Periodic solutions for certain systems of planar complex polynomial equations

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1 Introduction

The present paper deals with the existence of T-periodic solutions of the T-periodic system of complex planar equations

$$\begin{cases} u' = p_1(t, u, v) \\ v' = p_2(t, u, v), \end{cases} \tag{P}$$

where p_1 and p_2 are second order polynomials whose coefficients are T-periodic continuous functions from $\mathbb{R}$ into $\mathbb{C}$.

Like it will be clear from the examples of the last section, (P) represents a generalization of the well known complex periodic Riccati equation

$$u' = u^2 + g(t),$$

where $g : \mathbb{R} \to \mathbb{C}$ is a continuous T-periodic function.

The existence of periodic solutions of equations of the type

$$u' = \sum_{j=0}^{n} c_j(t) u^j \tag{E}$$

is an extensively investigated subject.

For example, in 1973 Lloyd [5] studied the problem when the polynomial at the right hand-side has real-valued coefficients.

More recently several authors have been concerned with complex-valued polynomial equations depending also by the conjugate of the solution (see Mawhin [7], Srzednicki [11] and the references therein).

(E) is considered in Campos-Ortega [4] and Miklaszewski [8], whose papers deal with the non-existence of periodic solutions, and in Campos [3], who analizes the possible dynamics of the complex periodic Riccati equation, e.g. the case $n = 2$.

As far as we know the only papers dealing with the existence of periodic solutions of (E) with complex-valued coefficients are those of Borisovich-Marzantovich [1] and [2].

Following their idea, we consider the system (P) under the further assumption that the coefficients of the polynomials have an holomorphic extension to a neighbourhood of the unit disc, i.e. that they are developable in Fourier series with all the coefficients corresponding to negative indexes equal to zero. Moreover we look for solutions having the same property.

This stronger assumption allows us to pose the problem in a smaller space, at the risk of loosing some solutions, but simplifying the problem, because of the multiplicability of the mean value functional.

In section 2 we introduce the normed space in which the coefficients are taken, proving some properties we will use in the sequel. In section 3 we present the topological method upon which the existence result is based. Like in the works of Borisovich-Marzantovich it is a continuation theorem, but slightly different from the one used there. The main theorems are contained in section 4, where we also show that when (P) is reduced to a second order polynomial equation we get the same existence result obtained in [2] and we apply the existence theorem to some generalization of the complex periodic Riccati equation.

2 A space of functions

First of all we need to introduce some notations and to recall some basic properties of the spaces of functions we will use in the following.

Given a real number $T > 0, S^1(T)$ and $D(T)$ will represent as usual the sets of complex numbers of module respectively equal to and less than $\frac{T}{2\pi}$. More precisely

$$S^1(T) = \left\{ z \in \mathbb{C} : |z| = \frac{T}{2\pi} \right\}$$

and

$$D(T) = \left\{ z \in \mathbb{C} : |z| < \frac{T}{2\pi} \right\}.$$

Let us now consider the maps

$$\varphi : \{u : \mathbb{R} \to \mathbb{C} \, T\text{-periodic}\} \quad \to \{\tilde{u} : S^1(T) \quad \to \mathbb{C}\}$$
$$u \qquad\qquad\qquad \to \quad \tilde{u} : S^1(T) \quad \to \mathbb{C}$$
$$\frac{T}{2\pi}e^{i\theta} \quad \to u\left(\frac{T}{2\pi}\theta\right)$$

and

$$\psi : \{\tilde{u} : S^1(T) \to \mathbb{C}\} \quad \to \{u : \mathbb{R} \quad \to \mathbb{C} \, T\text{-periodic}\}$$
$$\tilde{u} \qquad\qquad \to \quad u : \mathbb{R} \quad \to \mathbb{C}$$
$$t \quad \to \tilde{u}\Big(\tfrac{T}{2\pi}e^{i\frac{t}{T}2\pi}\Big).$$

It immediately follows that

$$(\varphi \circ \psi)(\tilde{u})\Big(\frac{T}{2\pi}e^{i\theta}\Big) = \psi(\tilde{u})\Big(\frac{T}{2\pi}\theta\Big) = \tilde{u}\Big(\frac{T}{2\pi}e^{i\frac{T}{2\pi}\theta\frac{2\pi}{T}}\Big) = \tilde{u}\Big(\frac{T}{2\pi}e^{i\theta}\Big)$$

and conversely

$$(\psi \circ \varphi)(u)(t) = \varphi(u)\Big(\frac{T}{2\pi}e^{i\frac{t}{T}2\pi}\Big) = u\Big(\frac{T}{2\pi}\frac{t}{T}2\pi\Big) = u(t),$$

which means that φ is bijective and $\varphi^{-1} = \psi$. In what follows we will identify each $T-$periodic function $u : \mathbb{R} \to \mathbb{C}$ with its image $\tilde{u}$ through the map φ. Besides, we will also identify a function $u : [0, T] \to \mathbb{C}$, such that $u(0) = u(T)$, with its continuous extension to the whole real line.

