Submanifolds in a hyperbolic space form with flat normal bundle

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Abstract

In this paper we give some rigidity results for compact submanifolds in a hyperbolic space form with flat normal bundle to be totally umbilical.

1 Introduction

Let $M^{n+p}(c)$ be an $(n+p)$-dimensional Riemannian manifold with constant sectional curvature c. We also call it a space form. When $c > 0$, $M^{n+p}(c) = S^{n+p}(c)$ (i.e. $(n+p)$-dimensional sphere space); when $c = 0$, $M^{n+p}(c) = R^{n+p}$ (i.e. $(n+p)$-dimensional Euclidean space); when $c < 0$, $M^{n+p}(c) = H^{n+p}(c)$ (i.e. $(n+p)$-dimensional hyperbolic space). We simply denote $H^{n+p}(-1)$ by H^{n+p}. Let M^n be an n-dimensional submanifold in $M^{n+p}(c)$. As it is well known, there are many rigidity results for minimal submanifolds or submanifolds with constant mean curvature H in $M^{n+p}(c)$ $(c \geq 0)$ by use of J. Simons' method, for example, see [1], [4], [7], [12], etc., but less of that were obtained for submanifolds immersed into a hyperbolic space from. Walter [13] gave a classification for non-negatively curved compact hypersurfaces in a space form under the assumption that the rth mean curvature is constant. Morvan-Wu [6], Wu [14] also proved some rigidity theorems for complete hypersurfaces M^n in a hyperbolic space form $H^{n+1}(c)$ under the assumption that the mean curvature is constant and the Ricci curvature is non-negative. Moreover, they proved that M^n is a geodesic distance sphere in $H^{n+1}(c)$ provided that it is compact.

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On the other hand, Cheng-Yau [2] firstly studied the rigidity problem for a hypersurface with constant scalar curvature in a space form by introducing a self-adjoint second order differential operator. Later, Hou [3] extended Cheng-Yau's technique to higher codimensional cases and studied the rigidity problem for closed submanifolds with constant scalar curvature in a hyperbolic space form.

In the present paper, we would like to use Cheng-Yau's technique to study the rigidity problem for compact submanifolds in a hyperbolic space form with flat normal bundle.

2 Preliminaries

Let M^n be an n-dimensional compact submanifold immersed in an $(n+p)$-dimensional Riemannian manifold $M^{n+p}(c)$ of constant curvature c. We choose a local field of orthonormal frames $e_1, \ldots, e_{n+p}$ in $M^{n+p}(c)$ such that at each point of M^n, $e_1, \ldots, e_n$ span the tangent space of M^n and form an orthonormal frame there. Let $\omega_1, \ldots, \omega_{n+p}$ be its dual frame field. In this paper, we use the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq n+p; \quad 1 \leq i, j, k, \ldots \leq n; \quad n+1 \leq \alpha, \beta, \gamma \leq n+p.$$

Then the structure equations of $M^{n+p}(c)$ are given by

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{1}$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D, \tag{2}$$

$$K_{ABCD} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{3}$$

Restrict these form to M^n, we have

$$\omega_\alpha = 0, \quad n+1 \leq \alpha \leq n+p. \tag{4}$$

From Cartan's lemma we can write

$$\omega_{\alpha i} = \sum_j h^\alpha_{ij} \omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji}. \tag{5}$$

From these formulas, we obtain the structure equations of M^n:

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{6}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \tag{7}$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}), \tag{8}$$

where R_{ijkl} are the components of the curvature tensor of M^n.

