Extremal Kähler $\mathcal{A}C^{\perp}$-surfaces

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Abstract

The aim of this paper is to give an example of a Kähler extremal metric with harmonic anti-self-dual Weyl tensor on the Hirzebruch surface F_1.

Introduction.

It is known that self-dual Kähler 4-manifolds (M, g, J) are Bochner-flat. M. Matsumoto and S. Tanno proved [M-T] that every Bochner flat Kähler manifold satisfies the condition

$$\nabla_X \rho(Y,Z) = \frac{1}{(2\dim M + 4)}(g(X,Y)Z\tau + g(X,Z)Y\tau + 2g(Y,Z)X\tau \qquad (*)$$
$$-g(JX,Y)(JZ)\tau - g(JX,Z)(JY)\tau),$$

where τ is the scalar curvature of (M, g). Consequently, Ricci tensor ρ of any Kähler Bochner-flat manifold satisfies the condition

$$\nabla_X \rho(X,X) = \frac{2}{n+2} X\tau g(X,X), \qquad (**)$$

where τ is the scalar curvature of (M, g) and $n = \dim M$. This property was studied by A. Gray in [G]. A. Gray called Riemannian manifolds satisfying $(**)$ the $\mathcal{A}C^{\perp}$ manifolds. In [J-1] we showed that every Kähler surface has a harmonic anti-self-dual part W^- of the Weyl tensor W (i.e. such that $\delta W^- = 0$) if and only if it is an $\mathcal{A}C^{\perp}$-manifold. We also proved that a Kähler manifold is an $\mathcal{A}C^{\perp}$-manifold if and only

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if it satisfies the condition (*). In that way we generalized the result of Sekigawa and Vanhecke [S-V] who proved that Kähler $\mathcal{AC}^{\perp}$ manifold with constant scalar curvature has parallel Ricci tensor. In [J-1] we have proved (in the real analytic case, for the general case see [A-C-G]):

Theorem. *Every compact Kähler surface (M, g, J) with harmonic anti-self-dual Weyl tensor ($\delta W^{-} = 0$) and non-zero signature $\sigma(M)$ is an Einstein manifold.*

In [J-1] I have also proved that the only such irreducible examples with $\sigma(M) = 0$ should be extremal metrics on ruled surfaces. The aim of the present paper is to construct an example of an extremal Kähler metric with non-constant scalar curvature and harmonic anti-self-dual Weyl tensor on the Hirzebruch surface F_1 which is a ruled surface of genus 0 and to show that this is the only such example in the class of Hirzebruch surfaces $F_k, k \in \mathbb{N}$. First we investigate the general construction of cohomogeneity one metrics on Hirzebruch surfaces. Using the methods of B. Bergery (see [B],[S],[P]) and our results from [J-1] we reduce the problem to a certain ODE of the second order. We show that this equation has a positive solution satisfying the appropriate boundary conditions only on the first Hirzebruch surface F_1. We show that on F_k with cohomogeneity one metric there exists a totally geodesic distribution $\mathcal{D}$ such that its orthogonal complement $\mathcal{D}^{\perp}$ in TM is umbilical. This enables us to verify that the tensor $C(X, Y) = \rho(X, Y) - \frac{2}{n+2} X\tau g(X, Y)$, where ρ is the Ricci tensor of the metric given by this construction, is Killing and we shall prove in this way the existence of an extremal metric with properties that interests us. We also investigate the opposite Hermitian structure $\overline{J}$ (described in [J-1]) of our example (F_1, g, J). We shall show that the Kähler structure J is the natural opposite almost Hermitian structure of a Hermitian structure $\overline{J}$. In this way we also give a new example of compact 4-dimensional $\mathcal{AC}^{\perp}$-manifold (compare [Be] p.433). We have divided the paper into three sections. In the first section we recall some basic properties of Hermitian manifolds, Killing tensors, distributions and foliations. In the section two we investigate the general construction of cohomogeneity one metrics on 4-manifolds $M = (a, b) \times P$ where $p : P \to N$ is a circle bundle over a Riemannian surface of constant sectional curvature. We show that on M there exist a Killing tensor with two-dimensional eigendistributions and two Hermitian structures, which commute with this tensor. The section three is devoted to the construction of the example of compact Kähler surface with harmonic anti-self-dual Weyl tensor.