Finally, given a continuous function u defined in [0,T], we will denote by c_k the k-th coefficient of the Fourier's development of u, i.e.

$$c_k = \int_0^T u(t)e^{-i\frac{t}{T}2\pi k}dt.$$

$\overline{C_H(T)}$ will denote the closure of the set of continuous T-periodic functions from $\mathbb{R}$ into $\mathbb{C}$ having an holomorphic extension in a neighbourhood of $D(T)$.

Proposition 2.1. $C_H(T) = \{u : \mathbb{R} \to \mathbb{C}$ *continuous* T*- periodic, developable in Fourier series:* $c_k = 0 \; \forall \, k < 0$ *and* $\limsup\limits_{k\to+\infty} \sqrt[k]{|c_k|} < 1\}$.

Proof. If u has an extension u_1 holomorphic in the open set $D(T + 2\pi\epsilon)$, its Taylor series centered at any point of $D(T + 2\pi\epsilon)$ converges to the function itself in the biggest open disc centered in the point and contained in the set, therefore

$$u_1(z) = \sum_{k=0}^{+\infty} \frac{u_1^{(k)}(0)}{k!}z^k \quad \forall \, z \in D(T + 2\pi\epsilon),$$

where $k!$ stands for the factorial of k.
Moreover the Cauchy's integral formula holds in any closed curve contained in the set and having index in the origin equal to 1, therefore

$$u_1^{(k)}(0) = \frac{k!}{2\pi i} \int_{S^1(T)} \frac{\tilde{u}(\xi)}{\xi^{k+1}}d\xi,$$

which implies that

$$u_1(z) = \sum_{k=0}^{+\infty} \frac{z^k}{2\pi i} \int_{S^1(T)} \frac{\tilde{u}(\xi)}{\xi^{k+1}}d\xi \quad \forall \, z \in D(T + 2\pi\epsilon).$$

Integrating now by substitution, we get

$$\int_{S^1(T)} \frac{\tilde{u}(\xi)}{\xi^{k+1}} d\xi = \int_0^{2\pi} \frac{\tilde{u}\left(\frac{T}{2\pi}e^{i\theta}\right)i}{\left(\frac{T}{2\pi}\right)^k e^{ik\theta}} d\theta = \int_0^T \frac{u(t)i}{\left(\frac{2\pi}{T}\right)^{k+1} e^{ik\frac{T}{2\pi}t}} dt = \frac{(2\pi)^{k+1}i}{T^k} c_k.$$

Hence

$$u_1(z) = \sum_{k=0}^{+\infty} c_k \left(\frac{2\pi z}{T}\right)^k \quad \forall z \in D(T + 2\pi\epsilon)$$

and, in particular,

$$u\left(\frac{T}{2\pi}\theta\right) = \sum_{k=0}^{+\infty} c_k e^{ik\theta} \quad \forall \theta \in [0, 2\pi],$$

which is equivalent to the developability in Fourier series of u with all the coefficients corresponding to negative indexes equal to zero.

Moreover the radius of the development in power series of u_1 is bigger than or equal to $\frac{1}{\frac{T}{2\pi}+\epsilon}$, therefore

$$\limsup_{k\to+\infty} \sqrt[k]{|c_k|\left(\frac{2\pi}{T}\right)^k} \leq \frac{1}{\frac{T}{2\pi} + \epsilon},$$

which implies that

$$\limsup_{k\to+\infty} \sqrt[k]{|c_k|} \leq \frac{1}{1 + \frac{2\pi}{T}\epsilon} < 1.$$

Viceversa if

$$u(t) = \sum_{k=0}^{+\infty} c_k e^{ik\frac{t}{T}2\pi} \quad \forall t \in [0, T]$$

and

$$\limsup_{k\to+\infty} \sqrt[k]{|c_k|} = \frac{1}{1 + \frac{2\pi}{T}\epsilon} < 1$$

then

$$\limsup_{k\to+\infty} \sqrt[k]{\left(\frac{2\pi}{T}\right)^k |c_k|} = \frac{\frac{2\pi}{T}}{1 + \frac{2\pi}{T}\epsilon} = \frac{1}{\frac{T}{2\pi} + \epsilon}.$$

Hence the radius of convergence of the power series $\sum_{k=0}^{+\infty} c_k(\frac{2\pi z}{T})^k$ is $\frac{T}{2\pi} + \epsilon$, which implies that the function

$$u_1(z) = \sum_{k=0}^{+\infty} c_k \left(\frac{2\pi z}{T}\right)^k$$

is an extension of u in the neighbourhood $D(T + 2\pi\epsilon)$ of $D(T)$, because

$$u_1\left(\frac{T}{2\pi}e^{i\theta}\right) = \sum_{k=0}^{+\infty} c_k e^{ik\theta} = u\left(\frac{T}{2\pi}\theta\right)$$

and the thesis follows by the developability in Taylor'series of the sum of each power's series in the interior of its set of convergence.   ■

