Invariant subspace for a singular integral operator on Ahlfors David surfaces

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Abstract

In this paper the singular integral operator on Ahlfors-David surfaces is shown to have an invariant subspace of the generalized Hölder continuous function space. We study the problem in the context of Quaternionic Analysis. Two equivalent norms on certain quaternionic monogenic functions subspace are treated.

1 Introduction

It is well known that when an operator or class of operators is shown to have invariant subspaces, a general structure theory usually emerges. We refer the reader to [2] for more information about the invariant subspace problem. To show that the singular integral operators over surfaces in $\mathbf{R}^n$ has nontrivial invariant subspace has received a lot of attention lately. In [21] by using results on Ten space by Coifman, Meyer and Stein, see [6], Murray proved the L^2 boundedness of the singular integral of Cauchy's type over Lipschitz graphs with small Lipschitz constant. For all Lipschitz constant, McIntosh in [18] proved also such a result. Recently, the L^2 boundedness of the singular Cauchy integral involving Ahlfors -David surfaces and rectifiability has been studied by a prominent group of mathematicians: David [7]; Mattila [19];[20] and Semmes [22];[23]. The notion of rectifiability arises in nature in connection with L^p estimates for singular integral operator cf. [8];[9].
In Hölder continuous spaces the boundedness of the singular integral of Cauchy type

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on Liapunov surfaces in the case $n = 3$ was proved by Gegelia in [12]. Singular integral of Cauchy type over compact Liapunov surfaces in $\mathbf{R}^n$ can be found in [17], also in spaces of Hölder continuous functions.
The boundedness of the singular integral operator in Hölder spaces in the context of the Quaternionic analysis were presented by the authors in [1], attempts were made regarding Ahlfors David surfaces.
This work can be considered as a continuation and a refinement of that in [1] and also of those obtained in [4], where we have involved ourselves with the study of the behavior of the Cauchy type singular integral operators over non-smooth surfaces in generalized Hölder spaces.
The purpose of the present paper is aimed at the constructing of a proper invariant subspace of the generalized Hölder spaces relative to the singular integral operator on Ahlfors-David surfaces in the Quaternionic analysis setting. Such invariant subspace on Ahlfors-David curves in the complex plane was first proved in 1977 by Issa [16] (for curves with additional condition) and in 1985 by Gonzalez and Bustamante [13] in the general case.

2 Some known definitions and results

Let e_1, e_2, e_3 be unit vectors in the real quaternionic skew field $\mathbf{H}$. Assume the generating relations $e_ie_j + e_je_i = -2\delta_{ij}$, $i, j = 1, 2, 3$ with δ_{ij} the Kronecker delta. Furthermore, let $e_1e_2 =: e_3$. The unit element of the algebra will be denoted by e_0. An arbitrary element $a \in \mathbf{H}$ is given by $a = a_0e_0 + \sum_{j=1}^{3} a_je_j$ and the conjugated quaternion $\overline{a} := a_0e_0 - \sum_{j=1}^{3} a_je_j$. For each $a \in \mathbf{H}$ we have the norm $|a|^2 = \overline{a}a = a\overline{a} = a_0^2 + a_1^2 + a_2^2 + a_3^2$.
We suppose $\Omega \subset \mathbf{R}^3$ is a bounded simply connected domain with an Ahlfors-David boundary $\Gamma = \partial\Omega$, i.e. there exists a positive number c such that

$$c^{-1}r^2 \leq \mathcal{H}^2(\Gamma \cap B(z, r)) \leq cr^2,$$

for all $z \in \Gamma$, $0 < r \leq diam\Gamma$, where $\mathcal{H}^2(F)$, $F \subset \mathbf{R}^3$ is the 2-dimensional Hausdorff measure of the set F and $B(z, r)$ stands for the closed ball with center z and radius r.
We use a first order differential operator, the Dirac operator $D = \sum_{j=1}^{3} e_j \frac{\partial}{\partial x_j}$, which has a fundamental solution $E(x) = \frac{1}{4\pi} \frac{\overline{x}}{|x|^3}$.
Any function $u : \Omega \to \mathbf{H}$ has the representation $u = \sum_{j=0}^{3} u_je_j$, with $\mathbf{R}$-valued coordinates u_j. The notation $u \in C^k(\Omega, \mathbf{H})$, $k \in \mathbf{N} \cup \{0\}$, might be understood both coordinate wisely and directly. We consider in Ω the equation $Du = 0$ and look for its $C^1(\Omega, \mathbf{H})$ solutions in term of (left) monogenic functions in Ω.
The reader is referred to [5;14;15] for more information about these topics and general quaternionic analysis. Using the function $E(x)$ we are going to deal with the following integral operators

