$|\tilde{\mathcal{M}} \cap H| = \frac{(q^{m+1}-1)(q^n+1)}{q-1} - q^n, \text{ respectively}$$
$$|\tilde{\mathcal{M}} \cap H| = \frac{(q^{m+1}-1)(q^n+1)}{q-1}.$$

(b) If $\mathcal{P} = Q^-(2n+1,q)$ and H is the tangent hyperplane of the quadric $Q^-(2n+1,q)$ at a point $p \in \tilde{\mathcal{M}}$, respectively the tangent hyperplane of $Q^-(2n+1,q)$ at a point $p \notin \tilde{\mathcal{M}}$ or a non-tangent hyperplane of $Q^-(2n+1,q)$, then

$$|\tilde{\mathcal{M}} \cap H| = \frac{(q^{m+1}-1)(q^n+1)}{q-1} - q^n, \text{ respectively}$$
$$|\tilde{\mathcal{M}} \cap H| = \frac{(q^{m+1}-1)(q^n+1)}{q-1}.$$

(c) If $\mathcal{P} = H(2n,q^2)$ and H is the tangent hyperplane of the hermitian variety $H(2n,q^2)$ at a point $p \in \tilde{\mathcal{M}}$, respectively the tangent hyperplane of $H(2n,q^2)$ at a point $p \notin \tilde{\mathcal{M}}$ or a non-tangent hyperplane of $H(2n,q^2)$, then

$$|\tilde{\mathcal{M}} \cap H| = \frac{(q^{2m+2}-1)(q^{2n-1}+1)}{q^2-1} - q^{2n-1}, \text{ respectively}$$
$$|\tilde{\mathcal{M}} \cap H| = \frac{(q^{2m+2}-1)(q^{2n-1}+1)}{q^2-1}.$$

Theorem 1.1 has the following corollary.

Corollary 1.2 (Shult and Thas [5]). *Every m-system of a polar space $\mathcal{P} \in \{W_{2n+1}(q), Q^-(2n+1,q), H(2n,q^2)\}$ defines a strongly regular graph and a two-weight code.*

2 A connection between m-systems and SPG reguli

An *SPG regulus* of $\mathsf{PG}(n,q)$ is a set R of m-dimensional subspaces $\pi_1,\dots,\pi_r$, $r>1$, of $\mathsf{PG}(n,q)$, satisfying:

SPG1 $\pi_i \cap \pi_j = \emptyset$ for all $i \neq j$.

SPG2 If $\mathsf{PG}(m+1,q)$ contains $\pi_i \in R$, then it has a point in common with either 0 or α ($\alpha > 0$) spaces in $R \setminus \{\pi_i\}$. If $\mathsf{PG}(m+1,q)$ has no point in common with $\pi_j \in R$ for all $j \neq i$, then it is called a tangent $(m+1)$-space of R at π_i.

SPG3 If the point x of $\mathsf{PG}(n,q)$ is not contained in an element of R, then it is contained in a constant number θ ($\theta \geq 0$) of tangent $(m+1)$-spaces of R.

In [6], Thas shows that for $n \neq 2m+1$, **SPG3** holds if conditions **SPG1** and **SPG2** are satisfied, and if also the following two conditions hold:

SPG3' At each $\pi_i \in R$, the union of all tangent $(m+1)$-spaces is a $\mathsf{PG}(n-m-1,q)$.

SPG4' $r = q^{(n+1)/2}+1$.

We now prove that for certain polar spaces, every m-system is an SPG regulus.

