Criteria for univalent, starlike and convex functions

S. Ponnusamy V. Singh

Abstract

Let $\mathcal{A}$ denote the space of analytic functions in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with the normalization $f(0) = f'(0) - 1 = 0$. We are interested in the following problems: Find conditions on $\alpha \in \mathbb{C}$ ($\mathrm{Re}\,\alpha > -1$) and $\mu > 0$ so that the subordination condition

$$zf''(z) + \alpha f'(z) \prec \alpha + \mu z, \quad z \in \Delta,$$

implies that f is starlike or convex in Δ. Define

$$P(\alpha, \delta) = \{f \in \mathcal{A} : \text{ there exists a } |\gamma| < \pi/2 \text{ such that } \mathrm{Re}\, e^{i\gamma}(f'(z) + \alpha z f''(z) - \delta) < 0\}$$

for some $\alpha \in \mathbb{C}$ with $\mathrm{Re}\,\alpha \leq -1$. For a given α, we find a precise condition on δ so that $f \in P(\alpha, \delta)$ is univalent in Δ. Further, in this paper we also prove several sufficient conditions for starlikeness and convexity for the convolution $f * g$ when both (or one) of f, g belong(s) to the class

$$\widetilde{\mathcal{R}}(\alpha, \lambda) = \{f \in \mathcal{A} : f'(z) + \alpha z f''(z) \prec 1 + \lambda z, \ z \in \Delta\},$$

where $\alpha \in \mathbb{C} \backslash \{-1\}$ with $\mathrm{Re}\,\alpha \geq -1$.

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1 Introduction and main results

Let $\mathcal{H}$ denote the space of analytic functions in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, with the topology of local uniform convergence. We use two kinds of normalization from the space $\mathcal{H}$, namely

$$\mathcal{A} = \{f \in \mathcal{H} : f(0) = f'(0) - 1 = 0\}, \quad \mathcal{A}' = \{f \in \mathcal{H} : f(0) = 1\}.$$

Let $\mathcal{B}$ denote the class of all functions $\omega \in \mathcal{H}$ such that $\omega(0) = 0$ and $|\omega(z)| < 1$ on Δ. The concept of subordination has proved to be very useful in studies of the range of values of analytic functions, see for example [L, p. 163-171]. Recall that for $f, g \in \mathcal{H}$, we say that the function f is *subordinate* to g, written $f \prec g$, or $f(z) \prec g(z)$, if and only if there exists a $\omega \in \mathcal{B}$ such that $f(z) = g(\omega(z))$ on Δ. It is a well-known result that this implies in particular $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$, and that these two conditions are also sufficient for $f(z) \prec g(z)$ whenever g is univalent in Δ. We remark that if $f \in \mathcal{H}$, $f(0) = 0$ and satisfies $|f(z)| \le M$ on Δ, then this can equivalently be expressed in the form

$$f(z) = M\omega(z), \quad \omega \in \mathcal{B},$$

and write

$$f(z) \prec Mz, \quad z \in \Delta.$$

We shall use either of these equivalent formulations according to our convenience.

For α real, $0 \le \alpha < 1$, a function $f \in \mathcal{A}$ is said to be in $\mathcal{S}^*(\alpha)$, the space of *starlike functions of order* α, if and only if

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + (1 - 2\alpha)z}{1 - z}, \quad z \in \Delta.$$

For $\alpha = 0$, $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ is the well-known space of normalized functions starlike (univalent) with respect to origin. A function $f \in \mathcal{A}$ is said to be *strongly starlike of order* α, $\alpha > 0$, if and only if

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}, \; z \in \Delta,$$

and is denoted by $\mathcal{S}(\alpha)$. If $\alpha = 1$, $\mathcal{S}(\alpha)$ coincides with $\mathcal{S}^*$ and if $0 < \alpha < 1$, $\mathcal{S}(\alpha)$ consists only of bounded starlike functions [BK] and therefore, the inclusion $\mathcal{S}(\alpha) \subset \mathcal{S}^*$ is proper. In fact, even starlike functions of *higher order* are not bounded. A function $f \in \mathcal{A}$ is said to be in $\mathcal{K}(\alpha)$ if and only if $zf'(z) \in \mathcal{S}^*(\alpha)$. As usual $\mathcal{K}(0) \equiv \mathcal{K}$ denotes the family of all convex functions in Δ. Also, for $0 < \alpha \le 1$, let

$$\mathcal{S}_\alpha = \{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \alpha z, \; z \in \Delta\}$$

and

$$\mathcal{K}_\alpha = \{f \in \mathcal{A} : zf'(z) \in \mathcal{S}_\alpha\}.$$

We need the following space of functions for our first result, namely, Theorem 1.3: For $-1 \ne \alpha \in \mathbb{C}$ with $\mathrm{Re}\, \alpha \ge -1$, let

$$\widetilde{\mathcal{R}}(\alpha, \lambda) = \{f \in \mathcal{A} : f'(z) + \alpha z f''(z) \prec 1 + \lambda z, \; z \in \Delta\}.$$

For convenience, we set $\widetilde{\mathcal{R}}(0, \lambda) := \mathcal{R}_\lambda$ and recall the following result.

Lemma 1.1. *For $0 < \alpha \leq 1$, the inclusion*

$$\mathcal{R}_\lambda \subset \mathcal{S}(\alpha)$$

holds whenever

$$0 < \lambda \leq \frac{2\sin(\pi\alpha/2)}{\sqrt{5+4\cos(\pi\alpha/2)}}.$$

This lemma is a special case of Corollary 1.7 from [PS1]. The interesting feature of this result is the sharpness part which was left open in the authors paper [PS1]. However, the sharpness part was recently verified in [RRS, Corollary 1.2]. Another result which will of interest in this connection is the following from [P3].

Lemma 1.2. *For $0 < \lambda \leq 1/2$, we have the inclusion $\mathcal{R}_\lambda \subset \mathcal{S}_{3\lambda/(2-\lambda)}$.*

A more general forms of these lemmas along with several interesting applications may be found from the work of Ponnusamy [P3] and, Ponnusamy and Singh [PS1].

The basic operations that we shall encounter frequently is the usual Hadamard product (or convolution) $f * g$ of two analytic functions $f, g \in \mathcal{H}$:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,\ g(z) = \sum_{k=0}^{\infty} b_k z^k \quad \Rightarrow \quad (f*g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

Note that $f * g$ is in $\mathcal{H}$. We use the notation $\mathcal{H}_1 * \mathcal{H}_2$ to denote the set of all $f * g$ where $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$. Here $\mathcal{H}_1$ and $\mathcal{H}_2$ are two subspaces of $\mathcal{H}$. Now we are in a position to state our preliminary but basic results on the class $\widetilde{\mathcal{R}}(\alpha, \lambda)$.

Theorem 1.3. *We have*

(i) *The inclusion $\widetilde{\mathcal{R}}(\alpha,\lambda) \subset \widetilde{\mathcal{R}}(\alpha',\lambda')$ holds whenever $\alpha \neq 0$ (unless $\alpha' = 0$) and*

$$\lambda' = \frac{\lambda}{|\alpha|\,|\alpha+1|}\left[|\alpha-\alpha'| + |\alpha'|\,|\alpha+1|\right].$$

In particular, if $|\alpha-\alpha'| \leq |\alpha+1|(|\alpha|-|\alpha'|)$, then

$$\widetilde{\mathcal{R}}(\alpha,\lambda) \subset \widetilde{\mathcal{R}}(\alpha',\lambda),$$

and if $0 \leq \alpha' \leq \alpha$, then

$$\widetilde{\mathcal{R}}(\alpha,\lambda) \subset \widetilde{\mathcal{R}}(\alpha',\lambda).$$

Moreover, if $\operatorname{Re}\alpha > -1$, we have

$$\widetilde{\mathcal{R}}(\alpha,\lambda) \subset \mathcal{R}_{\lambda'}, \quad \lambda' = \frac{\lambda}{|1+\alpha|}.$$

In addition, for $\operatorname{Re}\alpha > -1$, we have $\widetilde{\mathcal{R}}(\alpha,\lambda) \subset \mathcal{S}(\beta)$, if

$$0 < \lambda \leq \frac{2|1+\alpha|\sin(\pi\beta/2)}{\sqrt{5+4\cos(\pi\beta/2)}}. \tag{1.1}$$

(ii) *The inclusion*

$$\widetilde{\mathcal{R}}(\alpha,\lambda) * \widetilde{\mathcal{R}}(\alpha',\lambda') \subset \widetilde{\mathcal{R}}(\alpha,\lambda)$$

holds whenever $\lambda' \leq 2|1+\alpha'|$. *In particular, the space* $\widetilde{\mathcal{R}}(\alpha,\lambda)$ *is closed under convolution when* $\lambda \leq 2|1+\alpha|$.