$\overline{C_H^1(T)}$ will represent the set of the functions of class C^1 belonging to $\overline{C_H(T)}$, i.e.

$$C_H^1(T) = \{u \in C_H(T) \text{ of class } C^1\}.$$

The spaces just defined are linear subspaces of the spaces of continuous and continuously differentiable T-periodic functions, therefore are normed with the usual norms of those spaces, i.e. respectively

$$||u||_0 = \max_{t \in [0,T]} |u(t)|$$

and

$$||u||_1 = \max\{||u||_0, ||u'||_0\}.$$

Finally, given $u \in \overline{C_H(T)}$ we will denote by $\overline{u}$ its mean value in $[0, T]$, e.g.

$$\overline{u} = \frac{1}{T} \int_0^T u(t)dt.$$

For the spaces introduced above rather known properties hold. We refer to [10] for any details on the theory of functions of a complex variable.

Proposition 2.2.

1) *The derivative of each function of $\overline{C_H^1(T)}$ belongs to $\overline{C_H(T)}$ and conversely the primitive of each $u \in \overline{C_H(T)}$*

$$U : [0, T] \to \mathbb{C}$$
$$t \to \int_0^t u(s)ds$$

belongs to $\overline{C_H^1(T)}$;

2) *the mean value is a multiplicative functional in $\overline{C_H(T)}$, i.e. $\overline{f \cdot g} = \overline{f} \cdot \overline{g} \, \forall \, f, g$.*

Proof.

1) Let u be a function of class C^1 belonging to $\overline{C_H(T)}$. Then u' is a continuous T-periodic function.

   Moreover the properties of the power series in the interior of their disc of convergence imply that the derivative of the holomorphic extension of u in a neighbourhood of the unit disc is holomorphic in that neighbourhood. Therefore the properties of the uniform convergence yield that u' has a continuous extension to the same neighbourhood of the unit disc which is precisely the derivative of the extension of u.

   Conversely, given $u \in \overline{C_H(T)}$, by the previous computations it follows that

   $$c_k(U) = -\frac{Ti}{2\pi k} c_k(u).$$

   Hence $c_k(U) = 0$ when $k < 0$,

   $$\limsup_{k \to +\infty} \sqrt[k]{|c_k(U)|} = \limsup_{k \to +\infty} e^{\frac{1}{k} \log \frac{T}{2\pi k}} \sqrt[k]{|c_k(u)|} < 1$$

   and the thesis follows from the developability in Fourier series of each function of class C^1.

2) By the Cauchy's formula, the mean value of each function of $\overline{C_H(T)}$ is equal to the value in 0 of its extension to the neighbourhood of $D(T)$ and the conclusion follows.

∎

3   A continuation theorem

In this section we present the topological theorems upon which the existence result is based. This is a known theory (see, for example, [6] for any details on the topological degree and the continuation theorems), we just recall it briefly for completeness.

Given two normed spaces $(X_1, ||\cdot||_1)$ and $(X_2, ||\cdot||_2)$, there are many equivalent norms on the cartesian product $X_1 \times X_2$. We will consider the norm of the maximum, e.g.

$$||(x_1, x_2)|| = \max\{||x_1||_1, ||x_2||_2\}.$$

Definition 3.1.   *Let X,Z be normed spaces. A linear map L: dom $L \subset X \to Z$ is said to be a Fredholm map of index zero if*

1) dim ker L = codim Im L $< +\infty$

2) Im L is closed in Z.

Example 3.2 Recalling the properties of the normed spaces $C_H^1(T)$ and $C_H(T)$ defined in the previous paragraph, we consider the derivative map

$$L : C_H^1(T)^2 \to C_H(T)^2$$
$$(u, v) \to (u', v').$$

Then

$$\ker L = \{(u, v) \in C_H^1(T)^2 : \exists (z_1, z_2) \in \mathbb{C}^2 : u \equiv z_1, v \equiv z_2\} \simeq \mathbb{C}^2$$

and

$$\text{Im } L = \left\{(u, v) \in C_H(T)^2 : \int_0^T u(s)ds = \int_0^T v(s)ds = 0\right\} \equiv \mathbb{C}^2.$$

Therefore, since Im L is a closed subset of $C_H(T)^2$, L is a Fredholm map of index zero.

Given a Fredholm operator of index zero L, consider the continuous projection M of X onto $\ker L$, (obviously $L/_{\ker M}$ is bijective) and the continuous projection Q of Z into itself such that $\ker Q = \text{Im } L = \text{Im }(I - Q)$.

Definition 3.3.   *Let L:dom $L \subset X \to Z$ be a Fredholm map of index zero. A continuous function H defined on a normed space E and having values in Z is said to be L-compact in the closure of an open and bounded subset Ω of E if $((L/_{\ker M})^{-1} \circ (I - Q) \circ H)(\overline{\Omega})$ is compact.*

We now give sufficient conditions in order that a continuous homotopy of a normed space is compact.