Theorem 2.1. *If $\mathcal{P} \in \{W_{2n+1}(q), Q^-(2n+1,q), H(2n,q^2)\}$, then all m-systems of $\mathcal{P}$ are SPG reguli of the ambient space of $\mathcal{P}$.*

Proof.
Let $\mathcal{M}$ be an m-system of a polar space $\mathcal{P}$, with $\mathcal{P} \in \{W_{2n+1}(q), Q^-(2n+1,q), H(2n,q^2)\}$ and denote its ambient space by $\mathsf{PG}(k,t)$, where $(k,t) \in \{(2n+1,q),(2n+1,q),(2n,q^2)\}$. Let π_m be an element of $\mathcal{M}$. For $\pi_{m-1} \subseteq \pi_m$ consider an $(m+1)$-dimensional subspace π_{m+1} of $\mathsf{PG}(k,t)$ containing π_{m-1} and meeting $\mathcal{P}$ in $\pi_{m-1}\mathcal{P}_1$, where $\mathcal{P}_1$ is the polar space $W_1(q)$, $Q^+(1,q)$ or $H(1,q^2)$ in the respective cases. Denote by X the number of points of $\tilde{\mathcal{M}}$ contained in $\pi_{m-1}\mathcal{P}_1$ and by Y the number of points of $\tilde{\mathcal{M}} \cap (\pi_{m+1}^{\perp} \setminus \pi_{m-1})$, with $\pi_{m+1}^{\perp}$ the image of π_{m+1} with respect to the polarity defining $\mathcal{P}$. We now use Theorem 1.1 to count the number of pairs (H,x) with H a hyperplane containing π_{m+1} and x a point of $(H \cap \tilde{\mathcal{M}}) \setminus \pi_{m+1}$. This yields the following in the respective cases.

(a) For $\mathcal{P} = W_{2n+1}(q)$ and $\mathcal{P} = Q^-(2n+1,q)$ we obtain the same result:

$$\begin{aligned}\left(Y+\frac{q^m-1}{q-1}\right)&\left(\frac{(q^{m+1}-1)(q^n+1)}{q-1}-q^n-X\right)\\ &+\left(\frac{q^{2n-m}-1}{q-1}-\frac{q^m-1}{q-1}-Y\right)\left(\frac{(q^{m+1}-1)(q^n+1)}{q-1}-X\right)\\ &\qquad\qquad=\left((q^{n+1}+1)\frac{q^{m+1}-1}{q-1}-X\right)\frac{q^{2n-m-1}-1}{q-1},\end{aligned}$$

from which we obtain

$$(q-1)Y + q^{n-m-1}(q-1)X - q^n + q^{n-m-1} - q^{m+1} + q^m = 0.$$

(b) For $\mathcal{P} = H(2n, q^2)$ the result is:

$$\left(Y + \frac{q^{2m}-1}{q^2-1}\right)\left(\frac{(q^{2m+2}-1)(q^{2n-1}+1)}{q^2-1} - q^{2n-1} - X\right)$$
$$+\left(\frac{q^{4n-2m-2}-1}{q^2-1} - \frac{q^{2m}-1}{q^2-1} - Y\right)\left(\frac{(q^{2m+2}-1)(q^{2n-1}+1)}{q^2-1} - X\right)$$
$$= \left((q^{2n+1}+1)\frac{q^{2m+2}-1}{q^2-1} - X\right)\frac{q^{4n-2m-4}-1}{q^2-1},$$

which yields

$$(q^2-1)Y + q^{2n-2m-3}(q^2-1)X - q^{2n-1} + q^{2n-2m-3} - q^{2m+2} + q^{2m} = 0.$$

Now consider the special case where $\pi_m = \langle \pi_{m-1}, y\rangle \in \mathcal{M}$, for some $y \in \mathcal{P}_1$. In this case $Y = 0$ and we can determine X from the above equalities:

(a) For $\mathcal{P} = W_{2n+1}(q)$ and $\mathcal{P} = Q^-(2n+1, q)$ we obtain

$$X = \frac{q^{m+1}-1}{q-1} + q^{2m-n+1}, \tag{7}$$

here we put $\alpha := q^{2m-n+1}$.

(b) For $\mathcal{P} = H(2n, q^2)$ we find

$$X = \frac{q^{2m+2}-1}{q^2-1} + q^{4m-2n+3}, \tag{8}$$

and in this case $\alpha := q^{4m-2n+3}$.