(iii) *The inclusion* $\widetilde{\mathcal{R}}(\alpha,\lambda) * \widetilde{\mathcal{R}}(\alpha',\lambda') \subset \mathcal{S}(\beta)$ *holds whenever* $0<\beta\leq 1$ *and other parameters are related by the inequality*

$$0 < \frac{\lambda\lambda'}{|(1+\alpha)(1+\alpha')|} \leq \frac{4\sin(\pi\beta/2)}{\sqrt{5+4\cos(\pi\beta/2)}}.$$

In particular,

(a) $\widetilde{\mathcal{R}}(\alpha,\lambda) * \widetilde{\mathcal{R}}(\alpha,\lambda) \subset \mathcal{S}^*$ *if* $0 < \dfrac{\lambda}{|1+\alpha|} \leq \sqrt{\dfrac{4}{\sqrt{5}}}$

(b) $\widetilde{\mathcal{R}}(\alpha,\lambda) * \widetilde{\mathcal{R}}_\lambda \subset \mathcal{S}^*$ *if* $0 < \dfrac{\lambda}{\sqrt{|1+\alpha|}} \leq \sqrt{\dfrac{4}{\sqrt{5}}}$.

(iv) *The inclusion* $\widetilde{\mathcal{R}}(\alpha,\lambda) * \widetilde{\mathcal{R}}(\alpha',\lambda') \subset \mathcal{K}'(\beta)$ *holds whenever* $0<\beta\leq 1$ *and other parameters are related by the inequality*

$$0 < \frac{\lambda\lambda'}{|(1+\alpha)(1+\alpha')|} \leq \frac{2\sin(\pi\beta/2)}{\sqrt{5+4\cos(\pi\beta/2)}}.$$

Here $\mathcal{K}'(\beta)$ *denotes the family of functions* f *such that* $zf'(z) \in \mathcal{S}(\beta)$. (*Note then* $\mathcal{K}'(1) \equiv \mathcal{K}$). *In particular, one has the inclusions*

(a) $\widetilde{\mathcal{R}}(\alpha,\lambda) * \widetilde{\mathcal{R}}(\alpha,\lambda) \subset \mathcal{K}$ *if* $0 < \dfrac{\lambda}{|1+\alpha|} \leq \sqrt{\dfrac{2}{\sqrt{5}}}$

(b) $\widetilde{\mathcal{R}}(\alpha,\lambda) * \widetilde{\mathcal{R}}_\lambda \subset \mathcal{K}$ *if* $0 < \dfrac{\lambda}{\sqrt{|1+\alpha|}} \leq \sqrt{\dfrac{2}{\sqrt{5}}}$.

As a consequence of Theorem 1.3, we also obtain the following corollary and its proof will be outlined along with the proof of this theorem in Section 3.

Corollary 1.4. *Let* $\alpha \in \mathbb{C}$ *with* $\operatorname{Re}\alpha > -1$. *Then*

(a) $\widetilde{\mathcal{R}}(\alpha,\lambda) * \widetilde{\mathcal{R}}(\alpha',\lambda') \subset \mathcal{S}_1$ *whenever* $0 < \lambda\lambda' \leq |(1+\alpha)(1+\alpha')|$,

(b) $\widetilde{\mathcal{R}}(\alpha,\lambda) * \widetilde{\mathcal{R}}(\alpha',\lambda') \subset \mathcal{K}_1$ *whenever* $0 < 2\lambda\lambda' \leq |(1+\alpha)(1+\alpha')|$.

In order to state and prove our remaining results we need a number of observations. We recall these in the appropriate places. For example, if $p \in \mathcal{A}'$ is such that $\operatorname{Re} p(z) > 1/2$ in Δ, then using the Herglotz' representation for p it follows that for any analytic function F in Δ, the function $p * F$ takes values in the convex hull of the image of Δ under F. This observation immediately gives the following implication for $f, g \in \mathcal{A}$:

$$f \in \widetilde{\mathcal{R}}(\alpha,\lambda) \text{ and } \operatorname{Re}\left(\frac{g(z)}{z}\right) > \frac{1}{2} \;\Rightarrow\; f * g \in \widetilde{\mathcal{R}}(\alpha,\lambda).$$

Next, we recall the following simple result from [P1, Equation (16)]. If $p \in \mathcal{A}'$, then, for each $-1 \neq \alpha \in \mathbb{C}$ with $\mathrm{Re}\, \alpha \geq -1$, we have

$$p(z) + \alpha z p'(z) \prec 1 + \lambda z \implies p(z) \prec 1 + \frac{\lambda}{\alpha + 1} z, \quad z \in \Delta. \tag{1.2}$$

Lemma 1.5. *Let $p \in \mathcal{A}'$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}\, \alpha \leq -1$ $(\alpha \neq -1)$, $\delta > 1$ be such that*

$$\mathrm{Re}\, e^{i\gamma} \{p(z) + \alpha z p'(z) - \delta\} < 0, \quad z \in \Delta, \tag{1.3}$$

for some $|\gamma| < \pi/2$. Then

$$\mathrm{Re}\, e^{i\gamma} \{p(z) - \beta\} > 0, \quad z \in \Delta, \tag{1.4}$$

where

$$\beta = \beta(\delta, \mathrm{Re}\, \alpha) = 1 - 2(\delta - 1) \int_0^1 \frac{t^{\mathrm{Re}\, \alpha}}{1 + t^{\mathrm{Re}\, \alpha}} dt. \tag{1.5}$$

The estimate cannot be improved in general.

The proof of this lemma will be given in Section 3. Our next problem concerns sufficiency conditions for starlike functions.

Problem 1.6. *Let α be a complex number such that $\mathrm{Re}\, \alpha \geq -1$ $(\alpha \neq -1)$, and μ be a non-negative real number such that $\mu \leq 2|1 + \alpha|$. Find conditions on α and μ so that the subordination condition*

$$z f''(z) + \alpha f'(z) \prec \alpha + \mu z, \quad z \in \Delta, \tag{1.6}$$

implies that f is starlike or convex in Δ.

Affirmative answer to this problem is already known in the literature for example in [PS1, S3] and [P3, Theorem 2]. At this place, it is appropriate to recall the question of Mocanu [M] who showed that for $0 < \mu \leq 2/3$, each function f satisfying (1.6) with $\alpha = 0$ is starlike in Δ. He asked for the largest μ for which the subordination condition (1.6) with $\alpha = 0$ implies that f is starlike. This question was solved by V.Singh [S2, S3], see also Corollary 1.13 for a sharp result in a stronger form.