$$(\mathcal{C}_\Gamma u)(x) = \int_\Gamma E(x - y)n(y)u(y)d\mathcal{H}^2(y), \ x \notin \Gamma,$$

(Cauchy type operator)

$$(\mathcal{S}_\Gamma u)(x) = 2\int_\Gamma E(x-y)n(y)(u(y)-u(x))d\mathcal{H}^2(y) + u(x),\ x \in \Gamma,$$

(singular integral operator)

where $n(y)$ is the outward pointing normal vector to the boundary Γ at the point y due to Federer [11]. The integral, which defines the operator $\mathcal{S}_\Gamma$ is understood in the sense of Cauchy's principle value. The operator $\mathcal{P}_\Gamma^+ := \frac{1}{2}(I + \mathcal{S}_\Gamma)$ denotes the projection into the space of all $\mathbf{H}$-valued functions, which allows, a monogenic extension onto the domain Ω. $\mathcal{P}_\Gamma^- := \frac{1}{2}(I - \mathcal{S}_\Gamma)$ denotes the projection into the space of all $\mathbf{H}$-valued functions, which allows, a monogenic extension into the domain $\mathbf{R}^3\backslash\overline{\Omega}$ and vanishes at infinity. I stands for the identity operator.
In order to state the results, we require some more notation. As in [3], let $S(\Gamma, \mathbf{H})$ denotes the space of continuous $\mathbf{H}$-valued functions on Γ such that

$$\int_{\Gamma_\epsilon(z):=\Gamma\cap B(z,\epsilon)} E(z-y)n(y)(u(y)-u(z))d\mathcal{H}^2(y) \to 0,$$

when $\epsilon \to 0$, uniformly for $z \in \Gamma$.
Given a positive real number d, a continuous function $w : (0, d] \to \mathbf{R}_+$ with $w(0+) = 0$, will be called a majorant if $w(\delta)$ is increasing and $w(\delta)/\delta$ is non increasing for $\delta > 0$. If, in addition, there is a constant c_w such that

$$\int_0^\delta \frac{w(\tau)}{\tau} d\tau + \delta \int_\delta^d \frac{w(\tau)}{\tau^2} d\tau \le c_w w(\delta),$$

whenever $0 < \delta < d$, then we say that w is a regular majorant.
Here and in the sequel, notations c_w, c, etc will be used for positive constants, which may vary from one occurrence to the next. Subscripts, such as w in c_w, are used to stress dependence of constants.
If F is a closed subset of $\mathbf{R}^3$ and u is a bounded quaternionic-valued function on F we define the modulus of continuity w_u by

$$w_u(\delta) := \delta \sup_{r\ge\delta} r^{-1} \sup_{|z_1-z_2|\le r,\, z_1,z_2\in F} |u(z_1) - u(z_2)|,$$

whenever $\delta \ge 0$. Given a majorant w, then we define the generalized Hölder space of functions being quaternionic-valued continuous on F by

$$\Lambda_w(F) := \{u : F \to \mathbf{H},\, \exists c > 0; w_u(\delta) \le c w(\delta), \delta \ge 0\}.$$

One can endows this space with the norm

$$\|u\|_{\Lambda_\omega(\Gamma)} := \|u\|_{C(\Gamma,\mathbf{H})} + \sup_{0<\tau\le d} \frac{w_u(\tau)}{w(\tau)}.$$

We shall be concerned also with the space

$$S_w(\Gamma, \mathbf{H}) := S(\Gamma, \mathbf{H}) \cap \Lambda_w(\Gamma).$$

Throughout the paper, Γ denotes an Ahlfors David surface with diameter d.
The following fundamental statements, which are well know in the classical function theory of one complex variable were proved by the authors in previously works, see the references [1;3 and 4].

