

Retracting spreads

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Abstract

The concept of spread-retraction is introduced by which certain spreads in $PG(4m+1, q)$ or $PG(4m-1, q)$ may be ‘retracted’ to either Baer subgeometry partitions of $PG(2m, q^2)$ or to mixed partitions of $PG(2m-1, q^2)$ respectively. This characterizes the spreads produced by such partitions abstractly and furthermore allows a vast number of new mixed partitions to be recognized.

1 Introduction

In this article, we shall be discussing partitions of the points of finite projective geometries Σ over $GF(q^2)$. When Σ is isomorphic to $PG(2m, q^2)$, the partition components are Baer subgeometries isomorphic to $PG(2m, q)$. When Σ is isomorphic to $PG(2n-1, q^2)$, it is possible to have a so-called ‘mixed’ partition of β $PG(n-1, q^2)$'s and α $PG(2n-1, q)$'s. The configuration is such that $\alpha(q+1) + \beta = q^{2n} + 1$.

The interest in such partitions lies in the fact they may be used to construct spreads and hence translation planes. Baer subgeometry partitions produce translation planes of order q^{2m+1} where mixed partitions produce translation planes of order q^{2n} . These constructions are applications of the theory of Segré varieties and are given in Hirschfeld and Thas [8], p. 206. In particular, all Baer subgeometries produce $GF(q)$ -reguli in the associated spread. When the partition is a Baer subgeometry partition, the spread is a union of mutually disjoint $GF(q)$ -reguli. Furthermore, mixed partitions of $PG(2m-1, q^2)$ by $PG(m-1, q^2)$'s and $PG(2m-1, q)$'s produce spreads in $PG(4m-1, q)$ which contain d $GF(q)$ -reguli provided there are d $PG(2m-1, q)$'s in the mixed partition.

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Recently, there has been considerable work by Baker, Dover, Ebert and Wantz (see [1] and [2]) on flag-transitive translation planes using Baer subgeometry partitions of $PG(2m, q^2)$; partitions of $PG(2m, q^2)$ by $PG(2m, q)$'s. Such Baer subgeometry partitions produce spreads in $PG(4m + 1, q)$ which are unions of $GF(q)$ -reguli. One of their main results shows that any flag-transitive translation plane of order q^{2m+1} with spread in $PG(4m + 1, q)$ admitting a collineation group G such that acting on the spread is cyclic and regular must come from a Baer subgeometry partition of $PG(2m, q^2)$.

Until recently, there were no known non-trivial mixed partitions. In Mellinger's thesis [15], there are several new interesting infinite classes. This article came about by trying to understand what sort of spreads Mellinger's mixed partitions produce and the ideas had their primitive conception during a visit to University of Delaware in May of 2000.

One comes then naturally to the problem of deciding abstractly what translation planes these partitions can produce. Hence, in this article, we give a 'retraction' procedure which produces partitions of projective geometries. We then show that this retraction procedure is equivalent to the construction method of Hirschfeld and Thas. Mellinger has been calling this process of the construction of a spread from the projective partition 'lifting'. There is an algebraic process which we review in this article also called lifting where a spread in $PG(3, q)$ is 'lifted' to a spread in $PG(3, q^2)$. To distinguish between these two concepts, in this article, we shall call these processes 'algebraic lifting' and 'geometric lifting' which refer to the lifting of a spread to a related spread and the lifting of a projective partition respectively.

The fundamental theorem allows an identification of known various spreads satisfying the requirements for retraction which then produce a variety of new mixed partitions.¹

2 Spread-Retraction.

We begin by describing spreads which 'retract' suitably to partitions of projective geometries.

Theorem 1. *Let π be a translation plane with spread in $PG(4m - 1, q)$. Suppose the associated vector space may be written over a field K isomorphic to $GF(q^2)$ which extends the indicated field $GF(q)$ as a $2m$ -dimensional K -vector space.*

If the scalar mappings with respect to K over V_{2m}/K act as collineations of π , assume that the orbit lengths of components are either 1 or $q + 1$ under the scalar group of order $q^2 - 1$.

Let δ denote the number of components of orbit length 1 and let $(q + 1)d$ denote the number of components of orbit length $q + 1$.

Then there is a mixed partition of $PG(2m - 1, q^2)$ of δ $PG(m - 1, q^2)$'s and d $PG(2m - 1, q)$'s.

Definition 2. *Under the above conditions, we shall say that the mixed partition of $PG(2m - 1, q^2)$ is a 'retraction' of the spread of π or a 'spread-retraction'.*

¹The author is indebted to Keith Mellinger and Gary Ebert of the University of Delaware both for the introduction to this subject and for making their work available prior to publication.

Theorem 3. *Let π be a translation plane of order q^{2m+1} with kernel containing $GF(q)$, with spread in $PG(4m+1, q)$, whose underlying vector space is a $GF(q^2)$ -space and which admits as a collineation group the scalar group of order $q^2 - 1$. If all orbits of components have length $q + 1$ corresponding to $K - \{0\}$, then a Baer subgeometry partition of $PG(2m, q^2)$ may be constructed.*

Definition 4. *A Baer subgeometry partition produced from a spread as above is called a ‘spread-retraction’.*

Proof. The main idea of the proof is the following two points: (1) Any orbit of size $q + 1$ of $GF(q)$ -components under the scalar group forms a $GF(q)$ -regulus whose subplanes of order q incident with the zero vector are $GF(q^2)$ -1-dimensional vector subspaces.