The value of X tells us that every $(m+1)$-dimensional subspace of $\mathsf{PG}(k,t)$, containing $\pi_m \in \mathcal{M}$ and not contained in $\pi_m^\perp$, has exactly α points in common with $\tilde{\mathcal{M}} \setminus \pi_m$. From the definition of an m-system, it is known that every $(m+1)$-dimensional subspace of $\pi_m^\perp$ which contains π_m, has an empty intersection with all elements of $\mathcal{M} \setminus \{\pi_m\}$. Hence the union of all tangent $(m+1)$-spaces of $\mathcal{M}$ at π_m is exactly $\pi_m^\perp$ and thus has the dimension required in **SPG3'** of the alternative definition of an SPG regulus. As the number of elements of an m-system, see (1), (4) and (5), is exactly the value required in **SPG4'**, it follows that $\mathcal{M}$ satisfies **SPG1**, **SPG2**, **SPG3'** and **SPG4'**, so it is an SPG regulus in $\mathsf{PG}(k,t)$ with parameters

(a) for $\mathcal{P} = W_{2n+1}(q)$ or $\mathcal{P} = Q^-(2n+1, q)$:

$$r = q^{n+1}+1, \quad \alpha = q^{2m-n+1} \quad \text{and} \quad \theta = q^{n-m}+1;$$

(b) for $\mathcal{P} = H(2n, q^2)$:

$$r = q^{2n+1}+1, \quad \alpha = q^{4m-2n+3} \quad \text{and} \quad \theta = q^{2n-2m-1}+1. \qquad \blacksquare$$

We mention two interesting corollaries of the previous theorem.

Corollary 2.2. *For any m-system of $\mathcal{P} \in \{W_{2n+1}(q), Q^-(2n+1,q), H(2n,q^2)\}$ there holds that $2m+1 \geq n$.*

Proof.
In (7) and (8), $X \geq |\pi_m|$ must hold. The result follows. ■

Remark.
This inequality was already found by Hamilton and Mathon [2]. However the proofs are distinct.

Corollary 2.3. *If $\mathcal{M}$ is a 1-system of $Q^-(7,q)$, then every line of $Q^-(7,q)$ meets $\tilde{\mathcal{M}}$ in 0, 1, 2 or $q+1$ points. If a line of $Q^-(7,q)$ contains $q+1$ points of $\tilde{\mathcal{M}}$, then it is necessarily a line of $\mathcal{M}$.*

Proof.
This follows immediately from the proof of Theorem 2.1, applied to 1-systems of the quadric $Q^-(7,q)$. ■

3 semipartial geometries arising from the known m-systems

In [6], Thas shows that every SPG regulus gives rise to a semipartial geometry. Hence, by the previous theorem, every m-system of $W_{2n+1}(q)$, $Q^-(2n+1,q)$ or $H(2n,q^2)$ also gives rise to a semipartial geometry. For spreads of $H(2n,q^2)$ or $Q^-(2n+1,q)$, this was already observed by Thas in [6]. For arbitrary m-systems, the corresponding semipartial geometries have the following parameters:

(a) for $\mathcal{P} = W_{2n+1}(q)$ or $\mathcal{P} = Q^-(2n+1,q)$:

$$s = q^{m+1} - 1,\ t = q^{n+1},\ \alpha = q^{2m-n+1} \text{ and } \mu = q^{m+1}(q^{m+1}-1);$$

(b) for $\mathcal{P} = H(2n,q^2)$:

$$s = q^{2m+2} - 1,\ t = q^{2n+1},\ \alpha = q^{4m-2n+3} \text{ and } \mu = q^{2m+2}(q^{2m+2}-1).$$

For several values of m and n, these parameters are new. Unfortunately, most of the known m-systems of the considered polar spaces do not yield new semipartial geometries.

First we remark that a lot of examples of m-systems arise from a known m-system in a small polar space by applying the so-called "trace trick". This means that the trace map is used to reduce the field while at the same time increasing the dimension, see [3] for an algebraic approach to the trace trick and [5] for a geometric explanation of this method. The corresponding semipartial geometry is clearly isomorphic to the one arising from the initial m-system in the small polar space, so m-systems which are constructed with the trace trick never yield new semipartial geometries. This observation highly reduces the number of candidates for new semipartial geometries.