Denote $L_\alpha = (h^\alpha_{ij})_{n\times n}$ and $H_\alpha = (1/n)\sum_i h^\alpha_{ii}$ for $\alpha = n+1, \cdots, n+p$. Then the mean curvature vector field ξ, the mean curvature H and the square of the length of the second fundamental form S are expressed as

$$\xi = \sum_\alpha H_\alpha e_\alpha, \quad H = |\xi|, \quad S = \sum_{\alpha,i,j} (h^\alpha_{ij})^2,$$

respectively. Moreover, the normal curvature tensor $\{R_{\alpha\beta kl}\}$, the Ricci curvature tensor $\{R_{ik}\}$ and the normalized scalar curvature R are expressed as

$$\begin{aligned} R_{\alpha\beta kl} &= \sum_m (h^\alpha_{km} h^\beta_{ml} - h^\alpha_{lm} h^\beta_{mk}), \\ R_{ik} &= (n-1)\, c\, \delta_{ik} + n \sum_\alpha (H_\alpha) h^\alpha_{ik} - \sum_{\alpha,j} h^\alpha_{ij} h^\alpha_{jk}, \\ R &= c + \frac{1}{n(n-1)} (n^2 H^2 - S). \end{aligned} \tag{9}$$

Define the first and the second covariant derivatives of $\{h^\alpha_{ij}\}$, say $\{h^\alpha_{ijk}\}$ and $\{h^\alpha_{ijkl}\}$ by

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{kj}\omega_{ki} + \sum_k h^\alpha_{ik}\omega_{kj} + \sum_\beta h^\beta_{ij}\omega_{\beta\alpha}, \tag{10}$$

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} + \sum_m h^\alpha_{mjk}\omega_{mi} + \sum_m h^\alpha_{imk}\omega_{mj} + \sum_m h^\alpha_{ijm}\omega_{mk} + \sum_\beta h^\beta_{ijk}\omega_{\beta\alpha}. \tag{11}$$

Then, by exterior differentiation of (5), we obtain the Codazzi equation

$$h^\alpha_{ijk} = h^\alpha_{ikj}. \tag{12}$$

It follows from Ricci's identity that

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{mj} R_{mikl} + \sum_m h^\alpha_{im} R_{mjkl} + \sum_\beta h^\beta_{ij} R_{\beta\alpha kl}. \tag{13}$$

The Laplacian of h^α_{ij} is defined by $\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$. From (13), we have

$$\begin{aligned} \Delta h^\alpha_{ij} &= nH_{\alpha,ij} + \sum_{k,m} h^\alpha_{km} R_{mijk} + \sum_{k,m} h^\alpha_{im} R_{mkjk} + \sum_{k,\beta} h^\beta_{ik} R_{\beta\alpha jk} \\ &= nH_{\alpha,ij} + n\, c\, h^\alpha_{ij} - n\, c\, H_\alpha \delta_{ij} + n \sum_{\beta,m} H_\beta h^\alpha_{im} h^\beta_{mj} - \sum_\beta S_{\alpha\beta} h^\beta_{ij} \\ &\quad + 2 \sum_{\beta,k,m} h^\beta_{ik} h^\alpha_{km} h^\beta_{mj} - \sum_{m,k,\beta} h^\alpha_{im} h^\beta_{mk} h^\beta_{kj} - \sum_{\beta,k,m} h^\beta_{ik} h^\beta_{km} h^\alpha_{mj}, \end{aligned}$$

where $S_{\alpha\beta} = \sum_{i,j} h^\alpha_{ij} h^\beta_{ij}$ for all α and β. Define $N(A) = \sum_{i,j} a^2_{ij}$ for any real matrix $A = (a_{ij})_{n\times n}$. Then we have

$$\begin{aligned} \sum_{i,j} h^\alpha_{ij} \Delta h^\alpha_{ij} &= n \sum_{i,j} H_{\alpha,ij} h^\alpha_{ij} + n\, c\, S_\alpha - c\, n^2 H^2_\alpha + n \sum_\beta H_\beta Tr(L^2_\alpha L_\beta) \\ &\quad - \sum_\beta S^2_{\alpha\beta} - \sum_\beta N(L_\alpha L_\beta - L_\beta L_\alpha), \end{aligned} \tag{14}$$

where $S_\alpha = \sum_{i,j} (h^\alpha_{ij})^2$, for every α.

