The glueing of near polygons

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Abstract

In [7] a construction is given to derive so-called glued near polygons from spreads of symmetry in generalized quadrangles. We show here that this construction is also applicable to arbitrary near polygons and derive a similar theory as in the case of the generalized quadrangles. We also show that many new near polygons can be derived from a set of points in $\mathrm{PG}(5,3)$ discovered by Coxeter ([2]).

1 Basic definitions

A *near polygon* $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a partial linear space with the property that every line L contains a unique point $\pi_L(p)$ nearest to any given point p. Here distances are measured in the collinearity graph Γ. If d is the (finite) diameter of Γ, then the near polygon is called a near $2d$-gon. A near 0-gon consists of one point, a near 2-gon is a line, and the class of the near quadrangles coincides with the class of the generalized quadrangles (GQ's, [9]), which were introduced by Tits in [11]. Near polygons themselves were introduced by Shult and Yanushka in [10] because of their relationship with certain systems of lines in Euclidean spaces. For a point p and a line K of $\mathcal{A}$, let $\mathrm{d}(p,K)$ denote the minimal distance between p and a point of K. For two lines K and L of $\mathcal{A}$, let $\mathrm{d}(K,L)$ denote the minimal distance between two points on respectively K and L. There are two possibilities. Either there exist unique points $k \in K$ and $l \in L$ such that $\mathrm{d}(K,L) = \mathrm{d}(k,l)$, or, for every point $k \in K$ there exists a unique $l \in L$ such that $\mathrm{d}(K,L) = \mathrm{d}(k,l)$. In the latter case K and L are called *parallel* ($K \parallel L$). A subspace B of $\mathcal{A}$ is called *geodetically closed*, if

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every point on a shortest path between two points of B is as well contained in B. If every line of $\mathcal{A}$ is incident with at least three points, and if every two points at distance 2 have at least two common neighbours, then every two points at distance δ are contained in a unique geodetically closed sub near 2δ-gon, see Theorem 4 of [1]. The existence of geodetically closed sub near quadrangles, the so-called *quads*, was already proven in [10]. The direct product $\mathcal{A}\times\mathcal{B}$ of two near polygons $\mathcal{A}=(\mathcal{P},\mathcal{L},\mathrm{I})$ and $\mathcal{B}=(\mathcal{P}',\mathcal{L}',\mathrm{I}')$ is the near polygon whose point set is the cartesian product $\mathcal{P}\times\mathcal{P}'$, with two points (x,y) and (x',y') collinear if and only if ($x=x'$ and $y\sim y'$) or ($y=y'$ and $x\sim x'$).

In [7], it is explained how generalized quadrangles with a so-called spread of symmetry can be used to construct new near $2d$-gons. The case $d=3$ is treated more thoroughly in [6]. In the present paper, we show that the construction can be generalized to all near $2d$-gons with a spread of symmetry and derive a similar theory as in [3]. Examples of near polygons with a spread of symmetry include the near polygons with a linear representation. We take a closer look to these near polygons in the following section.

2 Linear representations

Let Π_∞ be a $\mathrm{PG}(n,q)$, $n\geq 0$, which is embedded as a hyperplane in $\Pi=\mathrm{PG}(n{+}1,q)$, and let $\mathcal{K}$ be a nonempty set of points of Π_∞. With every point x of Π_∞, we associate an element $i_{\mathcal{K}}(x)\in\mathbb{N}\cup\{+\infty\}$, called the *$\mathcal{K}$-index of x*:

- if $x\notin\langle\mathcal{K}\rangle$, then $i_{\mathcal{K}}(x)=+\infty$,
- if $x\in\langle\mathcal{K}\rangle$, then $i_{\mathcal{K}}(x)=m$, where m is the smallest integer with the property that there are m points of $\mathcal{K}$ generating a subspace containing x.

The *linear representation* $T_n^*(\mathcal{K})$ is the geometry with point set $\Pi\setminus\Pi_\infty$, with lines all the affine lines of Π through a point of $\mathcal{K}$, and with incidence the one derived from Π.

Theorem 1 ([8]).
- *If x and y are 2 different points of $T_n^*(\mathcal{K})$ and if z is the intersection of xy with Π_∞ then $d(x,y)=i_{\mathcal{K}}(z)$, where $d(\cdot,\cdot)$ denotes the distance in the collinearity graph of $T_n^*(\mathcal{K})$.*
- *$T_n^*(\mathcal{K})$ is a near polygon if and only if for every point $x\in\mathcal{K}$ and for every line L of Π_∞ through x, there is a unique point $y\in L\setminus\{x\}$ with smallest $\mathcal{K}$-index.*