Theorem 1.7. *Let $f \in \mathcal{A}$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}\, \alpha \geq -1$ $(\alpha \neq -1)$, and μ be such that $0 < \mu \leq 2|1 + \alpha|$. Suppose that $f \in \mathcal{A}$ satisfies the condition* (1.6). *Then we have the following:*

$$\left| \frac{z f'(z)}{f(z)} - a \right| < b(a, \alpha, \mu), \quad z \in \Delta,$$

where

$$|1 - a| < b(a, \alpha, \mu) := \frac{1}{2|1 + \alpha| - \mu} \{|1 - \alpha - a|\mu + |1 + \alpha|(2|1 - a| + \mu)\}.$$

In particular, if $\mu \leq \dfrac{2|1 + \alpha|}{1 + |\alpha| + |1 + \alpha|}$ then one has

$$\left| \frac{z f'(z)}{f(z)} - 1 \right| < 1.$$

For different choices of $a \in \mathbb{C}$, we get different conclusions for the range of $zf'(z)/f(z)$ in the right half plane. For example, we have

Corollary 1.8. *Let $\alpha > -1$, $0 < \mu \leq 2(1+\alpha)$ and that $f \in \mathcal{A}$ satisfy the condition* (1.6). *Then, we have the following*

(i) $\left|\dfrac{zf'(z)}{f(z)} - 1\right| < \dfrac{\mu(2\alpha+1)}{2(1+\alpha)-\mu}$. *In particular, if $\mu \leq 1$, then $f \in \mathcal{S}_1$*

(ii) $\left|\dfrac{zf'(z)}{f(z)} - \dfrac{2}{3}\right| < \dfrac{|1-3\alpha|\mu + 2(1+\alpha) + 3\mu(1+\alpha)}{3(2(1+\alpha)-\mu)}$. *In particular, if either $\alpha \in (-1, 1/3]$ with $0 < \mu \leq (1+\alpha)/3$ or $\alpha \in [1/3, \infty)$ with $0 < \mu \leq (1+\alpha)/(2+3\alpha)$ holds then*
$$\left|\frac{zf'(z)}{f(z)} - \frac{2}{3}\right| < \frac{2}{3}.$$

Now, we are in a position to provide an improved version of Theorem 1.7 which also provides solutions to Problem 1.6 in various forms as we see in the following results.

Theorem 1.9. *Assume the hypotheses of Theorem* 1.7. *Then f is convex in Δ whenever μ satisfies the inequality*
$$0 < \mu \leq \frac{2(1+\operatorname{Re}\alpha)}{2(1+\operatorname{Re}\alpha) + |2-\alpha| + |\alpha|}.$$

We have the following simple corollary when α is a real number.

Corollary 1.10. *Let $\alpha > -1$ and μ be such that*
$$0 < \mu \leq \begin{cases} \dfrac{1+\alpha}{2+\alpha} & \text{for } -1 < \alpha \leq 2 \\ \dfrac{1+\alpha}{2\alpha} & \text{for } 2 \leq \alpha < \infty. \end{cases}$$
If $f \in \mathcal{A}$ satisfies the condition (1.6) *then f convex in Δ.*

For $\alpha \neq 0$, (1.6) can be rewritten as
$$f'(z) + \frac{1}{\alpha}zf''(z) \prec 1 + \frac{\mu}{\alpha}z, \quad z \in \Delta,$$
and therefore, for example, if $\alpha > 0$, then (1.6) is equivalent to $f \in \widetilde{\mathcal{R}}(1/\alpha, \mu/\alpha)$. This observation, a simple computation and Corollary 1.10 yield the following

Corollary 1.11. *If*
$$0 < \lambda \leq \begin{cases} \dfrac{\alpha(1+\alpha)}{2} & \text{for } 0 < \alpha < \frac{1}{2} \\ \dfrac{\alpha(1+\alpha)}{1+2\alpha} & \text{for } \alpha \geq \frac{1}{2} \end{cases}$$
then we have $\widetilde{\mathcal{R}}(\alpha, \lambda) \subset \mathcal{K}$.

We can obtain some further bounds on μ from the known results as we see in the next theorem.

Theorem 1.12. *Assume the hypotheses of Theorem* 1.7. *Then* $f \in \mathcal{S}_1$ (*in particular, f is starlike*) *if*

$$0 \leq \mu \leq \frac{1 + \operatorname{Re} \alpha}{1 + |\alpha| + \operatorname{Re} \alpha}.$$

Moreover, $f \in \mathcal{K}_1$ (*in particular, f is convex*) *if*

$$0 \leq 2\mu \leq \frac{1 + \operatorname{Re} \alpha}{1 + |\alpha| + \operatorname{Re} \alpha}.$$

Corollary 1.13. *Let $f \in \mathcal{A}$ satisfy the condition $|zf''(z)| < \mu$ on Δ. Then we have*

(a) $f \in \mathcal{S}_1$ *for* $0 < \mu \leq 1$

(b) $f \in \mathcal{K}_1$ *for* $0 < \mu \leq 1/2$.

Both the results are sharp.

Proof. The desired bound follows if we choose $\alpha = 0$ in Theorem 1.12. Sharpness part follows if we consider the function $f(z) = z + (\mu/2)z^2$. ■

The following result improves Corollary 1.8.

Theorem 1.14. *Let $\alpha > -1$ and let $f \in \mathcal{A}$ satisfy the condition* (1.6). *If μ is such that*

$$0 < \mu \leq \begin{cases} \dfrac{2(1+\alpha)}{2 + \alpha^{2/(1-\alpha)}} & \text{for } \; -1 < \alpha \neq 1 < \infty \\ \dfrac{4e^2}{1 + 2e^2} & \text{for } \; \alpha = 1, \end{cases}$$

then f starlike in Δ. Moreover, if μ is such that

$$0 < \mu \leq \begin{cases} \dfrac{(1+\alpha)}{1 + \alpha^{2/(1-\alpha)}} & \text{for } \; -1 < \alpha \neq 1 < \infty \\ \dfrac{2e^2}{1 + e^2} & \text{for } \; \alpha = 1, \end{cases}$$

then we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1, \quad z \in \Delta.$$

Note that for $\alpha > 0$, (1.6) is equivalent to $f \in \widetilde{\mathcal{R}}(1/\alpha, \mu/\alpha)$. Therefore, as in the case of Corollary 1.11, a simple computation gives

Corollary 1.15. *We have*

(a) $\widetilde{\mathcal{R}}(\alpha, \lambda) \subset \mathcal{S}^*$ *if*

$$0 < \lambda \leq \begin{cases} \dfrac{2(1+\alpha)}{2 + \alpha^{2\alpha/(1-\alpha)}} & \text{for } \; 0 < \alpha \neq 1 < \infty \\ \dfrac{4e^2}{1 + 2e^2} & \text{for } \; \alpha = 1, \end{cases}$$

(b) $\widetilde{\mathcal{R}}(\alpha,\lambda) \subset \mathcal{S}_1$ *if*

$$0 < \lambda \leq \begin{cases} \dfrac{(1+\alpha)}{1+\alpha^{2\alpha/(1-\alpha)}} & \text{for } 0 < \alpha \neq 1 < \infty \\ \dfrac{2e^2}{1+e^2} & \text{for } \alpha = 1. \end{cases}$$

The proofs of Theorems 1.7, 1.9, 1.12 and 1.14 will be given in Section 3.

2 Sharp version of a problem of Ponnusamy

Let $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha \leq -1$ $(\alpha \neq -1)$ and δ be a positive real number such that $\delta > 1$. Define

$P(\alpha,\delta) = \{f \in \mathcal{A} :$ there exists a $|\gamma| < \pi/2$ such that $\operatorname{Re} e^{i\gamma}(f'(z)+\alpha z f''(z)-\delta) < 0\}$.

In [P2, p.185], the following result was proved, but without the rotation factor.

Theorem 2.1. *Let $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha < -2$, and $\delta > -\operatorname{Re}\alpha/2$. If f belongs to $P(\alpha,\delta)$, then $\operatorname{Re} e^{i\gamma} f'(z) > 0$ for all $z \in \Delta$. In particular, each function in the family $P(\alpha,\delta)$ is univalent in Δ.*

In [P2, p.185], it is also proved that (without the rotation factor)

$$P(\alpha,\delta) \subset P\left(\alpha', \frac{\delta(2+\alpha')+(\alpha-\alpha')}{2+\alpha}\right), \quad \text{for } -2 > \alpha > \alpha'. \tag{2.1}$$

Our first result gives a precise/sharp version of Theorem 2.1 in the following form.