1 Hermitian 4-manifolds.

Let (M, g, J) be an almost Hermitian manifold. We say that (M, g, J) is a Hermitian manifold if its almost Hermitian structure J is integrable. In the sequel we shall consider 4-dimensional Hermitian manifolds (M, g, J) which we shall also call Hermitian surfaces. Such manifolds are always oriented and we choose an orientation in such a way that the Kähler form $\Omega(X, Y) = g(JX, Y)$ is self-dual form (i.e. $\Omega \in \wedge^{+} M$). The vector bundle of self-dual forms admits a decomposition

$$\wedge^{+} M = \mathbb{R}\Omega \oplus LM, \tag{1.1}$$

where by LM we denote the bundle of real J-skew invariant 2-forms (i.e $LM = \{\Phi \in \wedge M : \Phi(JX, JY) = -\Phi(X, Y)\}$). The bundle LM is a complex line bundle over M with the complex structure $\mathcal{J}$ defined by $(\mathcal{J}\Phi)(X,Y) = -\Phi(JX,Y)$. For a 4-dimensional Hermitian manifold the covariant derivative of the Kähler form Ω is locally expressed by

$$\nabla\Omega = a \otimes \Phi + \mathcal{J}a \otimes \mathcal{J}\Phi, \tag{1.2}$$

where $\mathcal{J}a(X) = -a(JX)$. The Lee form θ of (M,g,J) is defined by the equality $d\Omega = \theta \wedge \Omega$. We have $\theta = -\delta\Omega \circ J$. A Hermitian manifold (M,g,J) is said to have Hermitian Ricci tensor ρ if $\rho(X,Y) = \rho(JX,JY)$ for all $X,Y \in \mathfrak{X}(M)$. The conformal scalar curvature κ of a Hermitian manifold (M,g,J) is defined by $\kappa = \tau - \frac{3}{2}(|\theta|^2 + 2\delta\theta)$.

An opposite (almost) Hermitian structure on a Hermitian 4-manifold (M,g,J) is an (almost) Hermitian structure $\overline{J}$ whose Kähler form (with respect to g) is anti-self-dual.

On a Riemannian manifold a distribution $\mathcal{D} \subset TM$ is called umbilical (see [J-3]) if $\nabla_X X_{|\mathcal{D}^{\perp}} = g(X,X)\xi$ for every $X \in \Gamma(\mathcal{D})$, where $X_{|\mathcal{D}^{\perp}}$ is the $\mathcal{D}^{\perp}$ component of X with respect to the orthogonal decomposition $TM = \mathcal{D} \oplus \mathcal{D}^{\perp}$. The vector field ξ is called the mean curvature normal of $\mathcal{D}$. An involutive distribution $\mathcal{D}$ is tangent to a foliation, which is called totally geodesic if its every leaf is a totally geodesic submanifold of (M,g) i.e. $\nabla_X X \in \mathcal{D}$ if X is a section of a vector bundle $\mathcal{D} \subset TM$. In the sequel we shall not distinguish between $\mathcal{D}$ and a tangent foliation and we shall also say that $\mathcal{D}$ is totally geodesic in such a case. A Riemannian metric on a manifold M is called bundle like with respect to a foliation $\mathcal{D} \subset TM$ (see [M]), if a geodesic perpendicular at one point to a leaf remains perpendicular to all the leaves it meets, equivalently if $X \in \Gamma(\mathcal{D}^{\perp})$ then $\nabla_X X \in \mathcal{D}^{\perp}$.

On any Hermitian non-Kähler 4-manifold (M,g,J) there are two natural distributions $\mathcal{D} = \{X \in TM : \nabla_X J = 0\}$, $\mathcal{D}^{\perp}$ defined in the open set $U = \{x : |\nabla J_x| \neq 0\}$. The distribution $\mathcal{D}$ we shall call the nullity distribution of (M,g,J). From (1.2) it is clear that $\mathcal{D}$ is J-invariant and that $\dim\mathcal{D} = 2$ in $U = \{x \in M : \nabla J_x \neq 0\}$. By $\mathcal{D}^{\perp}$ we shall denote the orthogonal complement of $\mathcal{D}$ in U. On U we can define the opposite almost Hermitian structure $\overline{J}$ by formulas $\overline{J}X = JX$ if $X \in \mathcal{D}^{\perp}$ and $\overline{J}X = -JX$ if $X \in \mathcal{D}$ which we shall call natural opposite almost Hermitian structure. It is not difficult to check that for the famous Einstein Hermitian manifold $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$ with D. Page's metric (see [P],[B],[S],[K],[LeB]) the opposite structure $\overline{J}$ is Hermitian and this structure extends to the global opposite Hermitian structure .