To see why (1) is valid, notice that we have subspaces of dimension q^k (for $k = 2m + 1$) that are actually covered by the point orbits under $GF(q^2)^*$. Each point orbit union the zero vector is isomorphic to $GF(q^2)$ which is a 2-dimensional $GF(q)$ -vector space. Since this vector space is decomposed naturally into $q + 1$ $GF(q)$ -spaces lying on components of the orbit of length $q + 1$, we obtain a spread of $GF(q^2)$. Hence, each point orbit within an orbit of components of length $q + 1$ is a Desarguesian affine subplane. Hence, we have a subplane covered net by Desarguesian subplanes of order q ; a $GF(q)$ -regulus net. (Note that this argument is independent of the value for k).

(2) If R is a regulus net of order q^{2m} and degree $q + 1$ then taking the subplanes incident with the zero vector as ‘points’, the structure becomes a projective geometry $PG(2m - 1, q)$.

To see this, use e.g. Liebler [13], Theorem (1.4).

Then starting from a vector space V_{2m} over $GF(q^2)$, to form the projective space by taking the lattice of $GF(q^2)$ -subspaces, it follows that the orbits of $GF(q)$ -subspaces under the group $GF(q^2)^*$ become $PG(2m - 1, q)$'s and the components of the spread which are $GF(q^2)$ -subspaces become natural $PG(m - 1, q^2)$'s.

Assume that we have a spread S of order q^{2m+1} with kernel containing $GF(q)$ and a corresponding translation plane π . Then π is a $2(2m + 1)$ -vector space over $GF(q)$. Assume that there is a field K isomorphic to $GF(q^2)$ such that π is also a $(2m + 1)$ - K -vector space and assume that the scalar group of order $q^2 - 1$ relative to K acts as a collineation group of S . Assume that each orbit of components has length $q + 1$.

We again note: Each orbit of length $q + 1$ is a $GF(q)$ -regulus and the subplanes incident with the zero vector of the regulus net are K -1-dimensional vector subspaces and each orbit of components corresponds in the projective geometry $PG(2m, K)$ to a $PG(2m, q)$.

This completes the proof. ■

3 Geometric Lifting.

In this section, we give a vector-based version of geometric lifting which ultimately will show that spread-retraction and geometric lifting are equivalent.

We begin with the embedding fundamentals associated with the construction process as developed in Hirschfeld and Thas [8], p. 206.

Consider Π isomorphic to $PG(4n-1, q)$ or $PG(4m+1, q)$ respectively and embed Π in Ω and isomorphic to $PG(4n-1, q^2)$ or $PG(4m+1, q^2)$ respectively. Choose a projective subspace Σ disjoint from Π in Ω where Σ is isomorphic to either $PG(2n-1, q^2)$ or $PG(2m, q^2)$ respectively as Ω is isomorphic to $PG(4n-1, q^2)$ or $PG(4m+1, q^2)$. For v a point in Ω , for any vector basis and $v = (x_i)$ where $x_i \in GF(q^2)$, define $v^q = (x_i^q)$.

We set-up with Π the set of points of Ω each of which is fixed under the q -mapping above. For more details, the reader is referred to Mellinger [15].

Lemma 5. (1) For any point v in Σ , $v^q \notin \Sigma$ and $\langle v, v^q \rangle \cap \Sigma$ is a Baer subline of $q+1$ points which may be taken in the form:

$$\{kv + v^q; k \in GF(q^2), |k| \text{ dividing } q+1\}.$$

(2) The set of points of Π is

$$\{sv + v^q; v \in \Sigma, s \text{ of order dividing } q+1\}.$$

Proof. Part (1), in the mixed case when Σ is $PG(2n-1, q^2)$ is in Mellinger [15]. The proof to the general case is virtually identical completing part (1).

To see that (2), is valid, note that $sv + v^q = tw + w^q$ if and only if $s = t$ and $v = w$ and then $(q+1)(q^{2(2n)} - 1)/(q^2 - 1) = (q^{4n} - 1)/(q - 1)$ and $(q+1)(q^{2(2m+1)} - 1)/(q^2 - 1) = (q^{4m+2} - 1)/(q - 1)$ which are the number of points in $PG(4n-1, q)$ and $PG(4m+1, q)$ respectively. ■

Lemma 6. let s, t be in the group C_{q+1} of order $q+1$ of $GF(q^2)^*$. Let $\theta_{s,t}$ be an element such that $\theta_{s,t}^q(s/t) = \theta_{s,t}$. Without loss of generality, $\theta_{s,t} = \theta_{ks,kt}$ for $k \in C_{q+1}$.

Proof. There exist exactly $q-1$ nonzero elements m such that $m^{q-1} = s/t$ and there exist exactly $q-1$ nonzero elements n such that $n^{q-1} = kt/ks = t/s$. ■

Lemma 7. Let $sv + v^q$ and $tw + w^q$ be distinct points of Π and let $L_{(s,v),(t,w)}$ denote the unique line of Π incident with these two points.