Suppose $H > 0$ on M^n and choose $e_{n+1} = \xi/H$. Then it follows that

$$H_{n+1} = H; \quad H_\alpha = 0, \qquad \alpha > n+1. \tag{15}$$

From (10) and (15) we can see

$$H_{n+1,k}\omega_k = dH, \quad H_{\alpha,k}\omega_k = H\omega_{n+1\alpha} \qquad \alpha > n+1. \tag{16}$$

From (11), (15) and (16) we have

$$H_{n+1,kl} = H_{kl} - \frac{1}{H}\sum_{\beta>n+1} H_{\beta,k}H_{\beta,l}, \tag{17}$$

where $dH = \sum_i H_i\omega_i$ and $\nabla H_k = \sum_l H_{kl}\omega_l \equiv dH_k + H_l\omega_{lk}$ for all k.

Using (14) and (17), we have

$$\begin{aligned} \sum_{i,j} h_{ij}^{n+1}\Delta h_{ij}^{n+1} &= n\sum_{i,j} H_{ij}h_{ij}^{n+1} - \frac{n}{H}\sum_{i,j}\sum_{\beta>n+1} H_{\beta,i}H_{\beta,j}h_{ij}^{n+1} \\ &\quad + n\,c\,S_{n+1} - c\,n^2H^2 + nHf_{n+1} - S_{n+1}^2 - \sum_{\beta>n+1} S_{n+1\beta}^2 \\ &\quad - \sum_{\beta>n+1} N(L_{n+1}L_\beta - L_\beta L_{n+1}). \end{aligned} \tag{18}$$

where $f_{n+1} = Tr(L_{n+1})^3$.

M. Okumura [8] established the following lemma (see also [1]).

Lemma 2.1. Let $\{a_i\}_{i=1}^n$ be a set of real numbers satisfying $\sum_i a_i = 0$, $\sum_i a_i^2 = t^2$, where $t \geq 0$. Then we have

$$-\frac{n-2}{\sqrt{n(n-1)}}t^3 \leq \sum_i a_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}t^3,$$

and the equalities hold if and only if at least $(n-1)$ of the a_i are equal.

Denote the eigenvalues of L_{n+1} by $\{\lambda_i^{n+1}\}_{i=1}^n$. Then we have

$$nH = \sum_i \lambda_i^{n+1}, \quad S_{n+1} = \sum_i (\lambda_i^{n+1})^2, \quad f_{n+1} = \sum_i (\lambda_i^{n+1})^3. \tag{19}$$

Set $\bar{L}_{n+1} = L_{n+1} - H\,I_n$, $\bar{f}_{n+1} = f_{n+1} - 3HS_{n+1} + 2nH^3$, $\bar{S}_{n+1} = S_{n+1} - nH^2$, and $\bar{\lambda}_i^{n+1} = \lambda_i^{n+1} - H$, where I_n denotes the identity matrix of degree n. Then (19) changes into

$$0 = \sum_i \bar{\lambda}_i^{n+1}, \quad \bar{S}_{n+1} = \sum_i (\bar{\lambda}_i^{n+1})^2, \quad \bar{f}_{n+1} = \sum_i (\bar{\lambda}_i^{n+1})^3. \tag{20}$$

By applying Okumura's Lemma to $\bar{f}_{n+1}$, we have

$$\bar{f}_{n+1} \geq -\frac{n-2}{\sqrt{n(n-1)}}\bar{S}_{n+1}\sqrt{\bar{S}_{n+1}} \Longleftrightarrow f_{n+1} \geq 3HS_{n+1} - 2nH^3 - \frac{n-2}{\sqrt{n(n-1)}}\bar{S}_{n+1}\sqrt{\bar{S}_{n+1}}.$$