Lemma 2.1 (Borel-Pompeiu's formula). *Let $u \in C^1(\Omega, \mathbf{H}) \cap C(\Omega \cup \Gamma, \mathbf{H})$. Then we have*

$$\int_{\Gamma} E(x-y)n(y)u(y)d\mathcal{H}^2(y) - \int_{\Omega} E(x-y)Du(y)d\mathcal{L}^3(y) = \\ = \begin{cases} u(x), & \text{if } x \in \Omega \\ 0, & \text{if } x \in \mathbf{R}^3 \setminus \overline{\Omega} \end{cases},$$

where $\mathcal{L}^3$ denotes the usual Lebesgue measure in $\mathbf{R}^3$.

Remark: The singular integral $\mathcal{S}_\Gamma$ and the limiting values of the Cauchy type integral $\mathcal{C}_\Gamma$ are connected by

Lemma 2.2 (Plemelj-Sokholzkij's formulas). *Let $u \in S(\Gamma, \mathbf{H})$. Then we have*

$$\textbf{i)} \lim_{\Omega \ni x \to z \in \Gamma} (\mathcal{C}_\Gamma u)(x) = (\mathcal{P}^+ u)(z)$$
$$\textbf{ii)} \lim_{\mathbf{R}^3 \setminus \overline{\Omega} \ni x \to z \in \Gamma} (\mathcal{C}_\Gamma u)(x) = -(\mathcal{P}^- u)(z),$$

for any $z \in \Gamma$.

Lemma 2.3. *Let w be a regular majorant. Then $\Lambda_\omega(\Gamma)$ represents an invariant subspace for the operator $\mathcal{S}_\Gamma$. Moreover it holds that*

$$\|\mathcal{S}_\Gamma u\|_{\Lambda_\omega(\Gamma)} \leq c_w \|u\|_{\Lambda_\omega(\Gamma)}.$$

We remark that if $u \in S(\Gamma, \mathbf{H})$, reviewing lemma 2.2 and lemma 2.3, it is possible to extend the function $\mathcal{C}_\Gamma u$ by continuity to $\Omega \cup \Gamma$ to a function of class $\Lambda_w(\Omega \cup \Gamma)$. As such an extension exists, it seems to be convenient to replace $\mathcal{C}_\Gamma u$ by $\mathcal{C}_\Gamma^+ u$, when $\mathcal{C}_\Gamma u$ is considered as a function defined inside and on the boundary of the domain Ω. In the sequel, we will identify $\mathcal{C}_\Gamma u$ and $\mathcal{C}_\Gamma^+ u$ when it does not lead to confusion. The next result is part of the folklore we shall need more than one time later, we included it without proof (see for instance [1, p.13]) for the sake of brevity.

Lemma 2.4. *Let $\varphi(t)$ be a nonnegative function that does not increase in $(0, d]$. Then, for every positive numbers r_1, $r_2 \in (0, d]$, $r_1 < r_2$, the following formula holds:*

$$\int_{\Gamma_{r_2}(x) \setminus \Gamma_{r_1}(x)} \varphi(|y-x|)d\mathcal{H}^2(y) = \int_{r_1}^{r_2} \varphi(\tau)d\mathcal{H}^2(\Gamma_\tau(x)).$$

3 Equivalent Norms on $S(\Gamma, \mathbf{H})$

Taking into account the Painlevé argument on removable singularities for continuous monogenic functions, see [4, p.139] and the theorem 2 in [3, p.85], it is not difficult to see that each $u \in S(\Gamma, \mathbf{H})$ admits a unique decomposition of the form $u = u^+ + u^-$, where $u^\pm \in ImP_\Gamma^\pm$, so that we can define the following norms in $S(\Gamma, \mathbf{H})$:

$$\|u\|_1 := \|u^+\|_{C(\Gamma,\mathbf{H})} + \|u^-\|_{C(\Gamma,\mathbf{H})},$$

$$\|u\|_2 := \|u\|_{C(\Gamma,\mathbf{H})} + \sup_{0<\epsilon<d} \| \int\limits_{\Gamma\setminus\Gamma_\epsilon(z)} E(z-y)n(y)(u(y)-u(z))d\mathcal{H}^2(y)\|_{C(\Gamma,\mathbf{H})}.$$