Theorem 2. (A) *Consider in Π_∞ two disjoint subspaces π_1 and π_2 of dimensions $n_1\geq 0$ and $n_2\geq 0$ respectively, such that $\Pi_\infty=\langle\pi_1,\pi_2\rangle$. Let $\mathcal{K}_i$, $i\in\{1,2\}$, be a set of points in π_i and put $\mathcal{K}=\mathcal{K}_1\cup\mathcal{K}_2$. If $T_{n_i}^*(\mathcal{K}_i)$, $i\in\{1,2\}$, is a near $2d_i$-gon, then $T_n^*(\mathcal{K})$ is a near $2(d_1+d_2)$-gon isomorphic to the direct product $T_{n_1}^*(\mathcal{K}_1)\times T_{n_2}^*(\mathcal{K}_2)$.*

(B) *Consider in Π_∞ two subspaces π_1 and π_2 of dimensions $n_1\geq 0$ and $n_2\geq 0$ respectively, such that $\pi_1\cap\pi_2=\{p\}$ and $\Pi_\infty=\langle\pi_1,\pi_2\rangle$. Let $\mathcal{K}_i$, $i\in\{1,2\}$, be a set of points in π_i containing p and put $\mathcal{K}=\mathcal{K}_1\cup\mathcal{K}_2$. If $T_{n_i}^*(\mathcal{K}_i)$, $i\in\{1,2\}$, is a near $2d_i$-gon, then $T_n^*(\mathcal{K})$ is a near $2(d_1+d_2-1)$-gon.*

Proof. Part (A) of the theorem was proved in [8]. We now prove part (B). Let L be a line of Π_∞ containing a point x of $\mathcal{K}_1 \cup \mathcal{K}_2$. We will prove that $L \setminus \{x\}$ contains a unique point with smallest $(\mathcal{K}_1 \cup \mathcal{K}_2)$-index. We may suppose that $x \in \mathcal{K}_1$ and $L \not\subseteq \pi_1$. For every point y of $L \setminus (\pi_1 \cup \pi_2)$ there exist points $a_1 \in \pi_1$ and $a_2 \in \langle L, \pi_1 \rangle \cap \pi_2$ such that $y \in \langle a_1, a_2 \rangle$ and $i_{\mathcal{K}_1 \cup \mathcal{K}_2}(y) = i_{\mathcal{K}_1}(a_1) + i_{\mathcal{K}_2}(a_2)$. Let b_2 be the unique point of $(\langle L, \pi_1 \rangle \cap \pi_2) \setminus \{p\}$ with smallest $\mathcal{K}_2$-index. By (A), $L \setminus \{x\}$ contains a unique point with smallest $(\mathcal{K}_1 \cup \{b_2\})$-index, and this point is also the unique point of $L \setminus \{x\}$ with smallest $(\mathcal{K}_1 \cup \mathcal{K}_2)$-index. ∎

A nonempty set of points in $\mathrm{PG}(n, q)$ is called *indecomposable* if it cannot be written as $\mathcal{K}_1 \cup \mathcal{K}_2$ with $\mathcal{K}_1$ and $\mathcal{K}_2$ as in (A) or (B) of the previous theorem. The following examples of indecomposable sets yield near polygons ([8]):

(1) the unique point of $\mathrm{PG}(0, q)$,

(2) a hyperoval in $\mathrm{PG}(2, 2^h)$,

(3) the set $\{(1,0,0,0,0,0), (0,1,0,0,0,0), (0,0,1,0,0,0), (0,0,0,1,0,0), (0,0,0,0,1,0), (0,0,0,0,0,1), (0,-1,-1,-1,-1,-1), (1,0,1,-1,-1,1), (1,1,0,1,-1,-1), (1,-1,1,0,1,-1), (1,-1,-1,1,0,1), (1,1,-1,-1,1,0)\}$ of 12 points in $\mathrm{PG}(5, 3)$ discovered by Coxeter in [2] as a set of points with certain ”nice” properties.

Successive application of Theorem 2, (B) to the the third example yields an infinite class of new near polygons. Concerning the classification of linear representations of near $2d$-gons, $d \leq 3$, we have the following result.

Theorem 3 ([8]). *Let $\mathcal{K}$ be an indecomposable set of points in $\mathrm{PG}(n, q)$, $q \geq 3$, different from (1), (2) and (3). If $T_n^*(\mathcal{K})$ is a near $2d$-gon, $d \leq 3$, then $d = 3$, $n \geq 7$ and $q = 2^h$ with $h \geq 4$.*

The case $d \geq 4$ has not yet been treated. The following theorem however suggests a recursive approach.