Theorem 3.4. *Let Ω be an open and bounded subset of X and $H : \overline{\Omega} \times [0,1] \to Z$ a continuous function. If $H(\cdot, \lambda)$ is L-compact for every λ in [0,1] and $(L/_{\ker M})^{-1} \circ \circ (I - Q) \circ H(x, \cdot)$ is uniformily continuous with respect to $x \in \overline{\Omega}$, then H is L-compact.*

Proof. Consider a sequence $\{x_k, \lambda_k\}_k$ of $\Omega \times [0, 1]$. Then, by the compactness of $[0, 1]$, eventually passing through a subsequence, $\{\lambda_k\}_k$ is converging to a certain $\overline{\lambda} \in [0, 1]$. Since $H(\cdot, \overline{\lambda})$ is L-compact, there exists $\overline{x} \in \overline{\Omega}$ such that

$$((L/_{\ker M})^{-1} \circ (I - Q) \circ H)(x_k, \overline{\lambda})_k \to \overline{x}$$

as k goes to $+\infty$. Therefore

$$||((L/_{\ker M})^{-1} \circ (I - Q) \circ H)(x_k, \lambda_k) - \overline{x}|| \le$$

$$\le ||((L/_{\ker M})^{-1} \circ (I - Q) \circ H)(x_k, \lambda_k) - ((L/_{\ker M})^{-1} \circ (I - Q) \circ H)(x_k, \overline{\lambda})|| +$$

$$+ ||((L/_{\ker M})^{-1} \circ (I - Q) \circ H)(x_k, \overline{\lambda}) - \overline{x}|| \to 0$$

when $k \to +\infty$, because $((L/_{\ker M})^{-1} \circ (I - Q) \circ H)(x, \cdot)$ is uniformily continuous with respect to x in Ω and the proof is complete. ∎

Example 3.5 For every $i, j = 1, 2, k = 1, ..., 4$, let c_{ik}^j be a function of $C_H(T)$ and consider the complex second order polynomials in two variables

$$p_i(t, u, v) = c_{i1}^2(t)u^2 + c_{i2}^2(t)v^2 + c_{i3}(t)uv + c_{i1}^1(t)u + c_{i2}^1(t)v + c_{i4}(t).$$

Consider now the homotopy

$$H : C_H(T)^2 \times [0, 1] \to C_H(T)^2$$

defined by

$$H(u, v, \lambda)(t) =$$

$$= \left((1 - \lambda)\overline{p_1(u, v)} + \lambda p_1(t, u(t), v(t)), (1 - \lambda)\overline{p_2(u, v)} + \lambda p_2(t, u(t), v(t)) \right).$$

Recalling the function L of the Example 3.2, let us show that H is L-compact on the closure of every open and bounded subset $\Omega_1 \times \Omega_2$ of $C_H^1(T)^2$.
The continuous projections of L are $M(u, v) = Q(u, v) \equiv (\overline{u}, \overline{v})$ and

$$(L/_{\ker M})^{-1}(u, v)(t) = \left(\int_0^t u(s)ds, \int_0^t v(s)ds \right).$$

Therefore

$$((L/_{\ker M})^{-1} \circ (I - Q) \circ H(u, v, \lambda))(t) =$$

$$= \left((1 - \lambda)\overline{p_1(u, v)}t + \lambda \int_0^t p_1(s, u(s), v(s))ds - t\overline{p_1(u, v)}, \right.$$

$$\left. (1 - \lambda)\overline{p_2(u, v)}t + \lambda \int_0^t p_2(s, u(s), v(s))ds - t\overline{p_2(u, v)} \right) =$$

$$= \lambda \left(\int_0^t p_1(s, u(s), v(s))ds, -t\overline{p_1(u, v)}, \int_0^t p_2(s, u(s), v(s))ds, -t\overline{p_2(u, v)} \right),$$

because, since $(1 - \lambda)\overline{p_i(u,v)}$ is a constant for each $i = 1, 2, \overline{H(u,v,\lambda)} = (\overline{p_1(u,v)},$
$\overline{p_2(u,v)}$. Hence

$$||((L/_{\ker M})^{-1} \circ (I - Q) \circ H(u,\lambda))|| \leq 2T\rho$$

where $\rho = \max\limits_{i=1,2} \max\limits_{\overline{\Omega_1} \times \overline{\Omega_2}} ||p_i(u,v)||_0$.

Moreover

$$((L/_{\ker M})^{-1} \circ (I - Q) \circ H(u,\lambda))'(t) =$$

$$= \lambda\Big(p_1(t,u(t),v(t)) - \overline{p_1(u,v)}, p_2(t,u(t),v(t)) - \overline{p_2(u,v)}\Big)$$

which implies

$$||((L/_{\ker M})^{-1} \circ (I - Q) \circ H(u,v,\lambda))'|| \leq 2\rho$$

and the compactness follows from the Ascoli-Arzelá theorem.
In the same way one gets that

$$||((L/_{\ker M})^{-1} \circ (I - Q) \circ H(u,v,\lambda_1)) - ((L/_{\ker M})^{-1} \circ (I - Q) \circ H(u,v,\lambda_2))|| \leq$$

$$\leq 2T\rho|\lambda_1 - \lambda_2|$$

and also the uniform continuity is proved.
Then H is $L-$compact in Ω by Theorem 3.4.