With each of the above norms the space $S(\Gamma, \mathbf{H})$ becomes a Banach space.
The space $S(\Gamma, \mathbf{H})$ is connected with singular integral operator S_Γ in the following way. If $u = u^+ + u^-$, then $u^+ = u + \tilde{u}$, where

$$\tilde{u}(z) = \int\limits_{\Gamma} E(z-y)n(y)(u(y)-u(z))d\mathcal{H}^2(y).$$

Theorem 3.1. *In $S(\Gamma, \mathbf{H})$ the norms $\|u\|_1$ and $\|u\|_2$ are equivalent.*

We proceed by estimating the norms involved. Let $z \in \Gamma$, we have

$$|u^+(z)| \leq \|u\|_{C(\Gamma,\mathbf{H})} + |\lim_{\epsilon\to 0} \int\limits_{\Gamma\setminus\Gamma_\epsilon(z)} E(z-y)n(y)(u(y)-u(z))d\mathcal{H}^2(y)| \leq$$

$$\|u\|_{C(\Gamma,\mathbf{H})} + \sup_{0<\epsilon\leq d} \| \int\limits_{\Gamma\setminus\Gamma_\epsilon(z)} E(z-y)n(y)(u(y)-u(z))d\mathcal{H}^2(y)\|_{C(\Gamma,\mathbf{H})} = \|u\|_2.$$

By analogy we obtain

$$\|u^-\|_{C(\Gamma,\mathbf{H})} \leq \|u\|_2,$$

whence $\|u\|_1 \leq \|u\|_2$. We proceed now by estimating the contrary inequality. To this end in view of theorem 3 in [3 p. 87] and the application of the maximum modulus theorem we obtain

$$\| \int\limits_{\Gamma\setminus\Gamma_\epsilon(z)} E(z-y)n(y)(u(y)-u(z))d\mathcal{H}^2(y)\|_{C(\Gamma,\mathbf{H})} \leq$$

$$\leq \| \int\limits_{\Gamma\setminus\Gamma_\epsilon(z)} E(z-y)n(y)(u^+(y)-u^+(z))d\mathcal{H}^2(y)\|_{C(\Gamma,\mathbf{H})} +$$

$$+\| \int\limits_{\Gamma\setminus\Gamma_\epsilon(z)} E(z-y)n(y)(u^-(y)-u^-(z))d\mathcal{H}^2(y)\|_{C(\Gamma,\mathbf{H})}.$$

Let us denote respectively by I_1 and I_2 the summands in the right side of the above inequality.
We have

$$I_1 = \| \int\limits_{\Gamma\setminus\Gamma_\epsilon(z)} E(z-y)n(y)(u^+(y)-u^+(z))d\mathcal{H}^2(y)\|_{C(\Gamma,\mathbf{H})} \leq$$

$$\leq \max_{x\in\partial B(z,\epsilon)\cap\overline{\Omega}} |u^+(x)-u^+(z)| \leq c\|u^+\|_{C(\Gamma,\mathbf{H})}.$$

In a similar way $I_2 \leq c\|u^-\|_{C(\Gamma,\mathbf{H})}$. Consequently,

$$\sup_{0<\epsilon\leq d} |\int\limits_{\Gamma\setminus\Gamma_\epsilon(z)} E(z-y)n(y)(u(y)-u(z))d\mathcal{H}^2(y)\|_{C(\Gamma,\mathbf{H})} \leq \|u\|_1.$$

Since $\|u\|_{C(\Gamma,\mathbf{H})} \leq \|u\|_1$, we conclude that $\|u\|_2 \leq c\|u\|_1$, which proves the equivalence of the norms. ■