Theorem 4 ([5]). *If $\mathcal{K}$ is a nonempty set of points in $\mathrm{PG}(n, q)$, such that $T_n^*(\mathcal{K})$ is a near polygon, then every geodetically closed sub near 2δ-gon, $\delta \neq 0$, of $T_n^*(\mathcal{K})$ is of the form $T_{n^*}^*(\mathcal{K}^*)$ with $\mathcal{K}^* \subseteq \mathcal{K}$, $\langle \mathcal{K}^* \rangle \cap \mathcal{K} = \mathcal{K}^*$ and $\dim(\langle \mathcal{K}^* \rangle) = n^*$.*

3 Spreads of near polygons

Let $\mathcal{A}$ be a near polygon. For two lines K and L of $\mathcal{A}$ with $K \parallel L$ and $\mathrm{d}(K, L) = 1$, we define $\{K, L\}^\perp$ as the set of all lines intersecting K and L, and $\{K, L\}^{\perp\perp}$ as the set of all lines meeting every line of $\{K, L\}^\perp$. If $\{K, L\}^\perp$ and $\{K, L\}^{\perp\perp}$ cover the same points of $\mathcal{S}$, then the pair $\{K, L\}$ is called *regular*. A spread S of $\mathcal{A}$ is called *admissible* if every two lines of S are parallel. An admissible spread has a nice property with respect to geodetically closed subgeometries.

Theorem 5. *Let S be an admissible spread of a near polygon $\mathcal{A}$, let $L \in S$, and let H be a geodetically closed subgeometry of $\mathcal{A}$ through L. Then every line of S which meets H is completely contained in H.*

Proof. Suppose there is a line M of S which meets H in exactly one point m, and let $l \in L$ such that $\mathrm{d}(l,m) = \mathrm{d}(m,L)+1$. The unique point of M at distance $\mathrm{d}(m,L)$ from l is then on a geodetically closed path from l to m, a contradiction. ∎

An admissible spread S of $\mathcal{A}$ is called *regular* if $\{K,L\}$ is regular and $\{K,L\}^{\perp\perp} \subseteq S$ for all $K,L \in S$ with $\mathrm{d}(K,L) = 1$. A spread S of $\mathcal{A}$ is a *spread of symmetry* if for every $K \in S$ and every $k_1, k_2 \in K$, there exists an automorphism of $\mathcal{A}$ fixing each line of S and mapping k_1 to k_2.

Theorem 6. *Every spread of symmetry is a regular spread.*

Proof. Let S be a spread of symmetry of $\mathcal{A}$. For every two lines $K, L \in S$, the distance $\mathrm{d}(k,L)$ is independent of the chosen point $k \in K$. Hence, K and L are parallel. Suppose now that $\mathrm{d}(K,L) = 1$, and let M be a line meeting K and L. If G denotes the full group of automorphisms fixing each line of S, then $\{K,L\}^{\perp} = \{M^g | g \in G\}$ and $\{K,L\}^{\perp\perp} = \{m^G | m \in M\}$ is a subset of S, proving that S is regular. ∎

We now consider two cases.

(A) If $\mathcal{A}$ is the direct product of a line L with a near polygon $\mathcal{B} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, then $S = \{L_x | x \in \mathcal{P}\}$ with $L_x := \{(y,x) | y \in L\}$ is a spread of $\mathcal{A}$. Every spread obtained this way is called *trivial*. Clearly, every trivial spread is a spread of symmetry.

(B) Let Π_∞ be a $\mathrm{PG}(n,q)$ which is embedded as a hyperplane in $\Pi = \mathrm{PG}(n+1,q)$, and let $\mathcal{K}$ be a nonempty set of points of Π_∞ such that $T_n^*(\mathcal{K})$ is a near polygon. For every point x of $\mathcal{K}$, the set of all affine lines through x determines a spread S_x of $T_n^*(\mathcal{K})$.

Theorem 7. *The spread S_x is a spread of symmetry.*

Proof. The points of $T_n^*(\mathcal{K})$ are the points of $\mathrm{AG}(n+1,q)$. Let $K \in S_x$ and $k_1, k_2 \in K$. There exists then a unique translation T of $\mathrm{AG}(n+1,q)$ mapping k_1 to k_2. Clearly T defines an isomorphism of $T_n^*(\mathcal{K})$ which fixes each line of S_x. ∎

Theorem 8. *If $q \geq 3$, then the spreads S_x, $x \in \mathcal{K}$, are the only regular spreads of $T_n^*(\mathcal{K})$.*

Proof. By Theorems 6 and 7, every spread S_x, $x \in \mathcal{K}$, is regular. Conversely, suppose that S is a regular spread. We prove that every two lines K and L of S determine the same point at infinity. If $\mathrm{d}(K,L) = 1$, then the lines K and L are contained in a quad Q. By Theorem 4 the points of Q are the points of $\alpha \setminus \Pi_\infty$ where α is a two- or threedimensional subspace. If $\dim(\alpha) = 2$, then $|\alpha \cap \mathcal{K}| = 2$, Q is a grid, and K and L determine the same point at infinity. If $\dim(\alpha) = 3$, then $\alpha \cap \mathcal{K}$ is a hyperoval and the result follows from Theorem 3.3.4 of [9]. If $\mathrm{d}(K,L) = k > 1$, then there exist lines $M_0, \ldots, M_k \in S$ such that $M_0 = K$, $M_k = L$ and $\mathrm{d}(M_{i-1}, M_i) = 1$ for every $i \in \{1, \ldots, k\}$. Since $M_{i-1} \cap \Pi_\infty = M_i \cap \Pi_\infty$ for every $i \in \{1, \ldots, k\}$, K and L determine the same point at infinity. ∎