So we have

$$\begin{aligned} & n\,c\,S_{n+1} - c\,n^2H^2 + nHf_{n+1} - S_{n+1}^2 \\ \geq \quad & \bar{S}_{n+1}\{n\,c - (\bar{S}_{n+1} - nH^2) - n(n-2)H\sqrt{\frac{\bar{S}_{n+1}}{n(n-1)}}\}. \end{aligned} \tag{21}$$

It follows from (15) that

$$\sum_{\beta>n+1} S_{n+1\beta}^2 = \sum_{\beta>n+1} \{\sum_{i,j}(h_{ij}^{n+1} - H\delta_{ij})h_{ij}^{\beta}\}^2. \tag{22}$$

Denote $S_I = \sum_{\beta>n+1} S_\beta$. From (22), we have

$$\sum_{\beta>n+1} S_{n+1\beta}^2 \leq \bar{S}_{n+1}S_I. \tag{23}$$

Let $T = \sum_{i,j} T_{ij}\omega_i\omega_j$ be a symmetric tensor on M^n defined by

$$T_{ij} = h_{ij}^{n+1} - nH\delta_{ij}. \tag{24}$$

We introduce an operator $\Box$ associated to T acting on $f \in C^2(M^n)$ by

$$\Box f = \sum_{i,j} T_{ij}f_{ij} = \sum_{i,j} h_{ij}^{n+1}f_{ij} - nH\Delta f,$$

where Δ is the Laplacian. Since (T_{ij}) is divergence-free, it follows from [2] that the operator $\Box$ is self-adjoint relative to the L^2-inner product of M^n.

Choosing $f = H$ in above expression, we have

$$\sum_{i,j} h_{ij}^{n+1}H_{ij} = \Box H + nH\Delta H. \tag{25}$$

Denote $\bar{S} = \bar{S}_{n+1} + S_I$. Substituting (21), (23) and (25) into (18), we get

$$\begin{aligned}
\sum_{i,j} h_{ij}^{n+1}\Delta h_{ij}^{n+1} \;\geq\; & n\Box H + \frac{1}{2}n^2\Delta(H^2) - n^2|\nabla H|^2 \\
& -\frac{n}{H}\sum_{\beta>n+1}\sum_{i,j} H_{\beta,i}H_{\beta,j}h_{ij}^{n+1} \\
& -\sum_{\beta>n+1} N(L_{n+1}L_\beta - L_\beta L_{n+1}) \\
& +\bar{S}_{n+1}\{n\,c + nH^2 - \bar{S} - n(n-2)H\sqrt{\frac{\bar{S}_{n+1}}{n(n-1)}}\}.
\end{aligned} \tag{26}$$

In codimension one case, Cheng-Yau [2] gave a lower estimation for $|\nabla\sigma|^2$, the square of the length of the covariant derivative of σ. They proved that, for a hypersurface in a space form of constant scalar curvature c, if the normalized scalar curvature R is constant and $R \geq c$, then $|\nabla\sigma|^2 \geq n^2|\nabla H|^2$.

In higher codimension cases, Hou [3] proved the following

Lemma 2.2. Let M^n be a connected submanifold in $M^{n+p}(c)$ with nowhere zero mean curvature H. If R is constant and $R \geq c$, then

$$|\nabla\sigma|^2 = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 \geq n^2|\nabla H|^2. \tag{27}$$

Moreover,

(i) when $R - c > 0$, if the equality in (27) holds on M^n, then H is constant.

(ii) when $R - c = 0$, if the equality in (27) holds on M^n, then either H is constant or $S_I = 0$ on M^n and M^n lies in a totally geodesic subspace $M^{n+1}(c)$ of $M^{n+p}(c)$.

3 Submanifolds with flat normal bundle

In this section, we propose to study the rigidity problem for submanifolds in H^{n+p}. We continue use the same notations as in section 2. Let M^n be a compact submanifold in H^{n+p}, suppose that the normalized mean curvature vector field ξ/H is parallel and choose $e_{n+1} = \xi/H$. Then $\omega_{n+1\alpha} = 0$ for all α. It follows from (11) and (16) that

$$H_{\alpha,k} = 0, \quad H_{\alpha,kl} = 0, \tag{28}$$

for all $\alpha > n+1$ and $k, l = 1, \cdots, n$.