If the surface Γ is such that the operator $\mathcal{S}_\Gamma$ is a bounded operator mapping from $\Lambda_w(\Gamma)$, into itself (for instance to be Ahlfors David and w is regular majorant) thus the decomposition

$$S_w(\Gamma,\mathbf{H}) = Im\mathcal{P}_\Gamma^+ \cap \Lambda_w(\Gamma) \oplus Im\mathcal{P}_\Gamma^- \cap \Lambda_w(\Gamma)$$

is actually a decomposition of $S_w(\Gamma,\mathbf{H})$ into the spaces $\mathcal{P}_\Gamma^\pm(\Lambda_w(\Gamma))$, and we can define a norm on it by the equality

$$\|u\|_3 := \|u^+\|_{\Lambda_w(\Gamma)} + \|u^-\|_{\Lambda_w(\Gamma)}.$$

We can no longer presume the boundedness of the singular integral operator $\mathcal{S}_\Gamma$ on the space $S_w(\Gamma,\mathbf{H})$ with the norm $\|u\|_3$, and assure that it is unitary.

4 Preliminarily Inequalities

In this section we will show some auxiliaries lemmas, which play a crucial role in what follows. Before stating them we define a special characteristic metric of $S(\Gamma,\mathbf{H})$ as

$$\Theta_u(\delta) = \delta \sup_{r\geq\delta} r^{-1} \sup_{0<\epsilon<r,\, z\in\Gamma} \|\int\limits_{\Gamma_\epsilon(z)} E(z-y)n(y)(u(y)-u(z))d\mathcal{H}^2(y)\|.$$

Lemma 4.1. *Let $x, z \in \Gamma$, $\epsilon = |z-x|$. Then*

$$\|\int\limits_{\Gamma\setminus\Gamma_\epsilon(z)\cup\Gamma_\epsilon(x)} E(x-y)n(y)d\mathcal{H}^2(y)\| \leq c,$$

and

$$\|\int\limits_{\Gamma\setminus\Gamma_\epsilon(z)\cup\Gamma_\epsilon(x)} (E(z-y)-E(x-y))n(y)(u(y)-u(z))d\mathcal{H}^2(y)\| \leq c\epsilon\int\limits_\epsilon^d \frac{w_u(\tau)}{\tau^2}d\tau$$

Proof : The proof is essentially a version of lemma 8.2 from [1, p.14]. It's last inequality can be verified by a similar calculation like the one shown in [1, p.16]. ■

Lemma 4.2. *Let $u \in S(\Gamma,\mathbf{H})$. Then for $z \in \Gamma$*

$$\sup_{|z-x|=\epsilon,\, x\in\Omega\cup\Gamma} |\mathcal{C}_\Gamma^+ u(z) - \mathcal{C}_\Gamma^+ u(x)| \leq c(w_u(\epsilon) + \Theta_u(\epsilon) + \epsilon\int\limits_\epsilon^d \frac{w_u(\tau)}{\tau^2}d\tau).$$

Proof : Let us distinguish two cases.
CASE 1. We assume that $|z-x|=\epsilon$, $x\in\Omega$. Denote by z_x a point of Γ such that $\delta=|x-z_x|=\inf_{z\in\Gamma}|x-z|$. It is easily seen that

$$|\mathcal{C}_\Gamma^+ u(z)-\mathcal{C}_\Gamma u(x)|\leq c(|\int\limits_{\Gamma_\delta(z_x)} E(x-y)n(y)(u(y)-u(z_x))d\mathcal{H}^2(y)\|+$$
$$+|\int\limits_{\Gamma\setminus\Gamma_\delta(z_x)}(E(x-y)-E(z_x-y)n(y)(u(y)-u(z_x))d\mathcal{H}^2(y)\|+$$
$$+|\int\limits_{\Gamma_\delta(z_x)} E(z_x-y)n(y)(u(y)-u(z_x))d\mathcal{H}^2(y)\|+$$
$$+w_{\mathcal{S}_\Gamma u}(|z_x-z|)+w_u(|z_x-z|)$$

By using the arguments similar to the proof of the theorem 1 of [3, p.85] we can derive the inequality

$$|\mathcal{C}_\Gamma^+ u(z)-\mathcal{C}_\Gamma u(x)|\leq c(w_u(\delta)+\Theta_u(\delta)+\delta\int\limits_\delta^d\frac{w_u(\tau)}{\tau^2}d\tau)\leq$$
$$\leq c(w_u(\epsilon)+\Theta_u(\epsilon)+\epsilon\int\limits_\epsilon^d\frac{w_u(\tau)}{\tau^2}d\tau).$$