Theorem 2.2. *Let $f \in \mathcal{A}$, $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha \leq -1$ $(\alpha \neq -1)$, $\delta > 1$ and that*

$$\operatorname{Re} e^{i\gamma}\{f'(z)+\alpha z f''(z)\} < \delta\cos\gamma, \quad z \in \Delta, \tag{2.2}$$

for some $|\gamma| < \pi/2$. Then

$$\operatorname{Re} e^{i\gamma} f'(z) > \beta\cos\gamma, \quad z \in \Delta,$$

where

$$\beta = 1 - 2(\delta-1)\int_0^1 \frac{t^{\operatorname{Re}\alpha}}{1+t^{\operatorname{Re}\alpha}}\,dt. \tag{2.3}$$

The estimate cannot be improved in general.

The proof of Theorem 2.2 is a consequence of Lemma 1.5 if we choose $p(z)$ in Lemma 1.5 as $f'(z)$. In particular, we have

Corollary 2.3. *Let $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha \leq -1$ $(\alpha \neq -1)$. Then functions in $P(\alpha,\delta)$ are univalent in Δ whenever*

$$1 < \delta \leq 1 + \frac{1}{2}\left(\int_0^1 \frac{t^{\operatorname{Re}\alpha}}{1+t^{\operatorname{Re}\alpha}}\,dt\right)^{-1}.$$

Proof. Let $f \in P(\alpha,\delta)$. Then, by Theorem 2.2, the condition on δ implies that the β in Theorem 2.2 satisfies $\beta \geq 0$ and therefore, we have that $\operatorname{Re} e^{i\gamma} f'(z) > 0$ in Δ. Thus, each $f \in P(\alpha,\delta)$ is univalent in Δ. ∎

Suppose that $-1 > \alpha \geq \alpha'$ and consider the equation

$$e^{i\gamma}[f'(z) + \alpha z f''(z)] = \left(1 - \frac{\alpha'}{\alpha}\right) e^{i\gamma} f'(z) + \frac{\alpha'}{\alpha} e^{i\gamma}[f'(z) + \alpha z f''(z)].$$

Now, we observe that if $f \in P(\alpha, \delta)$ then, by Theorem 2.2, the last equation implies that

$$\mathrm{Re}\, e^{i\gamma}\{f'(z) + \alpha' z f''(z)\} < \left(1 - \frac{\alpha'}{\alpha}\right) \beta \cos\gamma + \frac{\alpha'}{\alpha} \delta \cos\gamma,$$

where β is defined by (2.3) (note that $\mathrm{Re}\,\alpha = \alpha$). The last inequality is equivalent to state that

$$f \in P\left(\alpha', \frac{(\alpha - \alpha')\beta + \alpha'\delta}{\alpha}\right).$$

Using the definition of β, we find that

$$\begin{aligned}
\frac{(\alpha - \alpha')\beta + \alpha'\delta}{\alpha} &= \left(1 - \frac{\alpha'}{\alpha}\right) \beta + \frac{\alpha'}{\alpha}\delta \\
&= \delta - \left(1 - \frac{\alpha'}{\alpha}\right)(\delta - \beta) \\
&= \delta - \left(1 - \frac{\alpha'}{\alpha}\right)(\delta - 1)\left[1 + 2\int_0^1 \frac{t^\alpha}{1 + t^\alpha} dt\right].
\end{aligned}$$

This observation gives the following inclusion theorem which improves (2.1).

Theorem 2.4. *Let $\alpha, \alpha' \in \mathbb{R}$ and $-1 > \alpha \geq \alpha'$. Then, we have*

$$P(\alpha, \delta) \subseteq P(\alpha', \delta'), \quad \delta' = \delta - (\delta - 1)\left(1 - \frac{\alpha'}{\alpha}\right)\left[1 + 2\int_0^1 \frac{t^\alpha}{1 + t^\alpha} dt\right].$$

3 Proofs of main results

The proof of Theorem 1.3 relies on the following lemma.

Lemma 3.1. [*RSt*]. *If $f, g \in \mathcal{H}$, $F, G \in \mathcal{K}$ are such that $f \prec F$, $g \prec G$, then $f * g \prec F * G$.*

In this lemma, functions in $\mathcal{K}$ are not necessarily normalized.

Proof of Theorem 1.3 Let $f \in \mathcal{A}$ have the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then, by an elementary computation, we have

$$f'(z) + \alpha z f''(z) = 1 + \sum_{n=1}^{\infty} (n + 1)(n\alpha + 1) a_{n+1} z^n = f'(z) * \phi_\alpha(z)$$

where

$$\phi_\alpha(z) = \sum_{n=0}^{\infty} (n\alpha + 1) z^n.$$

It follows that

$$f \in \widetilde{\mathcal{R}}(\alpha, \lambda) \Longleftrightarrow f'(z) * \phi_\alpha(z) \prec 1 + \lambda z, \quad z \in \Delta. \tag{3.1}$$

We have the inverse map ψ_α of ϕ_α defined by

$$\psi_\alpha(z) := [\phi_\alpha(z)]^{-1} = \sum_{n=0}^{\infty} \frac{z^n}{n\alpha + 1}$$

so that

$$(\phi_\alpha * \psi_\alpha)(z) = \frac{1}{1-z}.$$

Note that the function $\psi_\alpha(z)$ for $\operatorname{Re}\alpha > -1$ and the function $1+\lambda z$ are both convex in the unit disc Δ. Now, we let $h(z) = (f * g)(z)$. The proof of this theorem essentially rely on the following equalities and the clever use of Lemma 3.1.

(a) $h'(z) = f'(z) * \dfrac{g(z)}{z}$

(b) $zh'(z) = zf'(z) * g(z)$

(c) $h'(z) + zh''(z) = f'(z) * g'(z)$, by (b),

(d) $(h'(z) + zh''(z)) * \phi_\alpha(z) * \phi_{\alpha'}(z) = [f'(z) * \phi_\alpha(z)] * [g'(z) * \phi_{\alpha'}(z)]$, by (c).

We use Lemma 3.1 frequently in this proof. Suppose that $f \in \widetilde{\mathcal{R}}(\alpha,\lambda)$ and $g \in \widetilde{\mathcal{R}}(\alpha',\lambda')$. Then, by (3.1), we have

$$f'(z) * \phi_\alpha(z) \prec 1 + \lambda z, \quad z \in \Delta.$$

Since $g \in \widetilde{\mathcal{R}}(\alpha',\lambda')$, by (1.2) with $p(z) = g'(z) - 1$, it follows that

$$g'(z) \prec 1 + \frac{\lambda'}{1+\alpha'} z, \quad z \in \Delta,$$

which in turn (again by (1.2) with $p(z) = (g(z)/z) - 1$) implies that

$$\frac{g(z)}{z} \prec 1 + \frac{\lambda'}{2(1+\alpha')} z, \quad z \in \Delta.$$

(i) Let $f \in \widetilde{\mathcal{R}}(\alpha,\lambda)$. Consider the identity

$$\alpha[f'(z) + \alpha' z f''(z) - 1] = (\alpha - \alpha')[f'(z) - 1] + \alpha'[f'(z) + \alpha z f''(z) - 1]$$

which holds for all α and α'. It follows that

$$|\alpha|\,|f'(z) + \alpha' z f''(z) - 1| < \lambda \left(\frac{|\alpha - \alpha'|}{|\alpha + 1|} + |\alpha'| \right)$$

and the desired inclusion is clear. Since $f \in \widetilde{\mathcal{R}}(\alpha,\lambda)$ implies that $f'(z) \prec 1 + \frac{\lambda}{1+\alpha} z$, we have

$$\widetilde{\mathcal{R}}(\alpha,\lambda) \subset \mathcal{R}_{\lambda/|1+\alpha|}.$$

Therefore, by Lemma 1.1, Theorem 1.3(i) follows.