By an $\mathcal{AC}^{\perp}$- manifold (see [G]) we mean a Riemannian manifold (M,g) satisfying the condition

$$\mathfrak{C}_{XYZ}\nabla_X\rho(Y,Z) = \frac{2}{(\dim M + 2)}\mathfrak{C}_{XYZ}X\tau g(Y,Z),$$

where ρ is the Ricci tensor of (M,g) and $\mathfrak{C}$ means the cyclic sum. A Riemannian manifold (M,g) is an $\mathcal{AC}^{\perp}$ manifold if and only if the Ricci endomorphism Ric of (M,g) is of the form $Ric = S + \frac{2}{n+2}\tau Id$ where S is a Killing tensor, τ is the scalar curvature and $n = \dim M$. Let us recall that a (1,1) tensor S on a Riemannian manifold (M,g) is called a Killing tensor if $g(\nabla S(X,X),X) = 0$ for all $X \in TM$. It is not difficult to prove the following lemma:

Lemma. *Let $S \in End(TM)$ be a (1,1) tensor on a Riemannian 4-manifold (M,g). Let us assume that S has exactly two everywhere different eigenvalues λ, μ*

of the same multiplicity 2, i.e. dim $\mathcal{D}_\lambda$= *dim* $\mathcal{D}_\mu = 2$, *where* $\mathcal{D}_\lambda, \mathcal{D}_\mu$ *are eigendistributions of* S *corresponding to* λ, μ *respectively. Then* S *is a Killing tensor if and only if both distributions* $\mathcal{D}_\lambda$ *and* $\mathcal{D}_\mu$ *are umbilical with mean curvature normal equal respectively*

$$\xi_\lambda = \frac{\nabla\mu}{2(\lambda-\mu)},\ \xi_\mu = \frac{\nabla\lambda}{2(\mu-\lambda)}.$$

2 Hermitian surfaces with Hermitian opposite natural almost Hermitian structure and Killing tensors.

Let us recall the following result proved in [J-1]:

Proposition 1. *Let* S *be a Killing tensor on a 4-dimensional Riemannian manifold* (M,g). *Let us assume that* S *has two 2-dimensional oriented eigendistributions* $\mathcal{D}_\lambda, \mathcal{D}_\mu$. *Then there exist two opposite Hermitian structures* $J, \overline{J}$ *on* M *which commute with* S.

We shall prove

Proposition 2. *Let* (M,g) *be a 4-dimensional Riemannian manifold. Let* $\mathcal{D}$ *be a two dimensional totally geodesic Riemannian foliation on* M *such that* g *is bundle like with respect to* $\mathcal{D}$. *Then* M *admits (up to 4-fold covering) two opposite Hermitian structures* $J, \overline{J}$ *such that* $J|\mathcal{D} = -\overline{J}|\mathcal{D}, J|\mathcal{D}^\perp = \overline{J}|\mathcal{D}^\perp$. *The nullity distribution of both* J *and* $\overline{J}$ *contains* $\mathcal{D}$.

Proof. To prove the first part of the Proposition it is enough to show that on M there exists a Killing tensor with eigendistributions $\mathcal{D}, \mathcal{D}^\perp$. Since the foliation $\mathcal{D}$ is totally geodesic and bundle-like it follows that $\nabla_X X \in \Gamma(\mathcal{D})$ (resp.) $\nabla_X X \in \Gamma(\mathcal{D}^\perp)$ if $X \in \Gamma(\mathcal{D})$ (resp.) $X \in \Gamma(\mathcal{D}^\perp)$. Consequently, a tensor S defined by

$$SX = \lambda X \text{ if } X \in \mathcal{D}$$
$$SX = \mu X \text{ if } X \in \mathcal{D}^\perp$$