Then

$$L_{(s,v),(t,w)} = \{s(v + \theta_{s,t}^q w) + (v + \theta_{s,t}^q w)^q; \forall \theta_{s,t}\} \cup \{sv + v^q, tw + w^q\}.$$

The $GF(q^2)$ -vector space generated by the two ‘vectors’ is

$$\langle sv + v^q, tw + w^q \rangle = l(sv + v^q) + m(tw + w^q) \forall l, m \in GF(q^2).$$

Note that

$$s(v + \theta_{s,t}^q w) + (v + \theta_{s,t}^q w)^q = sv + v^q + \theta_{s,t}(tw + w^q)$$

since $\theta_{s,t}^q = \theta_{s,t}t/s$. Hence, the indicated points are $q+1$ points of Π and are on the line generated by $sv + v^q$ and $tw + w^q$.

Theorem 8. Define $\sigma_k : sv + v^q \mapsto ksv + v^q$ for $k \in C_{q+1}$. Then σ_k is a collineation of Π .

Proof. It suffices to check that σ_k is a bijection on the points which maps lines to lines and preserves incidence.

It is claimed that $L_{(s,v),(t,w)}$ is mapped onto $L_{(ks,v),(kt,w)}$ by σ_k . We note that

$$L_{(ks,v),(kt,w)} = \left\{ ks(v + \theta_{ks,kt}^q w) + (v + \theta_{ks,kt}^q w)^q; \forall \theta_{ks,kt} \right\} \cup \{ksv + v^q, ktw + w^q\}.$$

Since $\theta_{ks,kt} = \theta_{s,t}$, it follows that

$$L_{(ks,v),(kt,w)} = L_{(s,v),(t,w)}\sigma_k. \quad \blacksquare$$

Lemma 9. Let B be a $PG(2n - 1, q)$ or a $PG(2m, q)$ of Σ respectively as Σ is $PG(2n - 1, q^2)$ or $PG(2m, q^2)$.

(1) Then for each $s \in C_{q+1}$,

$$B_s^+ = \{sv + v^q; v \in B\}$$

is a subspace of Π isomorphic to $PG(2n - 1, q)$ or $PG(2m + 1, q)$ respectively. Furthermore, $B_s^+\sigma_k = B_{ks}^+$. Thus, $\{B_t^+; t \in C_{q+1}\}$ is an orbit under $\langle \sigma_k; k \in C_{q+1} \rangle$.

(2) For a subgeometry S isomorphic to $PG(n - 1, q^2)$ when Σ is $PG(2n - 1, q^2)$, then

$$S^+ = \{sv + v^q; v \in S, \text{ for all } s \in C_{q+1}\}$$

is a subspace of Π is isomorphic to $PG(2n - 1, q)$.

Furthermore, S^+ is invariant under $\langle \sigma_k; k \in C_{q+1} \rangle$.

Proof. Note that the lines $L_{(s,v),(s,w)}$ have points of Π of the form:

$$L_{(s,v),(s,w)} = \left\{ s(v + \theta_{s,s}^q w) + (v + \theta_{s,s}^q w)^q; \forall \theta_{s,s} \right\} \cup \{sv + v^q, sw + w^q\}.$$

Thus, we see that

$$s(v + \theta_{s,s}^q w) + (v + \theta_{s,s}^q w)^q = sv + v^q + \theta_{s,s}(sw + w^q)$$

which is $s(v + \theta_{s,s}w) + (v + \theta_{s,s}w)^q$. Also, v and $w \in B$ implies that $v + \theta_{s,s}w \in B$. Hence, all points of the line generated by two points of B_s^+ are points of B_s^+ so that it follows that B_s^+ is a subspace. Since the number of points is the number of points of $PG(2n - 1, q)$ or $PG(2m + 1, q)$, we have the proof.

NOTE: This part also may be deduced using Segre varieties as in Hirschfeld and Thas.

The proof of part (2) is similar and left to the reader. ■

The following lemma is now immediate from our previous lemmas.

Lemma 10. For any partition of the projective geometry, the cyclic group

$$\langle \sigma_k; k \in C_{q+1} \rangle$$

is a collineation group of the projective spread in Π .

We now show the converse and prove:

Theorem 11. (1) Any mixed partition \mathcal{M} of $PG(2n-1, q^2)$ or Baer partition \mathcal{B} of $PG(2m, q^2)$ gives rise to a spread S in $PG(4n-1, q)$ or respectively in $PG(4m+1, q)$ defining a translation plane $\pi_{\mathcal{M}}$ or $\pi_{\mathcal{B}}$ respectively. In either case, the translation plane is a $GF(q^2)$ -vector space which admits the scalar group $GF(q^2)^*$ of order q^2-1 as a group of collineations.

(2) For mixed partitions with α $PG(2n-1, q)$'s and β $PG(m-1, q^2)$'s, there exist α orbits of components under $GF(q^2)^*$ of length $q+1$ each forming a $GF(q)$ -regulus and there exist β components which are fixed by $GF(q^2)^*$.