4 Two groups related to an admissible spread

Let S be an admissible spread of a near $2d$-gon $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$. The full group of automorphisms of $\mathcal{A}$ fixing each line of S is denoted by G_S. For every two lines K and L of S, one can define the projection P_L^K from the point set of K to the point set of L: $p_L^K(x)$ denotes the unique point on L nearest to $x \in K$. For a line $K \in S$, we call $\Pi_S(K) = \langle P_K^M \circ P_M^L \circ P_L^K | L, M \in S \rangle$ the *group of projectivities of K with respect to S*.

Theorem 9. (*a*) *The group $\Pi_S(K)$ is trivial if and only if S is trivial.*

(*b*) *If $\Pi_S(K)$ is not the trivial group, then for all $x_1, x_2 \in K$, there exists an element of $\Pi_S(K)$ mapping x_1 to x_2.*

Proof.

(a) Suppose that $\Pi_S(K)$ is trivial. For every point $x \in K$, the set $\Delta_x := \{y \in \mathcal{P} | \, \mathrm{d}(y, x) = \mathrm{d}(y, K)\}$ is a subspace of $\mathcal{A}$, i.e. a set of points of $\mathcal{A}$ intersecting each line in either the empty set, a singleton or the whole line. If x, y and z are points such that $x \in K$, $y \in \Delta_x$, $\mathrm{d}(y, z) = 1$ and $yz \notin S$, then there exists an element of $\Pi_S(K)$ mapping x to the unique point of K nearest to z, proving that $z \in \Delta_x$. Hence, for every point $y \in \mathcal{P} \setminus \Delta_x$, there exists a unique point $P_x(y) \in \Delta_x$ collinear with y. If $y_1, y_2 \in \mathcal{P} \setminus \Delta_x$ such that $\mathrm{d}(y_1, y_2) = 1$ and $y_1 y_2 \notin S$, then also $P_x(y_1)$ and $P_x(y_2)$ are collinear. Hence all subspaces Δ_x, $x \in K$, are isomorphic, and the result follows immediately.

(b) Let $x \in K$ and $\theta \in \Pi_S(K)$ such that $x^\theta \neq x$. It is sufficient to prove that the orbit of x under $\Pi_S(K)$ is equal to K. So, let $\tilde{x}$ be an arbitrary point of K. There exists a path $x = x_0, x_1, \ldots, x_k = x^\theta$ in $\mathcal{A}$ such that $\mathrm{d}(x_i, x_{i+1}) = 1$ and $x_i x_{i+1} \notin S$ for all $i \in \{0, \ldots, k-1\}$. Take now the smallest i such that x is not the unique point of K nearest to x_i, and let y be the unique point of $x_{i-1}x_i$ nearest to $\tilde{x}$. Since $x_{i-1}x_i$ and K are parallel, $\tilde{x}$ is the unique point of K nearest to y. If L and M are the elements of S through x_{i-1} and y, respectively, then $P_K^M \circ P_M^L \circ P_L^K$ maps x to $\tilde{x}$, proving the result. ■

Remark. If $\theta \in G_S$ fixes a point $x \in K$, then θ also fixes every point of $x^{\Pi_S(K)}$. Hence, if S is nontrivial, then only the trivial element of G_S has fixpoints.

The relation between the groups G_S and $\Pi_S(K)$ is the same as the one of Theorem 5.3 in [3].

Theorem 10. *If $\theta \in G_S$ then θ induces a permutation $\bar{\theta}$ on the point set of K that commutes with each element of $\Pi_S(K)$. Conversely, if a permutation ϕ on the point set of K commutes with each element of $\Pi_S(K)$, then $\phi = \bar{\theta}$ for some $\theta \in G_S$.*

Just as in Section 5 of [3], this implies the following result.

Theorem 11. *If K is a line of a nontrivial admissible spread S of a near polygon, then the following statements are equivalent:*

(1) *S is a spread of symmetry,*

(2) *$\Pi_S(K)$ acts regularly on the set of points of K,*

(3) *G_S acts regularly on the set of points of K.*

We also have the following result.