where $\lambda \neq \mu$ are two different real numbers is a smooth Killing tensor. We can assume (up to 4-fold covering) that the distributions $\mathcal{D}, \mathcal{D}^\perp$ are orientable. Let us denote by J the only almost hermitian structure which preserves $\mathcal{D}, \mathcal{D}^\perp$ and agrees with their orientations. We define $\overline{J}$ by $J|\mathcal{D} = -\overline{J}|\mathcal{D}, J|\mathcal{D}^\perp = \overline{J}|\mathcal{D}^\perp$. From Prop. 1. it follows that both J and $\overline{J}$ are hermitian. Now let $\{E_1, E_2\}$ be a local orthonormal frame on $\mathcal{D}$. Then it is clear that

$$\nabla J(E_1, E_1) + J(\nabla_{E_1} E_1) = \nabla_{E_1} E_2.$$

It follows that $\nabla J(E_1, E_1) \in \mathcal{D}$ and consequently, $\nabla_{E_1} J = 0$. Analogously $\nabla_{E_2} J = 0$. ∎

Corollary 1. Let us assume that $(M, g, \mathcal{D})$ with totally geodesic 2-dimensional distribution $\mathcal{D}$ admits two hermitian structures $J, \overline{J}$ constructed as above. Let us assume that a positive function $f \in C^\infty(M)$ satisfies a condition $df(\mathcal{D}^\perp) = 0$. Then $\mathcal{D}$ is also contained in the nullity of both $\overline{J}$ and J with respect to the metric fg and $\mathcal{D}^\perp$ is umbilical with respect to fg.

Proof. It is clear that $J, \overline{J}$ are Hermitian with respect to the metric $\overline{g} = fg$. We shall show that $\mathcal{D}$ is totally geodesic and $\mathcal{D}^{\perp}$ is umbilical with respect to $\overline{g}$. It is clear in view of the formula

$$\overline{\nabla}_X Y = \nabla_X Y + df(X)Y + df(Y)X - g(X,Y)\nabla f,$$

where $\overline{\nabla}$ is the Levi-Civita connection of $\overline{g}$ since $\nabla f \in \Gamma(\mathcal{D})$. ■

Let (M, g_0) be a compact Riemannian surface of constant curvature $K \in \mathbb{R}$ and let $p : P \to M$ be a principal circle bundle over M with a connection form θ such that $d\theta = cp^*\omega$ where ω is the volume form of (M, g) and $c \in \mathbb{R}$. The manifold P with the metric $g_P = \theta \otimes \theta + p^* g_0$ is a 3-dimensional $\mathcal{A}$-manifold. Let $\theta^\sharp$ be a vector field dual to θ with respect to g_P. Let us consider a local orthonormal frame $\{X_*, Y_*\}$ on (M, g_0) and let X, Y be the horizontal lifts of X_*, Y_* with respect to $p : P \to M$ (i.e. $\theta(X) = \theta(Y) = 0$ and $p(X) = X_*,\ p(Y) = Y_*$) and let $H = \frac{\partial}{\partial t}, Z = \theta^\sharp$. Now let us consider the manifold $Q = \mathbb{R} \times P$ with the metric

$$g_{f,h} = dt \otimes dt + f(t)^2 \theta \otimes \theta + h(t)^2 p^* g_0. \tag{2.1}$$

Let us define two almost Hermitian structures $J, \overline{J}$ on Q as follows

$$JH = \frac{1}{f} Z, JX = Y,\ \overline{J}H = -\frac{1}{f} Z, \overline{J}X = Y. \tag{2.2}$$

Proposition 3. *Let $\mathcal{D}$ be a distribution spanned by the fields $\{\theta^\sharp, H\}$. Then $\mathcal{D}$ is a totally geodesic foliation and $\mathcal{D}^{\perp}$ is umbilical with respect to the metric $g_{f,h}$. Both structures J and $\overline{J}$ are Hermitian and $\mathcal{D}$ is contained in the nullity of J and $\overline{J}$.*

Proof. Since $[H, \theta^\sharp] = 0$ it is clear that $\mathcal{D}$ is a foliation. We also have $\nabla^{f,h}_H H = 0, \nabla^{f,h}_{\theta^\sharp} \theta^\sharp = -\frac{1}{2} \operatorname{grad}_{g_{f,h}} f^2 \in \Gamma(\mathcal{D})$. Since $\theta^\sharp$ is a Killing field for $g_{f,h}$ is follows that $[X, \theta^\sharp] = 0$. We shall show that $g(\nabla_H \theta^\sharp, X) = 0$. In fact

$$2g(\nabla_H \theta^\sharp, X) = g([X, H], \theta^\sharp) - g([\theta^\sharp, X], H) - g([\theta^\sharp, H], X) = 0. \tag{2.3}$$