Suppose in addition that the normal bundle of M^n is flat. Then

$$\Omega_{\alpha\beta} = -\frac{1}{2} R_{\alpha\beta kl}\omega_k \wedge \omega_l = 0, \tag{29}$$

for all α and β on M^n. For all α and β we have $L_\alpha L_\beta = L_\beta L_\alpha$, which is equivalent to that $\{L_\alpha\}_{\alpha=n+1}^{n+p}$ can be diagonized simultaneously.

We denote the eigenvalues of L_α by $\{\lambda_1^\alpha, \cdots, \lambda_n^\alpha\}$ for every α. It follows from [15] that

$$\frac{1}{2}\Delta S = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + n \sum_{i,j,\alpha} H_{\alpha,ij} h_{ij}^\alpha + \sum_\alpha \sum_{i<j} K_{ij}(\lambda_i^\alpha - \lambda_j^\alpha)^2, \tag{30}$$

where $K_{ij} = -1 + \sum_\beta \lambda_i^\beta \lambda_j^\beta$ denotes the sectional curvature of M^n corresponding to the plane section spanned by $\{e_i, e_j\}$ for every pair of $i < j$.

Assume that R is constant and $R + 1 \geq 0$. From (25) and (28), we have

$$\sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + n \sum_{i,j,\alpha} H_{\alpha,ij} h_{ij}^\alpha = n \,\square H + \frac{1}{2}\Delta(n^2H^2) + \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 - n^2|\nabla H|^2.$$

Note that $\Delta S = \Delta(n^2H^2)$. Therefore (30) turns into

$$0 = n \,\square H + \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 - n^2|\nabla H|^2 + \sum_{i<j}\sum_\alpha K_{ij}(\lambda_i^\alpha - \lambda_j^\alpha)^2.$$

Integrating the both sides of above equality on M^n, we have

$$0 = \int_{M^n} \left(\sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 - n^2|\nabla H|^2 \right) * 1 + \sum_{i<j}\sum_\alpha \int_{M^n} K_{ij}(\lambda_i^\alpha - \lambda_j^\alpha)^2 * 1.$$

If $K_{ij} \geq 0$ on M^n, it follows from (27) and the above equality that

$$\sum_{(i,j,k,\alpha)} (h_{ijk}^\alpha)^2 \equiv n^2|\nabla H|^2; \qquad K_{ij}(\lambda_i^\alpha - \lambda_j^\alpha)^2 \equiv 0, \tag{31}$$

for every α and $i < j$. Hence we can prove the following theorem

Theorem 3.1. Let M^n be a compact submanifold with non-negative sectional curvature in H^{n+p}. Suppose that the normal bundle $N(M)$ is flat and the normalized mean curvature vector is parallel. If R is constant and $R + 1 \geq 0$, then either $M^n = M_1 \times M_2 \times \ldots \times M_k$ such that each M_i is a minimal submanifold of a totally umbilical submanifold N_i (with codimension > 0) and the N_i's are mutually perpendicular along their intersections; or M^n lies in a totally geodesic subspace H^{n+1} of H^{n+p}.

Proof. From the first equality of (31) and Lemma 2.2, we have that either H is constant or $S_I = 0$ on M^n. If H is constant on M^n, then ξ is parallel. Hence the proof follows from the result of Yau (Theorem 9, [16]). Otherwise, if $S_I = 0$ on M^n, then M^n lies in a totally geodesic subspace H^{n+1} of H^{n+p} and this completes the proof of the Theorem 3.1.