Of the hermitian polar space $H(2n, q^2)$, only one m-system is known, apart from those obtained by the trace trick from this one, namely the point set of $H(2, q^2)$ considered as an ovoid (or a spread) of $H(2, q^2)$. The associated semipartial geometry is well known and was introduced by Debroey and Thas in [1]; it is often denoted by $T_2^*(\mathcal{U})$.

For the elliptic quadric $Q^-(2n+1, q)$, the situation is similar. First we observe that for q even, every m-system of $Q^-(2n+1, q)$ is also an m-system of $W_{2n+1}(q)$. This can be seen as follows. It is possible to embed $Q^-(2n+1, q)$ in a parabolic polar space $Q(2n+2, q)$ such that the nucleus of $Q(2n+2, q)$ is not contained in the ambient space $\mathsf{PG}(2n+1, q)$ of $Q^-(2n+1, q)$. Clearly, every m-system of $Q^-(2n+1, q)$ is an m-system of $Q(2n+2, q)$ as well. If we project $Q(2n+2, q)$ from its nucleus onto $\mathsf{PG}(2n+1, q)$, we obtain a symplectic polar space $W_{2n+1}(q)$. Now it is easily seen that the projection of the m-system of $Q(2n+2, q)$ is an m-system of $W_{2n+1}(q)$. As this m-system is completely contained in $\mathsf{PG}(2n+1, q)$, it is projected onto itself and this shows that every m-system of $Q^-(2n+1, q)$, q even, is an m-system of $W_{2n+1}(q)$. Hence we may omit the q even case. If q is odd, m-systems are only known for the small dimensions, except for those which are constructed with the trace trick from the small ones. It is known that $Q^-(5, q)$ has several non-isomorphic spreads, but the case of spreads of elliptic quadrics was already discussed in [6]. Moreover, $Q^-(5, q)$ has no ovoids and the point set of $Q^-(3, q)$, considered as an ovoid of $Q^-(3, q)$, yields the well known semipartial geometry $T_3^*(\mathcal{O})$, with $\mathcal{O} = Q^-(3, q)$. Consequently, nothing new arises here.

Finally, we consider the known m-systems of $W_{2n+1}(q)$. The semipartial geometry corresponding to the regular spread of $W_{2n+1}(q)$, that is, a spread of $W_{2n+1}(q)$ which is regular considered as an n-spread of $\mathsf{PG}(2n+1, q)$, was given as an example in [6]. Other spreads of $W_{2n+1}(q)$ are known and they yield other semipartial geometries with the same parameters. Candidates for new semipartial geometries are given by the m-systems of $W_{2n+1}(2)$, $n \leq 4$, which were found by computer by Hamilton and Mathon in [2]. Some of these yield indeed new semipartial geometries, but their parameters are not new. Very recently, A. Offer ([4]) discovered a new class of spreads of the hexagon $\mathsf{H}(2^{2h})$, which yields a new class of 1-systems of the parabolic quadric $Q(6, 2^{2h})$. By projection from the nucleus of $Q(6, 2^{2h})$ onto a 5-dimensional subspace not containing the nucleus, a new class of 1-systems of $W_5(2^{2h})$ is obtained. These 1-systems are distinct from the only previously known 1-system of $W_5(q)$, which arises from $H(2, q^2)$ as described in [5, Theorem 14] and the semipartial geometry of which is isomorphic to $T_2^*(\mathcal{U})$. Hence this new class of spreads of $\mathsf{H}(2^{2h})$ implies the existence of a new class of semipartial geometries for $q = 2^{2h}$, but once again their parameters are not new. All other known m-systems of $W_{2n+1}(q)$ give rise to known semipartial geometries, as they are always obtained from an m-system in a small polar space, the semipartial geometry of which is well known.

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Ghent University,
Department of Pure Mathematics and Computer Algebra
Galglaan 2, B-9000 Gent
Belgium
email : dluyckx@cage.rug.ac.be