Note that if $h = 1$ then the mapping $p_Q : Q \to M$ given by $p_Q(t, x) = p_P(x)$ is a Riemannian submersion. It follows that in this case both distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are geodesic i.e. $\nabla_U U \in \Gamma(\mathcal{D})($ resp. $\in \Gamma(\mathcal{D}^{\perp}))$ if $U \in \Gamma(\mathcal{D})$ (resp. if $U \in \Gamma(\mathcal{D}^{\perp})$) or equivalently a metric $g_{f,1}$ is bundle like with respect to the foliation $\mathcal{D}$. Consequently, if λ, μ are two different real numbers, then the tensor S defined as follows:

$$SU = \lambda U \text{if } U \in \mathcal{D}, \tag{2.4a}$$

$$SU = \mu U \text{ if } U \in \mathcal{D}^{\perp}, \tag{2.4b}$$

is a Killing tensor. Thus both J and $\overline{J}$ are Hermitian if $h = 1$. Note however that the metric $g_{f,h}$ is conformally equivalent to the metric $g_{u,1}$ where $u = \frac{f}{h} \circ \phi^{-1}$ where $d\phi = \frac{1}{h} dt$ and the conformal factor h^2 satisfies the assumptions of Corollary 1. Thus the result holds in the general case also. ■

The following result easily follows from [D-1],[A-G] and [J-1].

Proposition 4. *Let (M, g) be a compact 4-manifold with even first Betti number admitting two opposite Hermitian structures $J, \overline{J}$ which commute with the Ricci tensor ρ of (M, g). Then M is a ruled surface or is locally a product of two Riemannian surfaces.*

Proof. Let us assume that J is non-Kähler structure i.e. $\nabla J \neq 0$. Then (M, g, J) is conformally Kähler, in particular admits a non-zero holomorphic vector field ξ with zeros. If $\overline{J}$ is Kähler then, since M is compact, ξ is also a holomorphic Killing vector field for $(M, g, \overline{J})$. In the other case $(M, g, \overline{J})$ admits, analogously as above, a holomorphic vector field η with zeros. From [C-H-K] it follows that both (M, g, J) and $(M, g, \overline{J})$ are blow-ups of a ruled surface or $\mathbb{CP}^2$ at a finite number of points. Thus the signature $\sigma(M)$ of M equals to zero and (M, J) is a ruled surface (see [J-1]). If both J and $\overline{J}$ are Kähler then it is obvious that M is locally a product of two Riemannian surfaces. ∎

3 An example of a Kähler metric with harmonic anti-self-dual Weyl tensor on the Hirzebruch surface F_1.

In my paper [J-1] I have proved the following:

Theorem 1. *Every compact real analytic Kähler surface (M, g, J) with harmonic anti-self-dual Weyl tensor ($\delta W^- = 0$) has constant scalar curvature (and thus is Einstein or is locally a product of two Riemannian surfaces with constant sectional curvatures) or is a ruled surface with extremal Kähler metric and non-constant scalar curvature which admits an opposite Hermitian structure $\overline{J}$ such that $(M, g, \overline{J})$ satisfies a (G_2) condition of A.Gray and is conformal to an extremal Kähler surface.*

Our present aim is to construct an example of a compact Kähler surface (M, g, J) with harmonic anti-self-dual Weyl tensor ($\delta W^- = 0$) and non-constant scalar curvature on the Hirzebruch surface F_1. These metrics are extremal in view of Th.1. As in [M-S] by $L(k, 1)$, where $k \in \mathbb{N}$, we shall denote the Lens spaces. By F_k we denote the k-th Hirzebruch surface i.e. the holomorphic $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$ associated with the principal bundle $p : P(k) = L(k, 1) \to \mathbb{CP}^1$ (it is the space of cohomogeneity 1 under an action of $U(2)$ with principal orbit $L(k, 1)$ and two special orbits $\mathbb{CP}^1$ i.e. $[\mathbb{CP}^1 | L(k, 1) | \mathbb{CP}^1]$. The diffeomorphism type of F_k depends only on the parity of k: if k is even , then F_k is diffeomorphic to $S^2 \times S^2$, for k odd, F_k is diffeomorphic to $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$. Let $a, b \in \mathbb{R}$ be any two real numbers such that $a < b$. Let us consider the metric $g_{f,h}$ on a product $(a, b) \times P(k)$ given by the formula:

$$g_{f,h} = dt^2 + g_t, \tag{3.1}$$

where $g_t = f^2(t)\theta^2 + h(t)^2 p^* \frac{can}{4}$ is the metric on $P(k)$ parameterized by t, can denotes the canonical metric on S^2 of constant curvature 1, $d\theta = 2kp^*\omega_{FS}$ where ω_{FS} is the Kähler form of $(\mathbb{CP}^1, \frac{can}{4})$ and $f, h \in C^\infty(a, b)$ are positive functions defined on (a, b).

Note that the projection

$$p : (P(k), g_t) \to (S^2, h(t)^2 \frac{can}{4})$$

is a Riemannian submersion, when t is fixed. We have (see [B], [M-S], [S])

Proposition 5. *The metric $g_{f,h}$ defined on $(a,b) \times P(k)$ extends to the smooth metric on the Hirzebruch surface $F_k; k \in \mathbb{N}$ if the following conditions are satisfied:*
(a)$f(a) = f(b) = 0, f'(a) = 1, f'(b) = -1, f^{(2p)}(a) = f^{(2p)}(b) = 0$ for $p \in \mathbb{N}$,
(b)$h(a) \neq 0 \neq h(b), h'(a) = h'(b) = 0, h^{(2p+1)}(a) = h^{(2p+1)}(b) = 0$ for $p \in \mathbb{N}$.

Let $U = (a,b) \times P(k) \subset F_k$. Then on U we have two hermitian structures $J, \overline{J}$ defined by Prop.3. From the results of B. Bergery [B](compare also [J-2]) it follows that if $f = \frac{hh'}{k}$ then (with the careful choice of orientation) the structure $\overline{J}$ defined on U is Kähler. Thus we have

Proposition 6. Let a function $h \in C^{\infty}(a,b)$ satisfy the condition (b) of Prop.5 and let us assume that $h' > 0$ on (a,b) and $h(a)h''(a) = k, h(b)h''(b) = -k$. Then the metric

$$g_h = dt^2 + \frac{1}{k^2}(hh')^2\theta^2 + h^2 p^* \frac{can}{4} \tag{3.2}$$

extends to a smooth Kähler metric on the Hirzebruch surface F_k.

Proof. It follows from the formula

$$f^{(2q)}(a) = \sum_{p=0}^{2q} \binom{2q}{p} h^{(p)}(a) h^{(2q+1-p)}(a). \tag{3.3}$$

Remark. From the above Corollary and [B] it follows that there are many examples of metrics g on ruled surfaces F_k described in Prop.4 (i.e. which admit two opposite hermitian structures which commute with the Ricci tensor of (F_k, g)). One of them is of course the D.Page's Hermitian Einstein metric on F_1.

Now our aim is to prove the following theorem:

Theorem 2. *On the Hirzebruch surface $F_1 = \mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$ with a standard complex structure $\overline{J}$ there exist an extremal Kähler metric g with harmonic anti-self-dual Weyl tensor and this is the only such metric in the class of Hirzebruch surfaces F_k. The surface $(F_1, g, \overline{J})$ admits an opposite Hermitian structure J such that $\overline{J}$ is the natural opposite almost Hermitian structure for J.*

Proof. Let us recall the following results of B. Bergery. Let us assume that $f = \frac{hh'}{k}$. Then (U, g_h) is a Kähler surface (see [B]). By $\mathcal{D}$ we shall denote the nullity distribution of the opposite Hermitian structure J. Then the Ricci tensor of (U, g_h) has two eigenvalues λ, μ corresponding to the eigendistributions $\mathcal{D}_\lambda = \mathcal{D}, \mathcal{D}_\mu = \mathcal{D}^{\perp}$ which are given by the following formulas:

$$\lambda = -2\frac{h''}{h} - \frac{f''}{f} = -\frac{f''}{f} + 2(k^2\frac{f^2}{h^4} - \frac{f'h'}{fh}), \tag{3.4a}$$

$$\mu = -\frac{h''}{h} + (k^2\frac{f^2}{h^4} - \frac{f'h'}{fh}) + \frac{4}{h^2} - (\frac{h'}{h})^2 - 3k^2\frac{f^2}{h^4}. \tag{3.4b}$$