Proof. By the Fundamental Theorem of Projective Geometry, the preimage of $\langle \sigma_k; k \in C_{q+1} \rangle$ acts on the $GF(q)$ -vector space V and permutes the vector space spread. Hence, the preimage G is a subgroup of $\Gamma L(V, q)$. First assume that $\alpha = 1$. So, we have a cyclic group acting on the line at infinity of order $q+1$ that fixes each component of the spread. Thus, we have a kernel homology group of the translation plane so that the translation plane has kernel containing $GF(q^2)$. Now assume that $\alpha \neq 1$.

Since we know that each $sv + v^q$ for all $s \in C_{q+1}$ is a Baer subline; i.e. isomorphic to $PG(1, q)$, the preimage space is then a Desarguesian plane isomorphic to $GF(q^2)$. In other words, the group G must be $GF(q^2)^*$. In the orbits of length $q+1$, this Desarguesian plane becomes a natural subplane of order q which lies on the orbit of components. That is, we obtain a $GF(q)$ -regulus net from each $q+1$ orbit. Since we now have a group isomorphic to $GF(q^2)^*$, we may define a vector space on V over $GF(q^2)$ by taking $v \cdot g = vg$ for all $g \in GF(q^2)^*$. Since $GF(q^2)^*$ is fixed-point-free and the group action is the scalar action, we obtain a proper vector space admitting the group as maintained with the component orbits of lengths 1 or $q+1$. This completes the proof. ■

4 The Fundamental Theorem.

Theorem 12. *Geometric lifting and spread-retraction are equivalent.*

Furthermore, the partition obtained by spread-retraction geometrically lifts back to the original spread.

Proof. Given a spread which permits retraction, we obtain a partition of the projective space. Conversely, given a partition of the projective space, we have constructed by geometric lifting a spread which permits retraction. The remaining question is whether the translation plane obtained by geometric lifting is isomorphic to the original translation plane.

Suppose there is a spread admitting retraction. To fix the situation, suppose that we have a translation plane with spread in $PG(4n-1, q)$ admitting $GF(q^2)^*$ as a collineation group. Pull back to the vector space V_{2n}/q^2 and consider this as $PG(2n-1, q^2)$. We consider the mapping $v \mapsto kv + v^q$ for all $k \in C_{q+1}$. We have seen that we can make this set into the set of 1-dimensional $GF(q)$ -subspaces of a $GF(q^2)$ -subspace. As a 1-dimensional $GF(q^2)$ -subspace, call the image space W_{2n}/K where K is isomorphic to $GF(q^2)$ and denote the 1-dimensional K -subspace

$kv + v^q$ for all $k \in C_{q+1}$ as \tilde{v} . Hence, we have a mapping from V_{2n}/q^2 onto W_{2n}/K by $\Gamma : v \mapsto \tilde{v}$. This is clearly an isomorphism of $GF(q^2)$ -vector spaces. Now think of V_{2n}/q^2 as V_{4n}/q with distinguished subspaces as components of a spread which are either fixed by $GF(q^2)^*$ or in orbits of length $q + 1$ under $GF(q^2)^*$. It is clear that Γ will map orbits of length 1 to orbits of length 1 and orbits of length $q + 1$ to orbits of length $q + 1$. Hence, the spread $(S^-)^+$ obtained by geometric lifting from the mixed partition obtained by spread-retraction by S is clearly isomorphic to S .

The proof for the Baer subgeometry partition is virtually identical. ■

Using the fundamental theorem, we may also discuss the associated collineation groups and isomorphisms.

Definition 13. *Let π be a translation plane with spread which permits spread-retraction. The collineation group of π which normalizes the scalar group of order $q^2 - 1$ will be called the ‘inherited group’ which we denote by $\mathcal{I}(\pi)$. Note that the inherited group permutes the components fixed by the scalar group (if there are any) and permutes the $GF(q)$ -regulus nets.*

Theorem 14. *Let π be a translation plane with spread which permits spread-retraction. Let $\mathcal{F}_{\text{Proj}}$ denote the full automorphism group in the associated projective group of the constructed partition of the associated projective geometry.*

Then $\mathcal{I}(\pi)/GF(q^2)^ \simeq \mathcal{F}_{\text{Proj}}$.*

The following result now follows essentially immediately. We note the second part of the corollary was proved by Baker et al by different methods.

Corollary 15. (1) *Two mixed partitions of $PG(2m - 1, q^2)$ are projectively equivalent if and only if their geometric lifts are isomorphic.*

(2) *(see also Baker et al [1] (4.2))*

Two Baer subgeometry partitions of $PG(2m, q^2)$ are projectively equivalent if and only if their geometric lifts are isomorphic.

Proof. Clearly two isomorphic translation planes have spreads which retract to projectively equivalent partitions of the associated projective geometry which geometrically lift back to the original or mutually isomorphic spreads. ■

Now we now are able to use the fundamental theorem to provide various new mixed partitions. Before we do this, we show an easy way to construct spreads satisfying the retraction requirement.

5 Net Replacement Version

Theorem 16. *Let π be any spread of order q^{2m} with spread in $PG(2m-1, q^2)$. Let K denote the kernel homology group of order $q^2 - 1$. Suppose we take any replaceable net N by $GF(q)$ -subspaces such that $NK = N$ and such that the replaceable net N^* contains exactly δ $GF(q^2)$ -subspaces and the remaining subspaces are in orbits of length $(q + 1)$ under K . Suppose that the degree of N is $\delta + d(q + 1)$.*

Then there exists a mixed partition of $PG(2m - 1, q^2)$ of d $PG(2m - 1, q)$'s and $q^{2m} + 1 - d(q + 1)$ $PG(m - 1, q^2)$'s.