In [10], Ryan completely classified the complete hypersurfaces with at most two distinct constant principal curvatures in H^{n+1}, from this we know that the compact hypersurface in H^{n+1} with at most two distinct constant principal curvatures is totally umbilical. Using this fact and making the same process of the proof of Theorem 2 and Theorem 3 in [5], we can obtain the following theorem

Theorem 3.2. Let M^n be an n-dimensional ($n \geq 3$) compact hypersurface with constant normalized scalar curvature R in H^{n+1}. If

(1) $\bar{R} = R + 1 \geq 0$,

(2) the norm square S of the second fundamental form of M^n satisfies

$$n\bar{R} \leq S \leq \frac{n}{(n-2)(n\bar{R}-2)}[n(n-1)\bar{R}^2 - 4(n-1)\bar{R} + n], \tag{32}$$

then M^n is a totally umbilical hypersurface.

Now we want to extend the above theorem to higher codimensional case. For this purpose, we need the following

Lemma 3.1 [11]. Let A and B be $n \times n$-symmetric matrices satisfying $Tr\,A = 0$, $Tr\,B = 0$ and $A\,B - B\,A = 0$. Then

$$-\frac{n-2}{\sqrt{n(n-1)}}(Tr\,A^2)(Tr\,B^2)^{1/2} \leq Tr\,A^2B \leq \frac{n-2}{\sqrt{n(n-1)}}(Tr\,A^2)(Tr\,B^2)^{1/2}, \tag{33}$$

and the equality holds on the right (resp. left) hand side if and only if $n-1$ of the eigenvalues x_i of A and the corresponding eigenvalues y_i of B satisfy

$$|x_i| = \tfrac{(Tr\,A^2)^{1/2}}{\sqrt{n(n-1)}}, \quad x_i\,x_j \geq 0, \quad y_i = -\tfrac{(Tr\,B^2)^{1/2}}{\sqrt{n(n-1)}} \quad (\text{ resp.} \quad y_i = \tfrac{(Tr\,B^2)^{1/2}}{\sqrt{n(n-1)}}).$$

Choose a suitable normal frame field $\{e_\beta\}_{\beta=n+2}^{n+p}$ such that $S_{\alpha\beta} = 0$ for all $\alpha \neq \beta$. Then

$$\sum_{\alpha,\beta>n+1} S_{\alpha\beta}^2 = \sum_{\beta>n+1} S_\beta^2 \leq S_I^2, \tag{34}$$

where the equality holds if and only if at least $p-2$ numbers of S_α's are zero.

Taking sum with respect to $\alpha > n+1$ on both-sides of (14), we have

$$\begin{aligned}\sum_{i,j,\alpha>n+1} h_{ij}^\alpha \Delta h_{ij}^\alpha &= (-n + n\,H^2)S_I + nH \sum_{\alpha>n+1} Tr(L_\alpha^2 \bar{L}_{n+1}) \\ &\quad - \sum_{\alpha>n+1} S_{n+1\alpha}^2 - \sum_{\alpha>n+1} S_\alpha^2.\end{aligned} \tag{35}$$

Using the left hand side of (33) to $Tr(L_\alpha^2 \bar{L}_{n+1})$, we have

$$Tr(L_\alpha^2 \bar{L}_{n+1}) \geq -(n-2)S_\alpha \sqrt{\frac{\bar{S}_{n+1}}{n(n-1)}}.$$

Substituting this into (35) and using (23) and (34), we have

$$\sum_{i,j,\alpha>n+1} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} \geq S_I \left\{(-n+n\,H^2) - n(n-2)H\sqrt{\frac{\bar{S}_{n+1}}{n(n-1)}} - \bar{S}\right\}. \qquad (36)$$

Substituting (28) into (26), we have

$$\begin{aligned}\sum_{i,j} h_{ij}^{n+1}\Delta h_{ij}^{n+1} \quad &\geq n\Box H + \frac{1}{2}\Delta(n^2H^2) - n^2|\nabla H|^2 \\ &+\bar{S}_{n+1}\left\{(-n+nH^2) - n(n-2)H\sqrt{\frac{\bar{S}_{n+1}}{n(n-1)}} - \bar{S}\right\}. \qquad (37)\end{aligned}$$