Let us denote (see (2.2)) $E_1 = \frac{X}{h}, E_2 = \frac{Y}{h}, E_3 = \frac{Z}{f}, E_4 = H$. Then $\Gamma^4_{11} = -\frac{h'}{h} = \Gamma^4_{22}$. Note that since ρ is Hermitian with respect to Hermitian structure J and the natural opposite structure of J is Kähler it follows (see [J-2] where we take $H = -E_4$) that $\Gamma^4_{11} = -\frac{1}{2}\alpha$ where $\alpha = \frac{1}{2\sqrt{2}}|\nabla J|$. On the other hand $-\alpha = \frac{\kappa'}{3\kappa}$ where κ is the conformal scalar curvature of (M, g, J). Thus we have $\kappa = C_0 h^{-6}$ for a non-zero constant $C \in \mathbb{R}$. Let us assume that $\rho - \frac{1}{3}\tau g$ is a Killing tensor. Then $\kappa = C_1(\lambda - \mu)^{-3}$ (see [J-1]). It follows that

$$\lambda - \mu = Ch^2$$

for some constant $C \in \mathbb{R} - \{0\}$. Thus we obtain an equation

$$-\frac{f''}{f} + 4(\frac{h'}{h})^2 - \frac{4}{h^2} = Ch^2. \tag{3.5}$$

Let us write $h' = \sqrt{P(h)}$. Then equation (3.5) reads

$$h^2P''(h) + 3P'(h)h - 8P(h) + 8 + 2Ch^4 = 0. \tag{3.6}$$

Consequently,

$$P(h) = -\frac{C}{8}h^4 + Ah^{-4} + Bh^2 + 1, \tag{3.7}$$

where $A, B \in \mathbb{R}$ are arbitrary. Note that the polynomial Q of the fourth order such that $Q(h^2) = h^4P(h)$ can be obtained by methods of Calabi, who proved ([C-1]) that every extremal Kähler metric of cohomogeneity 1 is described by the polynomial of the fourth order. Let $x > 0$ satisfy an equation $P(x) = 0$ and let us consider the equation

$$\frac{d^2h}{dt^2} = \frac{1}{2}P'(h),\ h'(0) = 0,\ h(0) = x. \tag{3.8}$$

This equation is equivalent, if $t \in D = \{t \geq 0 : h'(t) \geq 0\}$, to the equation

$$\frac{dh}{dt} = \sqrt{P(h)}, \quad h(0) = x. \tag{3.9}$$

Let us assume that there exists $y > 0$ such that $0 < x < y$, $P(y) = 0$ and $P(t) > 0$ if $t \in (x, y)$. Then equation (3.8) admits a smooth periodic solution h defined on the whole of $\mathbb{R}$ and such that $\operatorname{im} h = [x, y]$. Note that $P'(h)h = -\frac{C}{2}h^4 - 4Ah^{-4} + 2Bh^2$ and $h'' = \frac{1}{2}P'(h)$. Let us assume that h satisfies the assumptions of Proposition 6. Then the following equations are satisfied:

$$-\frac{C}{8}x^4 + Ax^{-4} = -1 - Bx^2, \tag{3.10a}$$

$$-\frac{C}{8}y^4 + Ay^{-4} = -1 - By^2, \tag{1}$$

$$-\frac{C}{4}x^4 - 2Ax^{-4} = k - Bx^2, \tag{3.10b}$$

$$-\frac{C}{4}y^4 - 2Ay^{-4} = -k - By^2. \tag{2}$$

It is easy to see that equations (3.10) are equivalent to

$$-\frac{C}{8} = \frac{k(y^4 - x^4) + B(y^6 - x^6)}{x^8 - y^8}, A = \frac{k(y^4 - x^4) + Bx^2y^2(y^2 - x^2)}{x^8 - y^8}, \tag{3.11a}$$

$$-\frac{C}{8} = \frac{k(y^4 + x^4) - B(x^6 - y^6)}{2(x^8 - y^8)}, A = \frac{kx^4y^4(y^4 + x^4) + Bx^6y^6(x^2 - y^2)}{2(x^8 - y^8)}. \tag{3.11b}$$