6 André Constructions

In Mellinger's thesis, there is a construction of a mixed partition with 'many' fixed components under a cyclic group. The constructions in this section were motivated by Mellinger's construction.

Let π be a Desarguesian spread of order q^4 considered as a spread in $PG(3, q^2)$. Hence, the components are lines of $PG(3, q^2)$. Consider an André net A of degree $(q^4 - 1)/(q - 1)$. This net has a replacement net of components $y = x^{q^i}m$ where $y = xm$ is in the original André net. For i odd, the components $y = x^{q^i}m$ are not $GF(q^2)$ -subspaces; i.e. not lines of $PG(3, q^2)$. Now take the André net projectively as a lattice of vector $GF(q^2)$ -subspaces. The question is what happens to the components? The Desarguesian affine plane admits a kernel homology group H of order $q^4 - 1$ which acts transitively on the non-zero points of each component. Furthermore, there are exactly $q^2 + 1$ $GF(q^2)$ -1-subspaces on each component in an orbit under the group and each fixed by the group H^- of order $q^2 - 1$. The group H acts transitively on the $(q^4 - 1)/(q - 1)$ components of the form $y = x^{q^i}m$ and the group H^- permutes these components in orbits of length $(q + 1)$. Hence, when we go to the projective version, the $(q + 1)$ vector subspaces of dimension 4 over $GF(q)$ basically collapse to one such projective space $PG(3, q)$. So, the collapsing or retraction process identifies one $PG(3, q)$ for a set of $q + 1$ of these. The geometric lifting process will construct all of these back by constructing the subplanes (of dimension two) sitting on them.

Lemma 17. *Let A be the André net $y = xm$ such that $m^{(q^4-1)/(q-1)} = 1$. And let $y = x^q m$ for all such m 's denote the replacement set. Let the group G be generated by $(x, y) \mapsto (ex, ey)$ for $|e|$ of order $q^2 - 1$. Then the image sets have the form $\{y = x^q m e^{i(1-q)}\}$, for m fixed, each of which is a set of $q+1$ replacement components.*

- (1) *Each image set forms a $GF(q)$ -regulus*
- (2) *Modulo $GF(q^2)$, each such $GF(q)$ -regulus is a $PG(3, q)$.*

Proof. Without loss of generality, take $m = 1$. Note that for a given element e of order $q^2 - 1$, $\{e\alpha + \beta; \alpha, \beta \in GF(q)\}$ is a field isomorphic to $GF(q^2)$ so since the multiplicative group sits in a field, it follows that $\{e\alpha + \beta; \alpha, \beta \in GF(q)\} = \langle e \rangle \cup \{0\}$. Now $(1, 1) \in (y = x^q)$ so that $(e, e) \in (y = x^q e^{(1-q)})$. The subspace $\langle (1, 1), (e, e) \rangle$ generated over $GF(q)$ is $\{(e\alpha + \beta, e\alpha + \beta); \alpha, \beta \in GF(q)\}$. It then follows that of the image set each subspace intersects this indicated 'line'. It is then easy to see that the set indicated is a $GF(q)$ -regulus. Now each subspace of the set is a 4-dimensional $GF(q)$ -subspace. The group connects all of the 'points' of each such $PG(3, q)$ which corresponds to the $GF(q^2)$ -lattice. That is, the set of images of the group corresponds to a particular $PG(3, q)$ in $PG(3, q^2)$. ■

We note that we could have accomplished the same result by subspaces $y = x^{q^3}m$ instead of $y = x^q m$ since the images of $y = x^{q^i}$ under the indicated group are of the form $y = x^{q^i} e^{j(1-q^i)}$. Hence, if $(1 - q^i, 1 - q^2) = 1 - q^{(i,2)} = q - 1$, the previous proof applies. We call such André replacements, the 'odd' André replacements.

Hence, we obtain the following result:

Theorem 18. *Every Desarguesian plane of order q^4 produces, from each of its ‘odd’ André replacements, a mixed partition of $(q^2 + 1)$ $PG(3, q)$ ’s of $PG(3, q^2)$ and the remaining $q^4 + 1 - (q^2 + 1)(q + 1)$ lines of $PG(3, q^2)$.*

Since we may multiply André replace, replacing any subset of $q - 1$ André nets each in any of three possible ways, we obtain a variety of mixed partitions. In particular, replacing say $\lambda \leq q - 1$ André nets, we obtain

Corollary 19. *Replacement of λ André nets by ‘odd’ replacement produces a mixed partition of $\lambda(q^2 + 1)$ $PG(3, q)$ ’s and $q^4 + 1 - \lambda(q^2 + 1)(q + 1)$ lines of $PG(3, q^2)$.*

This generalizes as follows:

Theorem 20. *Let π be a Desarguesian plane of order q^{2n} with spread in $PG(2n - 1, q^2)$. Then any of the n odd André replacements lead to a mixed partition of $(q^{2n} - 1)/(q^2 - 1)$ $PG(2n - 1, q)$ ’s and the remaining $q^{2n} + 1 - (q^{2n} - 1)/(q^2 - 1)$ $PG(n - 1, q^2)$ ’s of $PG(2n - 1, q^2)$.*

Corollary 21. *Replacement of λ André nets all by odd André replacement produces a mixed partition of $\lambda(q^{2n} - 1)/(q^2 - 1)$ $PG(2n - 1, q)$ ’s and $q^{2n} + 1 - \lambda(q^{2n} - 1)/(q^2 - 1)$ $PG(n - 1, q^2)$ ’s.*

Remark 22. *Now there was nothing particular about a Desarguesian plane, merely that when the order is q^4 we have a spread in $PG(3, q^2)$ and we have a replacement partial spread of $GF(q)$ -subspaces admitting the kernel homology group H^- of order $q^2 - 1$. That is, we have a replacement net N such that $NH^- = N$. However, in our construction, we required that $q + 1$ divides $|N|$. Actually, this was merely to be able to count the component parts of the mixed partition. It is possible that some of the replacement components are $GF(q^2)$ -spaces; ‘lines’. This still produces a mixed partition.*

Moreover, for Desarguesian spreads of order q^{2n} also for ‘even’ André replacements, as long as one of the André replacements is odd, we obtain a truly ‘mixed’ partition. Hence, for each André net we may choose any of the $2n - 1$ possible replacements and as long as one of the replacements is odd, we still obtain a non-trivial mixed partition of $PG(2n - 1, q^2)$.

7 Algebraic and Geometric Lifting.

In Johnson [11], there is described a very general construction process called ‘lifting’ and which we further term ‘algebraic lifting’ (to properly distinguish from geometric lifting) which constructs from any spread in $PG(3, q)$ a derivable spread in $PG(3, q^2)$. This spread has a replaceable net which we can replace to construct a spread in $PG(7, q)$ satisfying the retraction process as presented in this article. In Johnson [11], the reverse process of construction of a spread in $PG(3, q)$ from a spread of a given form in $PG(3, q^2)$ is also called ‘retraction’. Hence, we might use the terms ‘algebraic retraction’ and ‘geometric retraction’ here as well to distinguish these two constructions. We can relate ‘algebraic lifting’ now to geometric lifting.

Theorem 23. *Let π be translation plane with spread S in $PG(3, q)$. Let F denote the associated field of order q and let K be a quadratic extension field with basis $\{1, \theta\}$ such that $\theta^2 = \theta\alpha + \beta$ for $\alpha, \beta \in F$. Choose any quasifield and write the spread as follows:*

$$x = 0, y = x \left[\begin{array}{cc} g(t, u) & h(t, u) - \alpha g(t, u) = f(t, u) \\ t & u \end{array} \right] \forall t, u \in F$$

where g, f and unique functions on $F \times F$ and h is defined as noted via α .

Define $F(\theta t + u) = -g(t, u)\theta + h(t, u)$.

Then

$$x = 0, y = x \left[\begin{array}{cc} \theta t + u & F(\theta s + v) \\ \theta s + v & (\theta t + u)^q \end{array} \right] \forall t, u, s, v \in F$$

is a spread S^L in $PG(3, q^2)$ called the spread ‘algebraically lifted’ from S .

We note that there is a derivable net

$$x = 0, y = x \left[\begin{array}{cc} w & 0 \\ 0 & w^q \end{array} \right] \forall w \in K \simeq GF(q^2)$$

with the property that the derived net (replaceable net) contains exactly two Baer subplanes which are $GF(q^2)$ -subspaces and the remaining $q^2 - 1$ Baer subplanes form $(q - 1)$ orbits of length $q + 1$ under the kernel homology group.

Hence, we obtain a mixed partition of $(q - 1)$ $PG(3, q)$'s and $q^4 - q$ lines of $PG(3, q^2)$.

So, from any quasifield, we obtain a spread in $PG(3, q)$ which lifts and derives to a spread permitting retraction which produces a mixed partition of $(q - 1)$ $PG(3, q)$'s and $q^4 - q$ lines of $PG(3, q^2)$.

Remark 24. *From the mixed partition of $(q - 1)$ $PG(3, q)$'s and $q^4 - q$ lines of $PG(3, q^2)$, we construct a translation plane back which has $q^4 - q$ what might be called ‘ordinary’ components and $(q - 1)$ orbit's of length $q + 1$ of components each forming a $GF(q)$ -regulus. Now since we ‘get back’, what must happen is that two of these ordinary components together with the $q - 1$ reguli must form a derivable net —that is, we get the derived side — which when derived gets back to the original spread in $PG(3, q^2)$.*

On the other hand, if we have a mixed partition of $q - 1$ $PG(3, q)$'s and $q^4 - q$ lines of $PG(3, q^2)$, it is not necessarily the case that the geometrically lifted spread is derivable.

8 Retractions containing Reguli.

Again there is an interesting construction in Mellinger [15] of a mixed partition in $PG(3, q^2)$ of a regulus and $PG(3, q)$'s.

The constructions in this section were motivated by trying to understand the nature of Mellinger's mixed partition.