Theorem 12. *If $\Pi_S(K)$ is commutative, then S is a spread of symmetry.*

Proof. We may suppose that $\Pi_S(K)$ is not trivial. Since $\Pi_S(K)$ is commutative, every element of $\Pi_S(K)$ can be extended to an element of G_S (Theorem 10). Since $\Pi_S(K)$ acts transitively on the set of points of K (Theorem 9), S is a spread of symmetry. ∎

We now consider the case when the near polygon has a linear representation. So, let Π_∞ be a $\mathrm{PG}(n,q)$ which is embedded as a hyperplane in $\Pi = \mathrm{PG}(n+1,q)$ and let $\mathcal{K}$ be a nonempty set of points of Π_∞ such that $T_n^*(\mathcal{K})$ is a near polygon. For every point $\langle \bar{x} \rangle \in \mathcal{K}$, the set of all affine lines through $\langle \bar{x} \rangle$ determines a spread $S_{\langle \bar{x} \rangle}$ of $T_n^*(\mathcal{K})$. Let $K_1 = \langle \bar{x}, \bar{a} \rangle$ be a fixed line of $S_{\langle \bar{x} \rangle}$.

Theorem 13. (*a*) *The group $\Pi_{S_{\langle \bar{x} \rangle}}(K_1)$ is either trivial or isomorphic to the additive group of the finite field* $\mathrm{GF}(q)$.

(*b*) *If $\Pi_{S_{\langle \bar{x} \rangle}}(K_1)$ is the trivial group, or equivalently, if $S_{\langle \bar{x} \rangle}$ is a trivial spread, then $\langle \mathcal{K} \setminus \{\langle \bar{x} \rangle\} \rangle$ is a hyperplane of Π_∞ which does not contain $\langle \bar{x} \rangle$.*

Proof.

(a) Let $K_2, K_3 \in S_{\langle \bar{x} \rangle}$ such that $K_1 \neq K_2 \neq K_3 \neq K_4 := K_1$. The plane $\langle K_i, K_{i+1} \rangle$, $i \in \{1,2,3\}$, intersects Π_∞ in a line $\langle \bar{x}, \bar{u}_i \rangle$. We may suppose that $\langle \bar{u}_i \rangle$ is the unique point of $\langle \bar{x}, \bar{u}_i \rangle \setminus \{\langle \bar{x} \rangle\}$ with smallest index. Let $\lambda_1, \lambda_2, \lambda_3 \in \mathrm{GF}(q)$ such that $\langle \bar{a} + \lambda_1 \bar{u}_1 + \ldots + \lambda_i \bar{u}_i \rangle \in K_{i+1}$ for all $i \in \{1,2,3\}$. If $\Phi := p_{K_1}^{K_3} \circ p_{K_3}^{K_2} \circ p_{K_2}^{K_1}$, then Φ maps the point $\langle \bar{a} + \delta \bar{x} \rangle$ of K_1 to the point $\langle \bar{a} + \delta \bar{x} + \lambda_1 \bar{u}_1 + \lambda_2 \bar{u}_2 + \lambda_3 \bar{u}_3 \rangle$ of K_1. Hence there exists a $\mu \in \mathrm{GF}(q)$ such that $\lambda_1 \bar{u}_1 + \lambda_2 \bar{u}_2 + \lambda_3 \bar{u}_3 = \mu \bar{x}$, and $\Phi(\langle \bar{a} + \delta \bar{x} \rangle) = \langle \bar{a} + (\delta + \mu) \bar{x} \rangle$ for all $\delta \in \mathrm{GF}(q)$. As a consequence $\Pi_{S_{\langle \bar{x} \rangle}}(K_1)$ is a subgroup of the additive group of $\mathrm{GF}(q)$. If $\Pi_{S_{\langle \bar{x} \rangle}}(K_1)$ is not the trivial group, then, by (b) of Theorem 9, $\Pi_{S_{\langle \bar{x} \rangle}}(K_1)$ is isomorphic to the additive group of the finite field $\mathrm{GF}(q)$.

(b) Suppose that $\Pi_{S_{\langle \bar{x} \rangle}}(K_1)$ is trivial. Let L_1, L_2 and L_3 be three different coplanar lines of Π_∞ through $\langle \bar{x} \rangle$. Let $\langle \bar{u}_i \rangle$, $i \in \{1,2,3\}$, be the unique point of $L_i \setminus \{\langle \bar{x} \rangle\}$ with smallest index. Choose now $\lambda_1, \lambda_2, \lambda_3, \mu \in \mathrm{GF}(q)$ such that $\lambda_1 \bar{u}_1 + \lambda_2 \bar{u}_2 + \lambda_3 \bar{u}_3 = \mu \bar{x}$. Put $K_2 = \langle \bar{x}, \bar{a} + \lambda_1 \bar{u}_1 \rangle$ and $K_3 = \langle \bar{x}, \bar{a} + \lambda_1 \bar{u}_1 + \lambda_2 \bar{u}_2 \rangle$, then $\Phi := p_{K_1}^{K_3} \circ p_{K_3}^{K_2} \circ p_{K_2}^{K_1}$ maps $\langle \bar{a} + \delta \bar{x} \rangle$ to $\langle \bar{a} + (\delta + \mu) \bar{x} \rangle$ for all $\delta \in \mathrm{GF}(q)$. Since $\Pi_S(K_1)$ is trivial, $\mu = 0$ and $\langle \bar{u}_1 \rangle$, $\langle \bar{u}_2 \rangle$ and $\langle \bar{u}_3 \rangle$ are collinear. Hence the set of all points $\langle \bar{r} \rangle \neq \langle \bar{x} \rangle$ of Π_∞ with the property that $\langle \bar{r} \rangle$ is the unique point of $\langle \bar{r}, \bar{x} \rangle \setminus \{\langle \bar{x} \rangle\}$ with smallest index, is a hyperplane Δ of Π_∞. Clearly $\mathcal{K} \setminus \{\langle \bar{x} \rangle\} \subseteq \Delta$ and hence $\langle \mathcal{K} \setminus \{\langle \bar{x} \rangle\} \rangle \subseteq \Delta$. Since $T_n^*(\mathcal{K})$ is connected, it follows by Theorem 1 that $\langle \mathcal{K} \rangle = \Pi_\infty$. This implies that $\langle \mathcal{K} \setminus \{\langle \bar{x} \rangle\} \rangle = \Delta$. ∎