These equations imply both that $B = \frac{(k+2)x^4+(k-2)y^4}{y^6-x^6}$ and also that $B = \frac{(k+2)x^4+(k-2)y^4}{3x^2y^2(y^2-x^2)}$. Consequently, equations (3.10) with unknown A, B, x, y have a solution if $B = 0$ and $(k+2)x^4+(k-2)y^4 = 0$. It is possible only if $k = 1, C > 0$ (it means that such metric can only be constructed on the first Hirzebruch surface) and $x = (\frac{2}{C})^{\frac{1}{4}}, y = (\frac{6}{C})^{\frac{1}{4}}$. Then $P(h) = -\frac{C}{8}h^4 - \frac{3}{2C}h^{-4} + 1$. Note that $P(x) = P(y) = 0$, $P(t) > 0$ if $t \in (x, y)$ and x, y are the only positive roots of P. It follows that equation (3.8) has a positive periodic solution h with $\mathrm{im} h = [x, y]$. Let b be the smallest positive number such that $h(b) = y$. Let us take $a = 0$. Then it is easy to check that $h(a)h''(a) = k$ and $h(b)h''(b) = -k$ since $P'(x)x = 2k$ and $P'(y)y = -2k$. Note also that $h'(a) = h'(b) = 0$ and consequently, $h^{(2p+1)}(a) = h^{(2p+1)}(b) = 0, p \in \mathbb{N}$. Thus the metric g_h extends to a smooth metric on the whole of the Hirzebruch surface F_1. Now one can check that $\lambda = 2Ch^2$ and $\mu = Ch^2$. It is easy to check using Lemma and Prop.1-5 that the tensor $\rho - \frac{\tau}{3}g$ is a Killing tensor with eigenvalues $0, -Ch^2$ corresponding to $\mathcal{D}, \mathcal{D}^{\perp}$ respectively. It follows from [J-1] that (F_1, g_h) is an extremal Kähler metric with harmonic anti-self-dual Weyl tensor. Note that the metrics corresponding to different values of C are homothetic and the natural opposite almost Hermitian structure for J coincides with $\overline{J}$ on an open and dense subset where it is defined (see Prop.2 and Prop.3). Calabi proved that extremal metrics on Hirzebruch surfaces are of cohomogeneity 1 (see [C-2]) i.e. are of the shape g_h for some h described in Proposition 6. We proved in [J-1] that a Kähler surface with co-closed anti-self-dual Weyl tensor is extremal. Thus the metric constructed above is (up to a homothety) the only Kähler metric with co-closed anti-self-dual Weyl tensor in the class of Hirzebruch surfaces $F_k; k \geq 1$. ■

Remark. In that way we have constructed an example of a compact $\mathcal{AC}^{\perp}$ 4-manifold which additionally admit a Kähler structure. Since our example does not have harmonic Weyl tensor (since τ is not constant clearly $\delta W^+ \neq 0$) it follows that it is a proper $\mathcal{AC}^{\perp}$ manifolds i.e. its Ricci tensor is not a Codazzi tensor (compare [Be]). After constructing the example the author has learned that it was also found by Apostolov, Calderbank and Gaudouchon [A-C-G], who also proved that this is the only Kähler metric with co-closed anti-self-dual Weyl tensor and non-constant scalar curvature on ruled surfaces. Note that other examples of $\mathcal{AC}^{\perp}$-manifolds are given in [M-S] (they have Ricci tensor with two eigenvalues of multiplicities 1 and 3, our examples have Ricci tensor with two eigenvalues of the same multiplicity 2). The (non-compact) examples of Kähler surfaces with harmonic anti-self-dual Weyl tensor (in fact self-dual) and non-constant scalar curvature were given by A. Derdziński in [D-2]. Matsumoto and Tanno (see [T],[M-T]) proved that a Kähler manifold with $\delta W = 0$ has parallel Ricci tensor (this result is local). The methods similar to those used by us in the paper can be used in construction of families of examples of bi-Hermitian non-Kähler $\mathcal{A}C^{\perp}$-metrics on all Hirzebruch surfaces $F_k : k \geq 0$. We shall do it in the subsequent paper.

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