Suppose that we have a spread of order q^4 with spread in $PG(3, K \simeq q^2)$ admitting two symmetric groups of affine homologies which fix a derivable net and

both groups act transitively on the non-axis/co-axis components. Assume that the derivable net is not a K -regulus.

Hence, we may represent the two groups with axis and coaxis (coaxis and axis) $x = 0$ and $y = 0$ as follows:

$$H_y = \left\langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u^\sigma \end{bmatrix}; u \in K - \{0\} \right\rangle, \text{ for some automorphism } \sigma \neq 1 \text{ of } K$$

and

$$H_x = \left\langle \begin{bmatrix} u & 0 & 0 & 0 \\ 0 & u^\sigma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; u \in K - \{0\} \right\rangle, \text{ for some automorphism } \sigma \neq 1 \text{ of } K.$$

So, the two homology groups are groups of order $q^2 - 1$ and both correspond to the multiplicative group of a field. We note the derivable partial spread

$$D : x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u^\sigma \end{bmatrix}; u \in K$$

is a ‘regulus’ in the associated projective geometry $PG(3, \begin{bmatrix} u & 0 \\ 0 & u^\sigma \end{bmatrix})$. We note that the group:

$$H = \left\langle \begin{bmatrix} u & 0 & 0 & 0 \\ 0 & u^\sigma & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u^\sigma \end{bmatrix}; u \in K - \{0\} \right\rangle$$

acts as a collineation group of the translation plane. Moreover, this group is fixed-point-free and decomposes the vector space into a 4-dimensional $\begin{bmatrix} u & 0 \\ 0 & u^\sigma \end{bmatrix}$ -vector space.

Now assume that $\sigma = q$ and multiply each term in H by $u^{-1}I_4$ to obtain a Baer group

$$B = \left\langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & u^{q-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & u^{q-1} \end{bmatrix}; u \in K - \{0\} \right\rangle.$$

Note that this Baer group fixes the derivable net componentwise and has remaining component orbits of lengths $q + 1$. It then follows immediately that the group H fixes exactly $q^2 + 1$ components of the derivable net and has $(q^4 - q^2)/(q + 1) = q^2(q - 1)$ orbits of length $q + 1$. We notice that we are now considering the situation that we have previously, we have a vector space over $\begin{bmatrix} u & 0 \\ 0 & u^q \end{bmatrix}$ with exactly $q^2 + 1$ fixed components. The remaining components are not $\begin{bmatrix} u & 0 \\ 0 & u^q \end{bmatrix}$ -subspaces but are

$GF(q)$ -subspaces (i.e. $\begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}$ -subspaces for $v \in$ the subfield of K isomorphic to $GF(q)$).

Moreover, it is almost immediate that the component orbits of length $q + 1$ are $GF(q)$ -reguli and the point orbits are the subplanes of the regulus net incident with the zero vector which are, of course, $\begin{bmatrix} u & 0 \\ 0 & u^q \end{bmatrix}$ -1-dimensional vector subspaces. Now, actually we do not require the individual homology groups, only the group H . Hence, we obtain:

Theorem 25. *Let π be a translation plane of order q^4 with spread in $PG(3, K \simeq GF(q^2))$ admitting a derivable net of the form:*

$$D : x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u^q \end{bmatrix}; u \in K$$

and a collineation group of order $q^2 - 1$ of the form:

$$H = \left\langle \begin{bmatrix} u & 0 & 0 & 0 \\ 0 & u^q & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u^q \end{bmatrix}; u \in K - \{0\} \right\rangle.$$

(1) *Then there is an associated mixed partition of $PG(3, \begin{bmatrix} u & 0 \\ 0 & u^q \end{bmatrix} \simeq GF(q^2))$ consisting of a regulus of $q^2 + 1$ lines and $q^2(q - 1)$ $PG(3, q)$'s.*

(2) *The translation plane obtained by derivation also retracts to a regulus of $q^2 + 1$ lines and $q^2(q - 1)$ $PG(3, q)$'s.*

Corollary 26. *Let π be a translation plane of order q^4 with spread in $PG(3, q^2)$ of the following form:*

$$x = 0, y = x \begin{bmatrix} u & t^q f \\ t & u^q \end{bmatrix}; u, t \in GF(q^2) \text{ and } f \in GF(q^2) - GF(q).$$

Then there are two associated mixed partitions of some $PG(3, q^2)$ consisting of a regulus and $q^2(q - 1)$ $PG(3, q)$'s obtained from the plane and its derived plane

Proof. Merely check that the group H listed above does act as a collineation group.

Remark 27. (1) *The form of the spread listed above forces the associated translation plane to be a semifield plane coordinatizable by a semifield of order q^4 where all nuclei are isomorphic to $GF(q^2)$ and the right and middle nucleus are identical. Hence, the translation plane is a Hughes-Kleinfeld plane.*

(2) *In Johnson [11], there is a condition given which describes the spreads in $PG(3, q^2)$ which may be algebraically retracted to spreads in $PG(3, q)$. The condition is that there is an elation group of order q^2 , E , and a Baer group of order $q + 1$, B such that $[E, B] \neq 1$. Hence, it follows that the above Hughes-Kleinfeld plane can be algebraically retracted. It is not difficult to check that the algebraic retraction is a Desarguesian plane.*

Hence, the Hughes-Kleinfeld planes may be algebraically lifted from Desarguesian planes. ■

9 Remarks on Baer Subgeometry Partitions

The non-solvable flag-transitive planes have been completely determined by Buekenhout et al [3]. These are the Desarguesian, Lüneburg-Tits, Hering of order 27 and Hall of order 9.