(ii) We want to show that $h \in \widetilde{\mathcal{R}}(\alpha, \lambda)$ whenever $\lambda' \leq 2|1+\alpha'|$. To do this, by (a), it suffices to observe that

$$\begin{aligned} h'(z) * \phi_\alpha(z) &= f'(z) * \phi_\alpha(z) * \frac{g(z)}{z} \\ &\prec (1+\lambda z) * \left(1 + \frac{\lambda'}{2(1+\alpha')}z,\right), \quad \text{since } f'(z) * \phi_\alpha(z) \prec 1 + \lambda z, \\ &= 1 + \frac{\lambda\lambda'}{2(1+\alpha')}z \end{aligned}$$

showing that $h \in \widetilde{\mathcal{R}}(\alpha, \lambda)$ whenever $\lambda' \leq 2|1+\alpha'|$.

(iii) From the last subordination result we obtain that

$$\begin{aligned} h'(z) &= h'(z) * \phi_\alpha(z) * \psi_\alpha(z) \\ &\prec \left(1 + \frac{\lambda\lambda'}{2(1+\alpha')}z\right) * \psi_\alpha(z) \\ &= 1 + \frac{\lambda\lambda'}{2(1+\alpha)(1+\alpha')}z \end{aligned}$$

and the desired conclusion follows from Lemma 1.1.

(iv) We want to show that $zh' \in \mathcal{S}(\beta)$, where β is as in Theorem 1.3(iv). To do this, we use the identity (d), the Lemma 3.1, and the assumption that $f \in \widetilde{\mathcal{R}}(\alpha, \lambda)$ and $g \in \widetilde{\mathcal{R}}(\alpha', \lambda')$. Because of these, by (d), it suffices to observe that

$$(h'(z) + zh''(z)) * \phi_\alpha(z) * \phi_{\alpha'}(z) \prec (1+\lambda z) * (1+\lambda' z) = 1 + \lambda\lambda' z$$

so that, by Lemma 3.1, this subordination implies that

$$(zh')'(z) = h'(z) + zh''(z) \prec (1+\lambda\lambda' z) * \psi_\alpha(z) * \psi_{\alpha'}(z) = 1 + \frac{\lambda\lambda'}{(1+\alpha)(1+\alpha')}z$$

and the desired conclusion follows from Lemma 1.1. ∎

Proof of Corollary 1.4 . Let $f \in \widetilde{\mathcal{R}}(\alpha, \lambda)$, $g \in \widetilde{\mathcal{R}}(\alpha', \lambda')$ and $h = f * g$. From the proof of Theorem 1.3(iii) and Lemma 1.2, it follows that $h \in \mathcal{S}_{3m/(2-m)}$ with

$$m = \frac{\lambda\lambda'}{2|(1+\alpha)(1+\alpha')|}$$

and the first part follows. Similarly, the second part follows from the proof of Theorem 1.3(iv) because

$$(zh')'(z) \prec 1 + \frac{\lambda\lambda'}{(1+\alpha)(1+\alpha')}z.$$

∎

Proof of Lemma 1.5. By hypothesis, we can write

$$e^{i\gamma}\{p(z) + \alpha z p'(z) - \delta\} = e^{i\gamma}(1-\delta)P(z)$$

where $\mathrm{Re}\, e^{i\gamma}P(z) > 0$, and therefore,

$$p(z) + \alpha z p'(z) = \delta + (1-\delta)P(z) \tag{3.2}$$

which can be equivalently written in terms of their Maclaurin series expansion as

$$1 + \sum_{n=1}^{\infty} a_n(p)z^n + \alpha\left(\sum_{n=1}^{\infty} n a_n(p)z^n\right) = \delta + (1-\delta)\left(1 + \sum_{n=1}^{\infty} a_n(P)z^n\right).$$

Comparing the coefficients of z^n on both sides yields that

$$a_n(p)(1+\alpha n) = (1-\delta)a_n(P), \quad n \geq 1,$$

and, by (3.2), it follows that

$$\begin{aligned} p(z) &= 1 + \sum_{n=1}^{\infty} a_n(p)z^n \\ &= 1 + \sum_{n=1}^{\infty}\left(\frac{1-\delta}{1+\alpha n}\right) a_n(P)z^n \\ &= \left[1 + (1-\delta)\sum_{n=1}^{\infty}\left(\int_0^1 t^{\alpha n} z^n dt\right)\right] * P(z) \\ &= \left[1 + (1-\delta)\int_0^1 \frac{t^\alpha z}{1-t^\alpha z}dt\right] * P(z). \end{aligned}$$

The last equality may be rewritten as

$$e^{i\gamma}\left(\frac{p(z)-\beta}{1-\beta}\right) = \left[1 + \frac{(1-\delta)}{1-\beta}\int_0^1 \frac{t^\alpha z}{1-t^\alpha z}dt\right] * (e^{i\gamma}P(z)).$$

Thus, $p(z) \neq \beta$ if and only if

$$\frac{1}{2} < \mathrm{Re}\left[1 - \frac{(1-\delta)}{1-\beta}\int_0^1 \frac{(-t^\alpha z)}{1-t^\alpha z}dt\right]. \tag{3.3}$$

This gives the condition that

$$\frac{\delta-1}{1-\beta}\int_0^1 \frac{t^{\mathrm{Re}\,\alpha}}{1+t^{\mathrm{Re}\,\alpha}}dt \leq \frac{1}{2} \tag{3.4}$$

or equivalently,

$$\beta \leq 1 - 2(\delta-1)\int_0^1 \frac{t^{\mathrm{Re}\,\alpha}}{1+t^{\mathrm{Re}\,\alpha}}dt \tag{3.5}$$

which is true by (1.5). Moreover, because $\mathrm{Re}\, e^{i\gamma}P(z) > 0$, (3.5) also guarantees that

$$\mathrm{Re}\, e^{i\gamma}\{p(z)-\beta\} > 0, \quad z \in \Delta,$$

and we complete the proof. ∎

Proof of Theorem 1.7. Let $f \in \mathcal{A}$ satisfy the condition

$$zf''(z) + \alpha f'(z) \prec \alpha + \mu z, \quad z \in \Delta. \tag{3.6}$$

By (1.2), it follows easily that

$$f'(z) \prec 1 + \frac{\mu}{1+\alpha} z, \qquad z \in \Delta, \tag{3.7}$$

which in turn implies that

$$\frac{f(z)}{z} \prec 1 + \frac{\mu}{2(1+\alpha)} z, \quad z \in \Delta.$$

Therefore,

$$\left| \frac{f(z)}{z} - 1 \right| \leq \frac{\mu |z|}{2|1+\alpha|}, \quad \text{and} \quad \left| \frac{f(z)}{z} \right| \geq 1 - \frac{\mu |z|}{2|1+\alpha|}. \tag{3.8}$$

Again, by the definition of subordination, we can rewrite (3.6) as

$$zf''(z) + \alpha f'(z) = \alpha + \mu \omega(z), \quad \omega \in \mathcal{B}, \tag{3.9}$$

so that, by integration, we obtain

$$zf'(z) + (\alpha - 1) f(z) = \alpha z + \mu z \int_0^1 \omega(tz) dt.$$

It follows that

$$zf'(z) - af(z) = (1 - \alpha - a)(f(z) - z) + (1-a)z + \mu z \int_0^1 \omega(tz) dt$$

and therefore,

$$|zf'(z) - af(z)| = |(1 - \alpha - a)| \, |(f(z) - z)| + |(1-a)z| + \mu \frac{|z|^2}{2}.$$

Hence, because of (3.8), the last inequality shows that

$$\begin{aligned}
\left| \frac{zf'(z)}{f(z)} - a \right| &\leq \left| \frac{z}{f(z)} \right| \left\{ |1 - \alpha - a| \left| \frac{f(z)}{z} - 1 \right| + |1-a| + \mu \frac{|z|}{2} \right\} \\
&\leq \frac{1}{1 - \frac{\mu |z|}{2|1+\alpha|}} \left\{ |1 - \alpha - a| \mu \frac{|z|}{2|1+\alpha|} + |1-a| + \mu \frac{|z|}{2} \right\} \\
&< \frac{1}{2|1+\alpha| - \mu} \{ |1 - \alpha - a| \mu + |1+\alpha| (2|1-a| + \mu) \}.
\end{aligned}$$