It is well known that, associated to every Fredholm map of index zero, there is a topological degree theory that gives a measure of the set of zeros of the difference between the map and a compact function in a fixed subset of the domain. In the follow, given such a map $L : \text{dom } L \subset X \to Z$, an open and bounded subset Ω of X and a L-compact function $H : \overline{\Omega} \to Z$,

$$D_L(L - H, \Omega)$$

will denote the topological degree of H with respect to Ω and L.

Theorem 3.6. *Let X,Z be normed spaces, L: dom $L \subset X \to Z$ a Fredholm map of index zero and Ω an open and bounded subset of X. If*

1) *$H : \overline{\Omega} \times [0,1] \to Z$ is L-compact,*

2) *$H(u,\lambda) \neq Lu$ for every (u,λ) in dom $L \cap \partial\Omega \times$ [0,1),*

3) *$D_L(L - H(\cdot,0),\Omega) \neq 0$,*

then there exists $u \in \text{dom } L \cap \overline{\Omega}$ such that $H(u,1) = Lu$.

Proof. If there exists $u \in \text{dom } L \cap \partial\Omega$ such that $H(u,1) = Lu$, then the theorem is proved.
Otherwise $H(u,\lambda) \neq Lu \forall (u,\lambda) \in \text{dom } L \cap \partial\Omega \times [0,1]$, therefore

$$D_L(L - H(\cdot,1),\Omega) = D_L(L - H(\cdot,0),\Omega),$$

because of the homotopy invariance property of the degree.
Now, by hypothesis, the latter quantity is different from zero, hence there exists $u \in \text{dom } L \cap \Omega$ such that $H(u,1) = Lu$, because of the existence property of the degree. ∎

4 Existence of periodic solutions

Given a real number $T > 0$, let $C_H(T)$ and $C_H^1(T)$ be the normed spaces defined in section 2. For every $i, j = 1, 2$ and $k = 1, ..., 4$ we consider a function c_{ik}^j of $C_H(T)$. In this section we will give sufficient conditions in order that the system

$$\begin{cases} u' = p_1(t, u, v) \\ v' = p_2(t, u, v) \\ u, v \in C_H^1(T) \end{cases} \tag{P*}$$

has a solution, where p_1 and p_2 are the second order polynomials defined by

$$p_i : \mathbb{R} \times C_H(T)^2 \to C_H(T)^2$$
$$(t, u, v) \to c_{i1}^2(t)u^2 + c_{i2}^2(t)v^2 + c_{i3}(t)uv + c_{i1}^1(t)u + c_{i2}^1(t)v + c_{i4}(t).$$

Of course we suppose that p_1 and p_2 are different one from the other and that for every $i = 1, 2$ at least one among c_{i1}^2, c_{i2}^2 and c_{i3} is not identically equal to zero, otherwise p_i would be a first order polynomial. It is evident from the way $C_H(T)$ has been defined that the solutions of (P*) are also T-periodic solutions of

$$\begin{cases} u' = p_1(t, u, v) \\ v' = p_2(t, u, v). \end{cases} \tag{P}$$

Remark 4.1 When $p_1 \equiv p_2$, then $u' = v'$, therefore there exists a constant $z_0 \in \mathbb{C}$ such that $v = u + z_0$ and $u' = p_1(t, u, u + z_0)$. Since p_1 is a second order complex polynomial in two variables, the polynomial

$$p : \mathbb{R} \times C_H(T) \to C_H(T)$$
$$u \to p_1(t, u, u + z_0)$$

is a second order complex polynomial in one variable, e.g. u must satisfy an equation of the following kind

$$u' = c_2(t)u^2 + c_1(t)u + c_0(t).$$

We remind to Remark 4.3 for sufficient conditions for the existence of a solution of this kind of equations.

Now we need to introduce some notations that will be used in the sequel of the section. $\overline{p}$ will denote the function

$$\overline{p} : \mathbb{C}^2 \to \mathbb{C}^2$$
$$(z_1, z_2) \to (\overline{p_1}(z_1, z_2) \overline{p_2}(z_1, z_2)),$$

where

$$\overline{p_i} : \mathbb{C}^2 \to \mathbb{C}$$
$$(z_1, z_2) \to \overline{c_{i1}^2} z_1^2 + \overline{c_{i2}^2} z_2^2 + \overline{c_{i3}} z_1 z_2 + \overline{c_{i1}^1} z_1 + \overline{c_{i2}^1} z_2 + \overline{c_{i4}}$$

for every $i = 1, 2$. By Prop. 2.2 it follows that $\overline{p_i(u, v)} = \overline{p_i}(\overline{u}, \overline{v})$. Moreover, since both $\overline{p_1}$ and $\overline{p_2}$ are second order polynomials in two variables, the system

$$\begin{cases} \overline{p_1}(z_1, z_2) = 0 \\ \overline{p_2}(z_1, z_2) = 0 \end{cases} \tag{$\overline{P}$}$$

has at most 4 solutions.