5 Glued near polygons

5.1 Construction

Let X denote a set with order $|X| \geq 2$. For every $i \in \{1,2\}$, consider the following objects:

(a) a near polygon $\mathcal{A}_i$,

(b) an admissible spread $S_i = \{L_1^{(i)}, \ldots, L_{\alpha_i}^{(i)}\}$ of $\mathcal{A}_i$,

(c) a bijection $\theta_i : X \to L_1^{(i)}$.

The line $L_1^{(i)}$, $i \in \{1,2\}$, is called the *base line* of S_i. For all $i \in \{1,2\}$ and all $j,k \in \{1,\ldots,\alpha_i\}$, let $p_{j,k}^{(i)}$ denote the projection from the line $L_j^{(i)}$ to the line $L_k^{(i)}$. We put $\Phi_{j,k}^{(i)} := p_{k,1}^{(i)} \circ p_{j,k}^{(i)} \circ p_{1,j}^{(i)}$. Consider the following graph Γ with vertex set $X \times S_1 \times S_2$. Two vertices $(x, L_{i_1}^{(1)}, L_{j_1}^{(2)})$ and $(y, L_{i_2}^{(1)}, L_{j_2}^{(2)})$ are adjacent if and only if exactly one of the following three conditions is satisfied:

(a) $L_{i_1}^{(1)} = L_{i_2}^{(1)}$, $L_{j_1}^{(2)} = L_{j_2}^{(2)}$ and $x \neq y$,

(b) $L_{j_1}^{(2)} = L_{j_2}^{(2)}$, $\mathrm{d}(L_{i_1}^{(1)}, L_{i_2}^{(1)}) = 1$ and $\Phi_{i_1,i_2}^{(1)} \circ \theta_1(x) = \theta_1(y)$,

(c) $L_{i_1}^{(1)} = L_{i_2}^{(1)}$, $\mathrm{d}(L_{j_1}^{(2)}, L_{j_2}^{(2)}) = 1$ and $\Phi_{j_1,j_2}^{(2)} \circ \theta_2(x) = \theta_2(y)$.

The diameter of Γ equals $d_1 + d_2 - 1$ if $\mathcal{A}_i$ is a near $2d_i$-gon, $i \in \{1,2\}$. Similarly as in Lemma 1 of [7], one can prove that every two adjacent vertices of Γ are contained in a unique maximal clique. Considering these maximal cliques as lines, we obtain a partial linear space $\mathcal{A}$. If $\mathcal{A}$ is a near polygon, then it is called a *glued near polygon*. This precisely happens when the condition in the following theorem is satisfied. The proof is similar as the one of Theorem 1 in [7].

Theorem 14. *The partial linear space $\mathcal{A}$ is a glued near hexagon if and only if $[\theta_1^{-1}\Pi_{S_1}(L_1^{(1)})\theta_1, \theta_2^{-1}\Pi_{S_2}(L_1^{(2)})\theta_2]$ is the trivial group.*

This condition is always satisfied if S_1 or S_2 is trivial. If S_1 is trivial, then $\mathcal{A}_1 \simeq L \times \mathcal{B}$ with L a line and $\mathcal{B}$ a near $2(d_1-1)$-gon. In that case $\mathcal{A} \simeq \mathcal{B} \times \mathcal{A}_2$.

Suppose now that S_1 and S_2 are not trivial and that the condition in the previous theorem is satisfied. By Theorem 10 every element of $\theta_2\theta_1^{-1}\Pi_{S_1}(L_1^{(1)})\theta_1\theta_2^{-1}$ extends to an automorphism of $\mathcal{A}_2$ fixing each line of S_2. By Theorem 9, S_2 is a spread of symmetry of $\mathcal{A}_2$. Similarly, S_1 is a spread of symmetry of $\mathcal{A}_1$. Summarizing we have the following result.