CASE 2. Let $x\in\Gamma$. Putting $\epsilon=|z-x|$ and taking into account that $\mathcal{C}_\Gamma u(x)=u(x)+\tilde{u}(x)$ we obtain

$$|\mathcal{C}_\Gamma^+ u(z)-\mathcal{C}_\Gamma^+ u(x)|\leq c(w_u(|z-x|)+w_{\tilde{u}}(|z-x|)).$$

Following certain estimates like those given in the proof of theorem 8.1 in [1 p.15], it is possible to prove that

$$|\mathcal{C}_\Gamma^+ u(z)-\mathcal{C}_\Gamma^+ u(x)|\leq c(w_u(\epsilon)+\Theta_u(\epsilon)+\epsilon\int\limits_\epsilon^d\frac{w_u(\tau)}{\tau^2}d\tau).$$

Since, for $x\in\Omega\cup\Gamma$

$$|\mathcal{C}_\Gamma^+ u(z)-\mathcal{C}_\Gamma^+ u(x)|\leq|\mathcal{C}_\Gamma^+ u(z)-\mathcal{C}_\Gamma^+ u(z_x)|+|\mathcal{C}_\Gamma^+ u(z_x)-\mathcal{C}_\Gamma^+ u(x)|,$$

the statement of the lemma 4.2 follows now from the above inequalities. ∎

Lemma 4.3. *Let w be a majorant, $u\in S(\Gamma,\mathbf{H})$ and $\epsilon\in(0,d]$. Then*

$$\Theta_{\mathcal{S}_\Gamma u}(\epsilon)\leq c(w_u(\epsilon)+\Theta_u(\epsilon)+\epsilon\int\limits_\epsilon^d\frac{w_u(\tau)}{\tau^2}d\tau).$$

Proof: For $z\in\Gamma$ we can write $(\mathcal{S}_\Gamma u)(z)=2(\mathcal{C}_\Gamma^+ u)(z)-u(z)$. Then

$$|\int\limits_{\Gamma_\epsilon(z)} E(z-y)n(y)(\mathcal{S}_\Gamma u(y)-\mathcal{S}_\Gamma u(z))d\mathcal{H}^2(y)|\leq c(\Theta_u(\epsilon)+$$
$$+|\int\limits_{\Gamma_\epsilon(z)} E(z-y)n(y)((\mathcal{C}_\Gamma^+ u)(y)-(\mathcal{C}_\Gamma^+ u)(z))d\mathcal{H}^2(y)|)$$

As a consequence of the theorem 3 in [3, p.87] we get

$$|\int_{\Gamma_\epsilon(z)} E(z-y)n(y)((\mathcal{C}_\Gamma^+u)(y)-(\mathcal{C}_\Gamma^+u)(z))d\mathcal{H}^2(y)|\leq$$
$$\leq \sup_{|z-x|=\epsilon,\, x\in\Omega\cup\Gamma}|(\mathcal{C}_\Gamma^+u)(z)-(\mathcal{C}_\Gamma^+u)(x)|$$

With the aid of lemma 4.2 the proof is therefore complete. ■

5 Invariant Subspace

Now we shall prove the main result of the paper.

Theorem 5.1. *Let w be a majorant such that $\epsilon\int\limits_\epsilon^d \frac{w(\tau)}{\tau^2}d\tau \leq c_w w(\epsilon)$. Then the singular integral operator $\mathcal{S}_\Gamma$ is bounded on the subspace*

$$Z_w(\Gamma) := \{u\in S_w(\Gamma,\mathbf{H}) : \sup_{0<\delta\leq d}\frac{\Theta_u(\delta)}{w(\delta)} < +\infty\}.$$

Therefore, $Z_w(\Gamma)$ represents an invariant subspace for the operator $\mathcal{S}_\Gamma$. The norm in $Z_w(\Gamma)$ is given by

$$\|u\|_{Z_w(\Gamma)} := [\|u\|_{\Lambda_w(\Gamma)} + \sup_{0<\delta\leq d}\frac{\Theta_u(\delta)}{w(\delta)}.$$

Before proving the theorem itself, let us look at some connections between the spaces $\Lambda_w(\Gamma)$ and $Z_w(\Gamma)$.
The following proposition is a version of lemma 1 in [10, p.148], see also [16].