Calling $(z_{11}, z_{12}), ..., (z_{s1}, z_{s2})$ the distinct zeros of $(\overline{P})$, we define

$$M = \max_{i=1,2} \max_{1 \le k \le s} |z_{ki}|.$$

Finally p_i^0 will denote the real polynomial

$$p_i^0 : \mathbb{R}^2 \to \mathbb{R}$$
$$(x, y) \to ||c_{i1}^2||_0 x^2 + ||c_{i2}^2||_0 y^2 + ||c_{i3}||_0 xy + ||c_{i1}^1||_0 x + ||c_{i2}^1||_0 y + ||c_{i4}||_0.$$

Theorem 4.2.   *Let $(z_{11}, z_{12}), ..., (z_{s1}, z_{s2})$ be the distinct zeros of the system of complex polynomials*

$$\begin{cases} \overline{p_1}(z_1, z_2) = 0 \\ \overline{p_2}(z_1, z_2) = 0 \end{cases} \tag{$\overline{P}$}$$

and $M = \max_{i=1,2} \max_{1 \le k \le s} |z_{ki}|$. If there exists $R > M$ such that the system

$$\begin{cases} M + \frac{T}{2} p_i^0(x, y) - x \ge 0 \\ M + \frac{T}{2} p_2^0(x, y) - y \ge 0 \end{cases} \tag{P^0}$$

has no solutions in $[0, R] \times \{R\} \cup \{R\} \times [0, R]$, then the system

$$\begin{cases} u' = c_{11}^2(t)u^2 + c_{12}^2(t)v^2 + c_{13}(t)uv + c_{11}^1(t)u + c_{12}^1(t)v + c_{14}(t) \\ v' = c_{21}^2(t)u^2 + c_{22}^2(t)v^2 + c_{23}(t)uv + c_{21}^1(t)u + c_{22}^1(t)v + c_{24}(t) \end{cases} \tag{P}$$

has at least s distinct solutions.

Proof. Since $(z_{11}, z_{12}), ..., (z_{s1}, z_{s2})$ are distinct, take $\delta > 0$ such that $\overline{D_{z_k}^\delta} \cap \overline{D_{z_h}^\delta} = \emptyset$ for each $k, h = 1, ..., s$ with $k \ne h$, where

$$D_{z_k}^\delta = \{(z_1, z_2) \in \mathbb{C}^2 : \max\{|z_1 - z_{k1}|, |z_2 - z_{k2}|\} < \delta\}.$$

Consider the map L and the homotopy H of Examples 3.2 and 3.5. Recalling the properties therein proved, it is now sufficient to show that for every $k = 1, ..., s$ the assumptions 2) and 3) of Theorem 3.6 are satisfied in the open and bounded subset of $C_H(T)$

$$\Omega_k = \{(u, v) \in C_H(T)^2 : \max\{||u||_0, ||v||_0\} < R, \max\{|\overline{u} - z_{k1}|, |\overline{v} - z_{k2}\} < \delta\}$$

and the existence of s distinct solutions of (P) belonging to $C_H^1(T)^2$ will follow from the void intersection of each couple of the sets written above.

We point out that $\Omega_k \ne \emptyset$, because $(u \equiv z_{k1}, v \equiv z_{k2}) \in \Omega_k \ \forall \, k = 1, ..., s$.

Fixing $k \in \{1, ..., s\}$, the boundary of Ω_k is the set

$$\partial\Omega_k = \{(u,v) \in C_H(T)^2 : \max\{||u||_0, ||v_0||\} \leq R, \max\{|\overline{u} - z_{k1}|, |\overline{v} - z_{k2}|\} = \delta\} \cup$$

$$\cup \{(u,v) \in C_H(T)^2 : \max\{||u||_0, ||v||_0\} = R, \max\{|\overline{u} - z_{k1}|, |\overline{v} - z_{k2}|\} < \delta\}.$$

Given $\lambda \in [0, 1)$, take a solution (u, v) of $L(u, v) = H(u, v, \lambda)$ belonging to $\partial\Omega_k$. Then $\overline{u'} = \overline{v'} = 0$, because u and v are T-periodic. Recalling the properties of the polynomials $\overline{p_i}$,

$$(\overline{u'}, \overline{v'}) = (\overline{p_1(u,v)}, \overline{p_2(u,v)}) = \overline{p}(\overline{u}, \overline{v}),$$

hence $(\overline{u}, \overline{v})$ is a solution of $(\overline{P})$, which, by the choice of δ, implies $\overline{u} = z_{k1}$ and $\overline{v} = z_{k2}$. Therefore $L(u, v) = H(u, v, \lambda)$ has no solutions in

$$\{(u,v) \in C_H(T)^2 : \max\{||u||_0, ||v||_0\} \leq R, \max\{|\overline{u} - z_{k1}|, |\overline{v} - z_{k2}|\} = \delta\}.$$