Note that $\Delta S = \Delta(n^2H^2)$ and

$$\frac{1}{2}\Delta S = \sum_{i,j,k,\alpha}\left(h_{ijk}^{\alpha}\right)^2 + \sum_{i,j} h_{ij}^{n+1}\Delta h_{ij}^{n+1} + \sum_{i,j,\alpha>n+1} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}. \qquad (38)$$

From (36), (37) and (38), we obtain

$$\begin{aligned}0 \quad &\geq n\Box H + \sum_{(i,j,k,\alpha)}\left(h_{ijk}^{\alpha}\right)^2 - n^2|\nabla H|^2 \\ &+\bar{S}\left\{(-n+nH^2) - n(n-2)H\sqrt{\frac{\bar{S}_{n+1}}{n(n-1)}} - \bar{S}\right\}. \qquad (39)\end{aligned}$$

Note that

$$\bar{S}_{n+1} \leq \bar{S}_{n+1} + S_I = \bar{S}. \qquad (40)$$

Substituting (40) into (39) and using (27), we have

$$0 \geq n\Box H + \bar{S}\left\{(-n+nH^2) - \frac{n-2}{\sqrt{n-1}}H\sqrt{n\bar{S}} - \bar{S}\right\}. \qquad (41)$$

Integrating the both sides of (41) on M^n, we have

$$0 \geq \int_{M^n} \bar{S}\left\{(-n+nH^2) - \frac{n-2}{\sqrt{n-1}}H\sqrt{n\bar{S}} - \bar{S}\right\} * 1. \qquad (42)$$

Therefore we can prove the following

Theorem 3.3. Let M^n $(n \geq 3)$ be a compact submanifold with parallel normalized mean curvature vector field immersed into H^{n+p}. Suppose that R is constant and $\bar{R} = R + 1 \geq 0$. If the normal bundle $N(M)$ is flat and

$$n\bar{R} \leq S \leq \frac{n}{(n-2)(n\bar{R}-2)}[n(n-1)\bar{R}^2 - 4(n-1)\bar{R} + n], \qquad (43)$$

then M^n is totally umbilical.

Proof. By (9), we know

$$\bar{S} = S - nH^2 = \frac{n-1}{n}(S - n\bar{R}). \qquad (44)$$

By use of (42) and (44), we get

$$0 \geq \int_{M^n} \frac{n-1}{n}(S-n\bar{R})[-n+2(n-1)\bar{R}-\frac{n-2}{n}S-\frac{n-2}{n}\sqrt{(n(n-1)\bar{R}+S)(S-n\bar{R})}]. \tag{45}$$

It is a direct check that our assumption condition (43), i.e.

$$S \leq \frac{n}{(n-2)(n\bar{R}-2)}[n(n-1)\bar{R}^2 - 4(n-1)\bar{R}+n], \tag{46}$$

is equivalent to

$$(-n+2(n-1)\bar{R} - \frac{n-2}{n}S)^2 \geq \frac{(n-2)^2}{n^2}(n(n-1)\bar{R}+S)(S-n\bar{R}). \tag{47}$$

But it is clear from (46) that (47) is equivalent to

$$-n+2(n-1)\bar{R} - \frac{n-2}{n}S \geq \frac{n-2}{n}\sqrt{(n(n-1)\bar{R}+S)(S-n\bar{R})}. \tag{48}$$

From (45) and (48), we have either

$$S = n\bar{R}, \tag{49}$$

and M^n is totally umbilical; or

$$S = \frac{n}{(n-2)(n\bar{R}-2)}[n(n-1)\bar{R}^2 - 4(n-1)\bar{R}+n]. \tag{50}$$

In the latter case, all of the inequalities concerned become equalities. From (40), we have $S_I = 0$. So M^n lies in a totally geodesic subspace H^{n+1} of H^{n+p}. The rest of the proof follows from Theorem 3.2. This completes the proof of Theorem 3.3.

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