Proposition 5.2. *Let w be a majorant such that $\int\limits_0^\delta \frac{w(\tau)}{\tau}d\tau \leq \infty$. Then w_1, defined by $w_1(\delta)=\int\limits_0^\delta \frac{w(\tau)}{\tau}d\tau$, $\delta>0$,*
is a majorant and $\Lambda_w(\Gamma)\subset Z_{w_1}(\Gamma)$.

Proof : w_1 is an increasing continuous differentiable function, moreover, $w(0+)=0$ and such that $w_1(\delta)\geq w(\delta)$, $\delta>0$, which can be directly seem from its definition. Furthermore, differentiating w_1 gives

$$\frac{d}{d\delta}(\frac{w_1(\delta)}{\delta}) = -\frac{1}{\delta^2}\int_0^\delta \frac{w(\tau)}{\tau}d\tau) + \frac{w(\delta)}{\delta},$$

it follows that

$$\frac{d}{d\delta}(\frac{w_1(\delta)}{\delta}) \leq -\frac{w(\delta)}{\delta^2} + \frac{w(\delta)}{\delta^2} = 0.$$

Thus, w_1 is indeed a majorant. Finally, if $u\in\Lambda_w(\Gamma)$, the last statement of the proposition is an immediate consequence of the inequalities:

$$w_u(\delta)\leq \|u\|_{\Lambda_w(\Gamma)}w(\delta);\quad \Theta_u(\delta)\leq c\|u\|_{\Lambda_w(\Gamma)}w_1(\delta),\ \delta>0.$$

■

Remark: In connection with the proposition above it is not hard to see that if w is a regular majorant, then $\Lambda_w(\Gamma) = Z_w(\Gamma)$.

Proof of the Theorem 5.1: According to lemma 4.3 we have

$$\Theta_{\mathcal{S}_\Gamma u}(\epsilon) \leq c(w_u(\epsilon) + \Theta_u(\epsilon) + \epsilon \int_{\epsilon}^{d} \frac{w_u(\tau)}{\tau^2} d\tau).$$

Let $u \in Z_w(\Gamma)$ and let $x, z \in \Gamma$, $\delta = |x - z|$. We can write

$$\begin{aligned}(\mathcal{S}_\Gamma u)(x) - (\mathcal{S}_\Gamma u)(z) = 2 \Bigg\{ & \int_{\Gamma_\delta(x)} E(x-y)n(y)(u(y) - u(x))d\mathcal{H}^2(y) - \\ & - \int_{\Gamma_\delta(z)} E(z-y)n(y)(u(y) - u(z))d\mathcal{H}^2(y) + \\ + \int_{\Gamma\setminus(\Gamma_\delta(x)\cup\Gamma_\delta(z))} & (E(z-y) - E(x-y))n(y)(u(y) - u(z))d\mathcal{H}^2(y) + \\ & + \int_{\Gamma\setminus(\Gamma_\delta(x)\cup\Gamma_\delta(z))} E(z-y)n(y)(u(y) - u(x))d\mathcal{H}^2(y) + \\ & + \int_{\Gamma_\delta(z)} E(x-y)n(y)(u(y) - u(x))d\mathcal{H}^2(y) + \\ & - \int_{\Gamma_\delta(x)} E(z-y)n(y)(u(y) - u(z))d\mathcal{H}^2(y) \Bigg\} + u(x) - u(z).\end{aligned}$$

Using the lemma 2.4 and lemma 4.1, we get

$$w_{\mathcal{S}_\Gamma u}(\delta) \leq c(w_u(\delta) + \Theta_u(\delta) + \delta \int_{\delta}^{d} \frac{w_u(\tau)}{\tau^2} d\tau).$$

Combining the above estimate we obtain that $\mathcal{S}_\Gamma u \in Z_w(\Gamma)$ and the operator $\mathcal{S}_\Gamma$ is bounded. ∎

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