On the other side, like proved in [1], Corollary 2.12, since u and v are T-periodic, it holds

$$||u||_0 \leq \overline{u} + \frac{T}{2}||u'||_0 \leq M + \frac{T}{2}||p_1(t, u, v)||_0$$

and likewise

$$||v||_0 \leq M + \frac{T}{2}||p_2(t, u, v)||_0,$$

that is to say that $(||u||_0, ||v||_0)$ is a solution of (P^0). Hence, by hypothesis, $L(u, v) = = H(u, v, \lambda)$ cannot have solutions neither in

$$\{(u,v) \in C_H(T)^2 : \max\{||u||_0, ||v||_0\} = R, \max\{|\overline{u} - z_{k1}|, |\overline{v} - z_{k2}|\} < \delta\}$$

and assumption 2) is proved.

By definition $H(u, v, 0) = (\overline{p_1(u,v)}, \overline{p_2(u,v)}) = \overline{p}(\overline{u}, \overline{v})$, therefore, by [6], Theorem 3.1,

$$D_L(L - H(\cdot, 0), \Omega_k) = \deg(\overline{p}, D_{z_k}^\delta, 0),$$

where deg represents the Brouwer's degree. The only zero of $\overline{p}$ in $\overline{D_{z_k}^\delta}$ is (z_{k1}, z_{k2}) and this implies that

$$\deg(\overline{p}, D_{z_k}^\delta, 0) \neq 0,$$

because it is the multiplicity of (z_{k1}, z_{k2}) as zero of $\overline{p}$ (see [9] for a detailed analysis of the Brouwer's degree of a map defined in $\mathbb{R}^n$) and also 3) is proved. ∎

Remark 4.3 When (P) reduces to a second order equation of one variable, e.g. to the following system

$$\begin{cases} u' = c_{11}^2(t)u^2 + c_{11}^1(t)u + c_{14}(t) \\ v' = 0, \end{cases}$$

Theorem 4.2 can not apply, because $(\overline{P})$ has infinitely many solutions given by a line or the union of two parallel lines of the complex plane.
However if we reduce $(\overline{P})$ only to its first equation

$$\overline{c_{11}^2}z_1^2 + \overline{c_{11}^1}z_1 + \overline{c_{14}} = 0 \tag{$\overline{E}$}$$

and we consider M equal to the maximum of the module of its zeros, there will be $R > M$ such that

$$\begin{cases} M + \frac{T}{2}||c_{11}^2||_0 x^2 + \frac{T}{2}||c_{11}^1||_0 x + \frac{T}{2}||c_{14}||_0 - x \geq 0 \\ M - y \geq 0 \end{cases}$$

has no solutions in $[0, R] \times \{R\} \cup \{R\} \times [0, R]$ if and only if there exists $R > M$ such that

$$M + \frac{T}{2}||c_{11}^2||_0 R^2 + \frac{T}{2}||c_{11}^1||_0 R + \frac{T}{2}||c_{14}||_0 - R < 0,$$

which is the same necessary and sufficient conditions obtained in [2], Theorem 1.1, in order that

$$u' = c_{11}^2(t)u^2 + c_{11}^1(t)u + c_{14}(t)$$

has at least the number of solutions equal to the number of distinct zeros of $(\overline{E})$.

We now apply Theorem 4.2 to two generalizations of the complex periodic Riccati equation.

Example 4.4 Let g and h be T-periodic continuous functions from $\mathbb{R}$ into $\mathbb{C}$. Suppose moreover that they are developable in Fourier series with all the coefficients corresponding to negative indexes equal to zero and consider the following system of planar equations

$$\begin{cases} u' = u^2 + v + g(t) \\ v' = v^2 + u + h(t), \end{cases} \tag{P_1}$$

which corresponds to (P) when $c_{11}^2 = c_{12}^1 = c_{22}^2 = c_{21}^1 \equiv 1, c_{14} = g, c_{24} = h$ and all the other coefficients are identically equal to zero.

Then the system corresponding to the mean values becomes

$$\begin{cases} z_1^2 + z_2 + \overline{g} = 0 \\ z_2^2 + z_1 + \overline{h} = 0, \end{cases} \tag{$\overline{P_1}$}$$

which has at most 4 solutions, because at every z_2 solution of

$$z_2^4 + 2\overline{h}z_2^2 + z_2 + \overline{h}^2 + \overline{g} = 0$$

corresponds only one z_1 such that

$$z_1 = -z_2^2 - \overline{h}.$$

Calling M the maximum of the module of the solutions of $(\overline{P_1})$, to apply Theorem 4.2 it is sufficient to find a positive constant R bigger than M such that

$$\begin{cases} \frac{T}{2}x^2 - x + \frac{T}{2}y + \frac{T}{2}||g||_0 + M \geq 0 \\ \frac{T}{2}y^2 - y + \frac{T}{2}x + \frac{T}{2}||h||_0 + M \geq 0 \end{cases} \tag{P_1^0}$$

has no solutions in $[0, R] \times \{R\} \cup \{R\} \times [0, R]$.