Thus for $a = 1$ this inequality simplifies to

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{\mu(|\alpha + |1+\alpha|}{2|1+\alpha| - \mu}$$

and we observe that

$$\frac{\mu(|\alpha + |1+\alpha|}{2|1+\alpha| - \mu} \leq 1$$

is equivalent to $\mu \leq 2|1+\alpha|/[1 + |\alpha| + |1+\alpha|]$. The desired conclusion follows. ■

Proof of Theorem 1.9. Assume that $f \in \mathcal{A}$ satisfies the condition (3.6). Our aim is to show that f is convex in Δ. Thus, for the proof of the convexity, by (3.9), we compute

$$f'(z) = 1 + \mu \int_0^1 t^{\alpha-1}\omega(tz)dt = 1 + \mu \sum_{n=1}^{\infty} \frac{1}{n+\alpha} a_n z^n \tag{3.10}$$

where

$$\omega(z) = \sum_{n=1}^{\infty} a_n z^n, \quad \phi(z) = \sum_{n=1}^{\infty} \frac{z^n}{n+\alpha}.$$

It follows that

$$\begin{aligned} (zf'(z))' &= 1 + \mu \sum_{n=1}^{\infty} \frac{n+1}{n+\alpha} a_n z^n \\ &= 1 + \mu \sum_{n=1}^{\infty} \frac{1}{n+\alpha} a_n z^n + \mu \sum_{n=1}^{\infty} \frac{n+\alpha-\alpha}{n+\alpha} a_n z^n \\ &= f'(z) + \mu \left(\omega(z) - \alpha \int_0^1 t^{\alpha-1}\omega(tz)dt \right). \end{aligned}$$

In particular,

$$zf''(z) = \mu\omega(z) - \mu\alpha \int_0^1 t^{\alpha-1}\omega(tz)dt \tag{3.11}$$

and

$$1 + \frac{zf''(z)}{f'(z)} = \frac{\mu[\omega(z) - \alpha \int_0^1 t^{\alpha-1}\omega(tz)dt]}{1 + \mu \int_0^1 t^{\alpha-1}\omega(tz)dt} + 1. \tag{3.12}$$

Thus, for the convexity of f, we need to show that

$$1 + \frac{zf''(z)}{f'(z)} \neq -iT, \quad \text{for all real } T.$$

By (3.12), it can be easily seen that, for the convexity of f, this is equivalent to verify that

$$\frac{\mu}{1+iT} \left\{ \omega(z) - \alpha \int_0^1 t^{\alpha-1}\omega(tz)dt + \int_0^1 t^{\alpha-1}\omega(tz)dt \right\} \neq -1$$

and, by a simple computation, this is indeed equivalent to

$$\frac{\mu}{2} \left\{ \left[\omega(z) + (2-\alpha) \int_0^1 t^{\alpha-1}\omega(tz)dt \right] + \frac{1-iT}{1+iT} \left[\omega(z) - \alpha \int_0^1 t^{\alpha-1}\omega(tz)dt \right] \right\} \neq -1.$$

For convenience, we define

$$M = \sup_{T\in\mathbb{R},\ \omega\in\mathcal{B}} \left| \left\{ \left[\omega(z) + (2-\alpha) \int_0^1 t^{\alpha-1}\omega(tz)dt \right] + \frac{1-iT}{1+iT} \left[\omega(z) - \alpha \int_0^1 t^{\alpha-1}\omega(tz)dt \right] \right\} \right|.$$

Then, in view of the rotation invariance of $\mathcal{B}$, f is convex if $\mu M \leq 2$. Now, for $\operatorname{Re}\alpha > -1$, we observe that

$$\begin{aligned} M &\leq \left| \omega(z) + (2-\alpha) \int_0^1 t^{\alpha-1}\omega(tz)dt \right| + \left| \omega(z) - \alpha \int_0^1 t^{\alpha-1}\omega(tz)dt \right| \\ &\leq 1 + \frac{|2-\alpha|}{1+\operatorname{Re}\alpha} + 1 + \frac{|\alpha|}{1+\operatorname{Re}\alpha} \\ &= \frac{2(1+\operatorname{Re}\alpha) + |2-\alpha| + |\alpha|}{1+\operatorname{Re}\alpha} \end{aligned}$$

which shows that

$$M=\frac{2(1+\operatorname{Re}\alpha)+|2-\alpha|+|\alpha|}{1+\operatorname{Re}\alpha}.$$

Therefore, f is convex whenever $\mu M\leq 2$. The desired conclusion follows from the hypotheses. ∎

Proof of Theorem 1.12 Recall (3.11)

$$zf''(z)=\mu\omega(z)-\mu\alpha\int_0^1 t^{\alpha-1}\omega(tz)dt$$

and therefore,

$$|zf''(z)|\leq\mu\left(1+\frac{|\alpha|}{1+\operatorname{Re}\alpha}\right)=\frac{\mu(1+|\alpha|+\operatorname{Re}\alpha)}{1+\operatorname{Re}\alpha}:=m,\quad \text{say.}$$

Thus, we can write

$$zf''(z)=mW(z),\quad W\in\mathcal{B},$$

so that, by integration, we obtain

$$f'(z)=1+m\int_0^1\frac{W(tz)}{t}dt$$

and

$$f(z)=z+mz\int_0^1\frac{W(tz)}{t}(1-t)dt.$$

Hence, as in [S3, Theorem 1], we compute

$$\left|\frac{zf''(z)}{f'(z)}\right|=\frac{m|W(z)|}{\left|1+m\int_0^1\frac{W(tz)}{t}dt\right|}\leq\frac{m}{1-m}$$

and

$$\left|\frac{zf'(z)}{f(z)}-1\right|=\frac{m\left|\int_0^1 W(tz)dt\right|}{\left|1+m\int_0^1\frac{W(tz)}{t}(1-t)dt\right|}\leq\frac{m/2}{1-m/2}.$$

Note that $m/(1-m)\leq 1$ if and only if $0<m\leq 1/2$. This observation shows that for $0\leq\mu\leq\frac{1+\operatorname{Re}\alpha}{1+|\alpha|+\operatorname{Re}\alpha}$, we have

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1$$

and in particular, f is starlike in Δ. Similarly, for $0\leq\mu\leq\frac{1+\operatorname{Re}\alpha}{2(1+|\alpha|+\operatorname{Re}\alpha)}$, we have

$$\left|\frac{zf''(z)}{f'(z)}\right|<1$$

and in particular, f is convex in Δ. ∎

Proof of Theorem 1.14 In view of the representation (3.10), namely

$$f'(z) = 1 + \mu \int_0^1 t^{\alpha-1}\omega(tz)dt,$$

it can be easily checked that for $\mathrm{Re}\,\alpha > -1$,

$$\frac{f(z)}{z} = \begin{cases} 1 + \dfrac{\mu}{\alpha-1}\displaystyle\int_0^1 (1-t^{\alpha-1})w(tz)dt & \text{for } \alpha \neq 1 \\ 1 + \mu \displaystyle\int_0^1 \log(1/t)w(tz)dt & \text{for } \alpha = 1. \end{cases}$$

We need to consider the two cases $\alpha = 1$ and $\alpha \neq 1$ separately. Using the last two equations we see that for $\alpha \neq 1$,

$$\frac{zf'(z)}{f(z)} = \frac{1 + \mu \int_0^1 t^{\alpha-1}w(tz)dt}{1 + \frac{\mu}{\alpha-1}\int_0^1 (1-t^{\alpha-1})w(tz)dt}. \tag{3.13}$$