Theorem 15. *If $\mathcal{A}$ is a near polygon and if S_1 and S_2 are not trivial, then S_1 and S_2 are spreads of symmetry in the respective near polygons.*

In the construction of $\mathcal{A}$ the lines $L_1^{(1)}$ and $L_1^{(2)}$ of the spreads S_1 and S_2 seem to play a special role. If $\mathcal{A}$ is a near polygon, then $\mathcal{A}$ can be obtained starting with two arbitrary base lines (one in each spread). The maps θ_i, $i \in \{1,2\}$, needed to obtain $\mathcal{A}$ will then depend on the chosen base lines.

5.2 Spreads of symmetry in glued near polygons

We use the same notations as in Section 5.1. Suppose that $\mathcal{A}$ is a glued near polygon. For every $(i,j) \in \{1,\ldots,\alpha_1\} \times \{1,\ldots,\alpha_2\}$, $L_{i,j} := \{(x, L_i^{(1)}, L_j^{(2)}) | x \in X\}$ is a line of $\mathcal{A}$. Clearly $T = \{L_{i,j} | 1 \leq i \leq \alpha_1, 1 \leq j \leq \alpha_2\}$ is a spread of $\mathcal{A}$.

Theorem 16. *The spread T is a spread of symmetry if and only if the following two conditions are satisfied:*

(*a*) *S_1 and S_2 are spreads of symmetry,*

(*b*) *if S_1 and S_2 are not trivial, then $\theta_1^{-1}\Pi_{S_1}(L_1^{(1)})\theta_1$ and $\theta_2^{-1}\Pi_{S_2}(L_1^{(2)})\theta_2$ are equal. Hence, by Theorem 14, both groups are commutative.*

Proof. Consider the line $L_{1,1}$ of T. One calculates that $\Pi_T(L_{1,1})$ is generated by $\theta_1^{-1}\Pi_{S_1}(L_1^{(1)})\theta_1$ and $\theta_2^{-1}\Pi_{S_2}(L_1^{(2)})\theta_2$. The theorem follows then from Theorem 11. ■

5.3 Glued near polygons with a linear representation

Part (B) of Theorem 2 can be improved as follows.

Theorem 17. *Let Π_∞ be a $\mathrm{PG}(n,q)$ which is embedded as a hyperplane in $\Pi = \mathrm{PG}(n+1,q)$. Consider in Π_∞ two subspaces π_1 and π_2 of dimensions $n_1 \geq 0$ and $n_2 \geq 0$ respectively, such that $\pi_1 \cap \pi_2 = \{p\}$ and $\Pi_\infty = \langle \pi_1, \pi_2 \rangle$. If $\mathcal{K}_i$, $i \in \{1,2\}$, is a set of points of π_i containing p such that $T_{n_i}^*(\mathcal{K}_i)$ is a near polygon, then $T_n^*(\mathcal{K}_1 \cup \mathcal{K}_2)$ is a glued near polygon.*

Proof.

(a) Let a be a fixed point of $\Pi \setminus \Pi_\infty$. For each $i \in \{1,2\}$, let $\mathcal{A}_i$ be the near polygon $T_{n_i}^*(\mathcal{K}_i)$ determined by the embedding $\pi_i \subseteq \langle a, \pi_i \rangle$, let S_i be the spread of $\mathcal{A}_i$ determined by the point p at infinity, let $L_1^{(i)}$ be the line $X := pa$, and let θ_i be the identical permutation of X. By Theorems 13 and 14, the associated incidence structure $\mathcal{A}$ is a glued near polygon.

(b) We prove that for every $\alpha \in \Pi \setminus \Pi_\infty$, there exists a unique $\tilde{\alpha} \in \langle a, \pi_1 \rangle$ such that $\mathrm{d}(\alpha,\gamma) = \mathrm{d}(\alpha,\tilde{\alpha}) + \mathrm{d}(\tilde{\alpha},\gamma)$ for every $\gamma \in \langle a, \pi_1 \rangle$. We may suppose that $\alpha \notin \langle a, \pi_1 \rangle$. The unique point $\bar{\alpha}$ of $(\langle a, \alpha, \pi_1 \rangle \cap \pi_2) \setminus \{p\}$ with smallest $(\mathcal{K}_1 \cup \mathcal{K}_2)$-index is also the unique point of $(\langle a, \alpha, \pi_1 \rangle \cap \Pi_\infty) \setminus \pi_1$ with smallest $(\mathcal{K}_1 \cup \mathcal{K}_2)$-index, proving that $\{\tilde{\alpha}\} := \alpha\bar{\alpha} \cap \langle a, \pi_1 \rangle$ is the unique point of $\langle a, \pi_1 \rangle$ nearest to α. For every other point $\gamma \neq \tilde{\alpha}$ of $\langle a, \pi_1 \rangle$, the points $\tilde{\alpha}\gamma \cap \Pi_\infty$, $\alpha\gamma \cap \Pi_\infty$ and $\bar{\alpha}$ are on the same line, implying that $i_{\mathcal{K}_1 \cup \mathcal{K}_2}(\alpha\gamma \cap \Pi_\infty) = i_{\mathcal{K}_1 \cup \mathcal{K}_2}(\bar{\alpha}) + i_{\mathcal{K}_1 \cup \mathcal{K}_2}(\tilde{\alpha}\gamma \cap \Pi_\infty)$ or $\mathrm{d}(\alpha,\gamma) = \mathrm{d}(\alpha,\tilde{\alpha}) + \mathrm{d}(\tilde{\alpha},\gamma)$.