The solutions of the first inequality of (P_1^0) are all the points of the plane which stand outside the parabola of equation $y = -x^2 + \frac{2}{T}x - ||g||_0 - \frac{2M}{T}$. It is easy to

verify that its vertex $V_1 = (\frac{1}{T}, \frac{1}{T^2} - ||g||_0 - \frac{2M}{T})$ belongs to the part of the first quarter of plane above the line $y = x$ if and only if

$$T < \frac{2}{\sqrt{1 + 4M + 4M^2 + 8||g||_0} + 1 + 2M}.$$

Likewise the solutions of the second one belong to the part of the plane situated outside of the parabola $x = -y^2 + \frac{2}{T}y - ||h||_0 - \frac{2M}{T}$, whose vertex $V_2 = (\frac{1}{T^2} - ||h||_0 - \frac{2M}{T}, \frac{1}{T})$ belongs to the part of the first quarter of plane below that line if and only if

$$T < \frac{2}{\sqrt{1 + 4M + 4M^2 + 8||h||_0} + 1 + 2M}.$$

Therefore if

$$T < \frac{2}{\sqrt{1 + 4M + 4M^2 + 8\max\{||g||_0, ||h||_0\}} + 1 + 2M},$$

one has that the segment

$$\left\{\frac{1}{T}\right\} \times \left[0, \frac{1}{T^2} - ||g||_0 - \frac{2M}{T}\right)$$

does not verifies the first inequality, while

$$\left[0, \frac{1}{T^2} - ||h||_0 - \frac{2M}{T}\right) \times \left\{\frac{1}{T}\right\}$$

does not satisfies the second. Hence the existence of a number of solutions of (P_1) at least equal to the number of distinct zeros of $(\overline{P_1})$ follows choosing

$$R = \frac{1}{T} > \frac{1}{2}\sqrt{1 + 4M + 4M^2 + 8\max\{||g||_0, ||h||_0\}} + \frac{1}{2} + M > M.$$

Example 4.5 Let g and h be as in the previous example and consider the system

$$\begin{cases} u' = v^2 + g(t) \\ v' = u^2 + h(t), \end{cases} \tag{P_2}$$

i.e. consider (P) with all the coefficients identically equal to zero, but $c_{12}^2 = c_{21}^2 \equiv 1$, $c_{14} = g$ and $c_{24} = h$.
Then we have

$$\begin{cases} z_2^2 + \overline{g} = 0 \\ z_1^2 + \overline{h} = 0. \end{cases} \tag{$\overline{P_2}$}$$

Set $M = \max\left\{\sqrt{|\overline{g}|}, \sqrt{|\overline{h}|}\right\}$, the system corresponding to the C^0 norms becomes

$$\begin{cases} \frac{T}{2}y^2 - x + \frac{T}{2}||g||_0 + M \geq 0 \\ \frac{T}{2}x^2 - y + \frac{T}{2}||h||_0 + M \geq 0. \end{cases} \tag{P_2^0}$$

The tangent to $x = \frac{T}{2}y^2 + \frac{T}{2}||g||_0 + M$ in $(\frac{1}{2T} + \frac{T}{2}||g||_0 + M, \frac{1}{T})$ is $y = x + \frac{1}{2T} - \frac{T}{2}||g||_0 - M$ that lays in the upper half-plane determined by $y = x$ if and only if

$$T < \frac{1}{\sqrt{M^2 + ||g||_0} + M}.$$

Likewise the tangent to $y = \frac{T}{2}x^2 + \frac{T}{2}||h||_0 + M$ in $(\frac{1}{T}, \frac{1}{2T} + \frac{T}{2}||h||_0 + M)$ is $y = x - \frac{1}{2T} + \frac{T}{2}||h||_0 + M$ that lays in the lower half-plane determined by $y = x$ if and only if

$$T < \frac{1}{\sqrt{M^2 + ||h||_0} + M}.$$

Therefore if

$$T < \frac{1}{\sqrt{M^2 + \max\{||g||_0, ||h||_0\}} + M}$$

it follows that $\left\{\frac{1}{T}\right\} \times \left[0, \frac{1}{T}\right]$ does not satisfy the first inequality, while $\left[0, \frac{1}{T}\right] \times \left\{\frac{1}{T}\right\}$ does not verify the second one.

Theorem 4.2 then applies choosing

$$R = \frac{1}{T} > \sqrt{M^2 + \max\{||g||_0, ||h||_0\}} + M > M.$$

We point out that the method described in this paper can be applied in the same way to get existence results for any system of complex polynomial equations with the order of the polynomial bigger or equal than 2. We prefered not to consider the very general case in this context, because the proof would have been made heavy by the computations, while our aim was to give prominence to the method.

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