For the starlikeness of f, it suffices to show that

$$\frac{zf'(z)}{f(z)} \neq iT, \quad T \in \mathbb{R}.$$

But, by (3.13), it can be easily seen that this is equivalent to verify that

$$\frac{\mu}{2}\left[\int_0^1 \left\{t^{\alpha-1} + \frac{1-t^{\alpha-1}}{\alpha-1}\right\} w(tz)dt + \frac{1+iT}{1-iT}\int_0^1 \left\{t^{\alpha-1} - \frac{1-t^{\alpha-1}}{\alpha-1}\right\} w(tz)dt\right] \neq -1$$

which is same as

$$\frac{\mu}{2}\left[\int_0^1 \left\{\frac{1+(\alpha-2)t^{\alpha-1}}{\alpha-1}\right\} w(tz)dt + \frac{1+iT}{1-iT}\int_0^1 \left\{\frac{\alpha t^{\alpha-1}-1}{\alpha-1}\right\} w(tz)dt\right] \neq -1.$$

Now, if we let

$$M = \sup_{T\in\mathbb{R},\ \omega\in\mathcal{B}} \left|\int_0^1 \left\{\frac{1+(\alpha-2)t^{\alpha-1}}{\alpha-1}\right\} w(tz)dt + \frac{1+iT}{1-iT}\int_0^1 \left\{\frac{\alpha t^{\alpha-1}-1}{\alpha-1}\right\} w(tz)dt\right|, \tag{3.14}$$

then, in view of the rotation invariance of the class $\mathcal{B}$, it is clear that $f \in \mathcal{S}^*$ whenever $M\mu \leq 2$. Thus our aim is to find the value of M. We consider the positiveness of the integrands in (3.14). If $\alpha > 1$, then, for all $t \in [0,1]$, we find that

$$\frac{1+(\alpha-2)t^{\alpha-1}}{\alpha-1} > 0 \Longleftrightarrow 1 - t^{\alpha-1} + (\alpha-1)t^{\alpha-1} > 0$$

and the later inequality clearly holds for all $t \in [0,1]$. Again, for $\alpha > 1$, we note that

$$\frac{\alpha t^{\alpha-1}-1}{\alpha-1} > 0 \Longleftrightarrow \alpha t^{\alpha-1} - 1 > 0, \quad \text{i.e. } t \geq \frac{1}{\alpha^{1/(\alpha-1)}}.$$

Similarly, if $-1 < \alpha < 1$, then, for all $t \in [0,1]$, we obtain that

$$\frac{1+(\alpha-2)t^{\alpha-1}}{\alpha-1} > 0 \Longleftrightarrow 1 - \frac{1}{t^{1-\alpha}} - \frac{1-\alpha}{t^{1-\alpha}} < 0$$

and the later inequality is clearly true for $t \in [0,1]$. Also, for $-1 < \alpha < 1$, we note that

$$\frac{\alpha t^{\alpha-1} - 1}{\alpha - 1} > 0 \Longleftrightarrow \alpha t^{\alpha-1} - 1 < 0, \quad \text{i.e.} \quad t < \alpha^{1/(1-\alpha)}.$$

Next we deal the case $\alpha = 1$. First we observe that

$$\lim_{\alpha\to 1} \frac{1 - t^{\alpha-1}}{\alpha - 1} = \lim_{\alpha\to 1} \frac{1 - e^{(\alpha-1)\log t}}{\alpha - 1} = \log(1/t).$$

Therefore, as $\alpha \to 1$, we have

$$\frac{1 - t^{\alpha-1}}{\alpha - 1} + t^{\alpha-1} \to 1 + \log(1/t) > 0 \quad \text{for all } t \in [0,1].$$

Similarly, as $\alpha \to 1$, one has

$$\frac{1 + (\alpha-2)t^{\alpha-1}}{\alpha - 1} = t^{\alpha-1} - \left(\frac{1 - t^{\alpha-1}}{\alpha - 1}\right) \to 1 + \log t \geq 0 \quad \text{for all } t \in [1/e, 1].$$

For $\alpha > 1$, using the above observations, we estimate that

$$\begin{aligned}
M &\leq \int_0^1 \frac{1 + (\alpha-2)t^{\alpha-1}}{\alpha - 1} t dt + \int_0^1 \left|\frac{\alpha t^{\alpha-1} - 1}{\alpha - 1}\right| t dt \\
&= \frac{1}{\alpha - 1}\left(\frac{1}{2} + \frac{\alpha - 2}{\alpha + 1}\right) + \int_0^{\alpha^{1/(1-\alpha)}} t \left(\frac{1 - \alpha t^{\alpha-1}}{\alpha - 1}\right) dt + \int_{\alpha^{1/(1-\alpha)}}^1 t \left(\frac{\alpha t^{\alpha-1} - 1}{\alpha - 1}\right) dt \\
&= \frac{1}{\alpha - 1}\left(\frac{1}{2} + \frac{\alpha - 2}{\alpha + 1}\right) + \frac{1}{\alpha - 1}\left(\frac{1}{2}\alpha^{2/(1-\alpha)} - \frac{\alpha}{\alpha + 1}\alpha^{(1+\alpha)/(1-\alpha)}\right) + \\
&\qquad \frac{1}{\alpha - 1}\left\{\frac{\alpha}{\alpha + 1}\left(1 - \alpha^{(1+\alpha)/(1-\alpha)}\right) - \frac{1}{2}\left(1 - \alpha^{2/(1-\alpha)}\right)\right\} \\
&= \frac{2}{\alpha + 1} + \frac{\alpha^{2/(1-\alpha)}}{1 + \alpha}
\end{aligned}$$

which shows that

$$M = \frac{2}{\alpha + 1} + \frac{\alpha^{2/(1-\alpha)}}{1 + \alpha}. \tag{3.15}$$

For $\alpha \to 1$, we can easily obtain that

$$M = \frac{2 + e^{-2}}{2},$$

because

$$\lim_{\alpha\to 1} \alpha^{2/(1-\alpha)} = \lim_{\alpha\to 1} \exp(2/(1-\alpha)\log(1 - (1-\alpha)) = \exp(-2).$$

For $-1 < \alpha < 1$ (so that $\alpha - 1 < 0$), it follows similarly that

$$\begin{aligned}
M &\leq \frac{1}{\alpha - 1}\left(\frac{1}{2} + \frac{\alpha - 2}{\alpha + 1}\right) + \int_0^{\alpha^{1/(1-\alpha)}} t \left(\frac{1 - \alpha t^{\alpha-1}}{\alpha - 1}\right) dt + \int_{\alpha^{1/(1-\alpha)}}^1 t \left(\frac{\alpha t^{\alpha-1} - 1}{\alpha - 1}\right) dt \\
&= \frac{2}{\alpha + 1} + \frac{\alpha^{2/(1-\alpha)}}{1 + \alpha}
\end{aligned}$$

and therefore, M is again given by (3.15). The first part follows.

Let us now proceed to the proof of our second part. To do this, using (3.13) and also with the help of the positiveness of the integrand as described above, we deduce that for $\alpha \neq 1$,

$$
\begin{aligned}
\left|\frac{zf'(z)}{f(z)} - 1\right| &\leq \frac{\mu\left|\int_0^1 \left(t^{\alpha-1} - \frac{1-t^{\alpha-1}}{\alpha-1}\right) w(tz)dt\right|}{\left|1 + \frac{\mu}{\alpha-1}\int_0^1 (1-t^{\alpha-1})w(tz)dt\right|} \\
&\leq \frac{\mu\int_0^1 t\left|\frac{\alpha t^{\alpha-1}-1}{\alpha-1}\right| dt}{1 - \frac{\mu}{\alpha-1}\int_0^1 (t-t^\alpha)dt} \\
&= \frac{\mu\left[\frac{1}{1-\alpha}\int_0^{\alpha^{1/(1-\alpha)}} (\alpha t^\alpha - t)dt + \int_{\alpha^{1/(1-\alpha)}}^1 \frac{t-\alpha t^\alpha}{1-\alpha}dt\right]}{1 - \frac{\mu}{\alpha-1}\left(\frac{1}{2} - \frac{1}{\alpha+1}\right)} \\
&= \frac{\mu}{1 - \frac{\mu}{2(1+\alpha)}}\left[\frac{1}{1-\alpha}\left\{\frac{\alpha}{\alpha+1}\alpha^{(\alpha+1)/(1-\alpha)} - \frac{\alpha^{2/(1-\alpha)}}{2}\right\}\right. \\
&\qquad \left. + \frac{1}{1-\alpha}\left\{\frac{1}{2} - \frac{\alpha^{2/(1-\alpha)}}{2} - \frac{\alpha}{\alpha+1} + \frac{\alpha}{\alpha+1}\alpha^{(\alpha+1)/(1-\alpha)}\right\}\right] \\
&= \frac{\mu}{1-\alpha}\left(\frac{1}{1 - \frac{\mu}{2(1+\alpha)}}\right)\left[\frac{\alpha^{2/(1-\alpha)}(1-\alpha)}{2(\alpha+1)} + \frac{1-\alpha}{2(\alpha+1)} + \frac{\alpha^{2/(1-\alpha)}(1-\alpha)}{2(\alpha+1)}\right] \\
&= \frac{\mu}{2(1+\alpha)}\left(\frac{1+2\alpha^{2/(1-\alpha)}}{1 - \frac{\mu}{2(1+\alpha)}}\right).
\end{aligned}
$$