(c) For every point x of $T_n^*(\mathcal{K}_1 \cup \mathcal{K}_2)$, let $\phi_0(x)$ denote the unique point of pa nearest to x, and let $\phi_i(x) = \langle x, \pi_{3-i} \rangle \cap \langle a, \pi_i \rangle$, $i \in \{1,2\}$. We now prove that the bijection $\theta : x \mapsto (\phi_0(x), \phi_1(x), \phi_2(x))$ is actually an isomorphism between $T_n^*(\mathcal{K}_1 \cup \mathcal{K}_2)$ and $\mathcal{A}$. Since both geometries have order $(q-1, |\mathcal{K}_1 \cup \mathcal{K}_2| - 1)$, it suffices to prove that θ preserves adjacency. So, let x_1 and x_2 be two different adjacent points in $T_n^*(\mathcal{K}_1 \cup \mathcal{K}_2)$. If p is on the line x_1x_2, then $\phi_1(x_1) = \phi_1(x_2)$ and $\phi_2(x_1) = \phi_2(x_2)$, proving that $\theta(x_1)$ and $\theta(x_2)$ are collinear. Suppose therefore that the line x_1x_2 meets Π_∞ in a point $u \in \mathcal{K}_1 \setminus \mathcal{K}_2$. We have $\phi_2(x_1) = \phi_2(x_2)$ and $\mathrm{d}(\phi_1(x_1), \phi_1(x_2)) = 1$ since the plane $\langle \phi_1(x_1), \phi_1(x_2) \rangle$ meets π_1 in the line pu. From $1 + \mathrm{d}(x_2, \tilde{x}_2) \geq \mathrm{d}(x_1, \tilde{x}_2) = \mathrm{d}(x_1, \tilde{x}_1) + \mathrm{d}(\tilde{x}_1, \tilde{x}_2)$ and $1 + \mathrm{d}(x_1, \tilde{x}_1) \geq \mathrm{d}(x_2, \tilde{x}_2) + \mathrm{d}(\tilde{x}_1, \tilde{x}_2)$, it follows $\mathrm{d}(\tilde{x}_1, \tilde{x}_2) \leq 1$. Since $\tilde{x}_1 \in \phi_1(x_1)$ and $\tilde{x}_2 \in \phi_1(x_2)$, $\mathrm{d}(\tilde{x}_1, \tilde{x}_2) = 1$, or equivalently "$\Phi^{(1)}_{i_1,i_2} \circ \theta_1(x) = \theta_1(y)$" with the notations of Section 5.1. ∎

Theorem 18. *Every glued near polygon with a linear representation is obtained as described in Theorem 17.*

Proof. Let $\mathcal{A}$ be a glued near polygon arising in the way as described in Section 5.1 from near polygons $\mathcal{A}_i$, spreads S_i and bijections θ_i, $i = 1, 2$. Suppose also that $\mathcal{A} \simeq T_n^*(\mathcal{K})$ where $\mathcal{K}$ is a set of points in $\Pi_\infty = \mathrm{PG}(n, q)$. Let K_i, $i \in \{1, 2\}$, denote a fixed line of S_i. All the points of $\mathcal{A}$ with K_i, $i \in \{1, 2\}$, as $(i + 1)$-th coordinate determines a geodetically closed sub near polygon $\mathcal{B}_{3-i} \simeq \mathcal{A}_{3-i}$. By Theorem 4, there exists a subspace π_i of Π_∞ such that $\mathcal{K}_i = \pi_i \cap \mathcal{K}$ and $\mathcal{B}_i \simeq T_{n_i}(\mathcal{K}_i)$ with $n_i = \dim(\pi_i)$. Since $\mathcal{B}_1$ and $\mathcal{B}_2$ intersect in a line L, $\pi_1 \cap \pi_2$ is a point of $\mathcal{K}$. Every line through a point $x \in L$ is contained in either $\mathcal{B}_1$ or $\mathcal{B}_2$, proving that $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$. By Theorem 1, $\langle \mathcal{K} \rangle = \Pi_\infty$ and hence also $\langle \pi_1, \pi_2 \rangle = \Pi_\infty$. ∎

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