Note that for $\alpha \neq 1$,

$$
\frac{\mu}{2(1+\alpha)}\left(\frac{1+2\alpha^{2/(1-\alpha)}}{1 - \frac{\mu}{2(1+\alpha)}}\right) < 1 \iff \mu \leq \frac{1+\alpha}{1+\alpha^{2/(1-\alpha)}}.
$$

The case for $\alpha = 1$ is essentially the limiting case and we complete the proof. ∎

4 Conclusion

We conclude the paper with following observations. As for the starlikeness of the class $\widetilde{\mathcal{R}}(\alpha, \lambda)$ for $\alpha > 0$ is concerned, the following informations are known: For $\alpha > 0$, we have

(a) $\widetilde{\mathcal{R}}(\alpha, \lambda) \subset \mathcal{S}^*$ if $0 < \lambda \leq 2(1+\alpha)/\sqrt{5}$, see Theorem 1.3(i) and (1.1).

(b) $\widetilde{\mathcal{R}}(\alpha, \lambda) \subset \mathcal{S}^*$ if

$$
0 < \lambda \leq \begin{cases} \dfrac{2(1+\alpha)}{2+\alpha^{2\alpha/(1-\alpha)}} & \text{for } 0 < \alpha \neq 1 < \infty \\ \dfrac{4e^2}{1+2e^2} \approx 1.873 & \text{for } \alpha = 1, \end{cases}
$$

see Corollary 1.15(a). For $\alpha = 1$ and $\alpha = 1/2$, it is easy to see that the inclusion (b) is better.

(c) $\widetilde{\mathcal{R}}(\alpha,\lambda) \subset \mathcal{S}_1$ if $0 < \lambda \leq (1+\alpha)/2$, see Theorem 1.3(i) and Lemma 1.2.

(d) $\widetilde{\mathcal{R}}(\alpha,\lambda) \subset \mathcal{S}_1$ if $0 < \lambda \leq \alpha$, see Theorem 1.3. We observe that for $\alpha \geq 1$, (d) is better than (c).

(e) $\widetilde{\mathcal{R}}(\alpha,\lambda) \subset \mathcal{S}_1$ if

$$0 < \lambda \leq \begin{cases} \dfrac{(1+\alpha)}{1+\alpha^{2\alpha/(1-\alpha)}} & \text{for } \; 0 < \alpha \neq 1 < \infty \\ \dfrac{2e^2}{1+e^2} & \text{for } \; \alpha = 1, \end{cases}$$

see Corollary 1.15(b). Note that for $\alpha = 1$, (e) is clearly better than (c) and (d).

Apart from the above mentioned results, one can get additional information by looking at the other possibilities. For example, if we add (3.10) and (3.11), we find that

$$(zf'(z))' - 1 = \mu \left(\omega(z) + (1-\alpha) \int_0^1 t^{\alpha-1}\omega(tz)dt \right) \tag{4.1}$$

so that

$$|(zf'(z))' - 1| \leq \mu \left(1 + \frac{|1-\alpha|}{1+\operatorname{Re}\alpha} \right). \tag{4.2}$$

This observation shows that if f satisfies the condition (1.6) then $f \in \mathcal{S}_1$ if

$$\mu \left(1 + \frac{|1-\alpha|}{1+\operatorname{Re}\alpha} \right) \leq 1.$$

Equivalently, for $\alpha > 0$, we have

$$\widetilde{\mathcal{R}}(\alpha,\lambda) \subset \mathcal{S}_1$$

if

$$0 < \lambda \leq \begin{cases} \dfrac{\alpha(1+\alpha}{2} & \text{if } 0 < \alpha \leq 1 \\ \dfrac{1+\alpha}{2} & \text{if } \alpha \geq 1. \end{cases}$$

A comparison with (c) and (d) above shows that the last inclusion is not better. On the other hand, if f satisfies the condition (1.6) then from (4.2) we see that f is convex if

$$0 < \mu \leq \frac{2(1+\operatorname{Re}\alpha)}{(1+\operatorname{Re}\alpha + |1-\alpha|)\sqrt{5}} \quad (\operatorname{Re}\alpha > -1).$$

In particular, functions f satisfying the condition (1.6) are convex if

$$0 < \mu \leq \begin{cases} \dfrac{\alpha+1}{\sqrt{5}} & \text{if } \; -1 \leq \alpha \leq 1 \\ \dfrac{(1+\alpha)}{\alpha\sqrt{5}} & \text{if } \alpha \geq 1. \end{cases} \tag{4.3}$$

In view of (4.3) and Corollary 1.10, a simple calculation shows the following:

Corollary 4.1. *Functions $f \in \mathcal{A}$ satisfying the condition* (1.6) *are convex if*

$$0<\mu\leq \begin{cases} \dfrac{1+\alpha}{2+\alpha} & \textit{for} \quad -1<\alpha\leq\sqrt{5}-2 \\ \dfrac{1+\alpha}{\sqrt{5}} & \textit{for} \quad \sqrt{5}-2\leq\alpha\leq 1 \\ \dfrac{1+\alpha}{\alpha\sqrt{5}} & \textit{for} \quad 1\leq\alpha\leq\frac{2}{\sqrt{5}-1} \\ \dfrac{1+\alpha}{2+\alpha} & \textit{for} \quad \frac{2}{\sqrt{5}-1}\leq\alpha\leq 2 \\ \dfrac{1+\alpha}{2\alpha} & \textit{for} \quad \alpha\geq 2. \end{cases}$$

In particular, we have

Corollary 4.2. *For $\alpha>0$, the inclusion $\widetilde{\mathcal{R}}(\alpha,\lambda)\subset\mathcal{K}$ holds whenever*

$$0<\lambda\leq \begin{cases} \dfrac{\alpha(1+\alpha)}{2} & \textit{for} \quad 0<\alpha\leq\frac{1}{2} \\ \dfrac{\alpha(1+\alpha)}{1+2\alpha} & \textit{for} \quad \frac{1}{2}\leq\alpha\leq\frac{\sqrt{5}-1}{2} \\ \dfrac{\alpha(1+\alpha)}{\sqrt{5}} & \textit{for} \quad \frac{\sqrt{5}-1}{2}\leq\alpha\leq 1 \\ \dfrac{1+\alpha}{\sqrt{5}} & \textit{for} \quad 1\leq\alpha\leq\frac{1}{\sqrt{5}-2} \\ \dfrac{\alpha(1+\alpha)}{1+2\alpha} & \textit{for} \quad \alpha\geq\frac{1}{\sqrt{5}-2}. \end{cases}$$

Similarly, one can list down the conditions on α and λ for functions satisfying the condition (1.6) or functions from $\widetilde{\mathcal{R}}(\alpha,\lambda)$ to be in $\mathcal{K}_1$.

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S. Ponnusamy
Department of Mathematics
Indian Institute of Technology
IIT-Madras, Chennai- 600 036, India
e-mail: `samy@acer.iitm.ernet.in`
Fax: +91-44-2350509

Vikramaditya Singh
3A/95, Azad Nagar,
Kanpur-208 002, India.