In Baker, Dover, Ebert, and Wantz [1], it is shown that cyclic and regular groups acting flag-transitively on odd order planes force the spreads to arise from Baer subgeometry partitions. Here, we provide a variation of this result based upon the spread-retraction ideas of the fundamental theorem. Basically, for such spread, we need only show that the vector space is a $GF(q^2)$ -space and that the component orbit lengths are all $q + 1$.

Hence, consider solvable flag-transitive planes of order q^n where n is odd > 1 and kernel containing $GF(q)$. The collineation group of any such plane is a subgroup of $\Gamma L(1, q^{2n})$ by Foulser (see [6] and [7]). Basically, the points of the plane are identified with the points of the Desarguesian affine plane (or of $GF(q^{2n})$). That is, in this instance, the vector space may be considered over $GF(q^2)$. Now assume that the collineation group contains $GL(1, q^2)$ so that there is a linear and cyclic subgroup of order $q^2 - 1$ that contains the kernel homology group of order $q - 1$. Let L be a component and consider $GL(1, q^2)_L$. Since the full group is transitive and normalizes any cyclic subgroup of $GL(1, q^2)$, we see that $GL(1, q^2)_L$ must fix each component of the translation plane and hence induces a kernel homology subgroup. If $GF(q)^*$ is properly contained in $GL(1, q^2)_L$ then the kernel of the translation plane contains the ring generated by $\{GL(1, q^2)_L, GF(q)\}$ which is clearly $GL(1, q^2)$. Hence, the kernel of the translation plane contains $GF(q^2)$. However, the translation plane is of order q^{2m+1} and is a $2k$ dimensional vector space over the kernel $GF(q^2)$ so is $2kz$ dimensional over $GF(q)$. Thus, $2kz = 2(2m + 1)$ so that kz is odd implying both k and z are odd a contradiction. Thus, $GL(1, q^2)_L = GF(q)^*$ for all components L .

Thus, we obtain the following theorem:

Theorem 28. *Let π be a non-Desarguesian flag-transitive translation plane of order q^{2m+1} and kernel containing $GF(q)$ with group G .*

- (1) *If G is non-solvable then π is the Hering plane of order 27.*
- (2) *If G is solvable then G is a subgroup of $\Gamma L(1, q^{2(2m+1)})$. If $GL(1, q^2) \subseteq G$ then π corresponds to a Baer subgeometry partition of $PG(2m, q^2)$.*

Proof. Our remarks above and arguments in the next subsection show that if the group is non-solvable and non-Desarguesian then only the Hering plane of order 27 is possible.

When the group is solvable, our previous arguments together with fundamental theorem complete the proof. ■

Corollary 29. *Let π be a non-Desarguesian flag-transitive plane of order q^{2m+1} , $m > 0$ which contains $GF(q)$ in the kernel and let $q = 2^r$.*

- (1) *If $(r(2m + 1), q + 1) = 1$ then π arises from a Baer subgeometry partition of $PG(2m, q^2)$*
- (2) *Hence, any non-Desarguesian flag-transitive plane of order 2^{2m+1} for m odd and not divisible by 3 arises from a Baer subgeometry partition of $PG(2m, 4)$.*

Proof. By the previous result, we may assume that the group is a subgroup of $GL(1, q^{2n})$. Let $q + 1 = \prod_{i=1}^k p_i^{\alpha_i}$ be the prime decomposition. Note that $(q^n + 1)/(q + 1) = \sum_{i=1}^{n-1} (-1)^{i+1} (q^{n-i} - (-1)^{i+1}) + n$. Hence, it follows that the Sylow p_i -subgroups S_{p_i} have order $p_i^{\alpha_i}$.

Suppose that $x \mapsto x^{p^t} a$ is an element of a Sylow p_i -subgroup S_{p_i} . Then since $(2rn, q + 1) = 1$, it follows that since $(x \mapsto x^{p^t} a)^{p_i^{\alpha_i}}$ is $x \mapsto x$ then $p^t = 1$. Hence, every Sylow p_i -subgroup S_{p_i} is in $GL(1, q^{2n})$ which, by uniqueness is then in $GL(1, q^2)$. Hence, there is a unique Sylow p_i -subgroup of G so that $\prod_{i=1}^k S_{p_i} \subseteq GL(1, q^2)$. Part (1) of the corollary now follows immediately. Since $GF(2)$ is a subkernel of any translation plane of even order, and since the group cannot be non-solvable if $(2m + 1, 3) = 1$, part (2) then follows from the previous theorem and part (1). ■

As an application to the above corollary, since $(2nr, q^n + 1) = 1$ implies that $(nr, q + 1) = 1$, we obtain a result of G.L. Ebert.

Corollary 30. (see Ebert [5]) *Let π be a non-Desarguesian flag-transitive plane of order q^n , $n > 1$ odd which contains $GF(q)$ in the kernel and let $q = 2^r$.*

If $(2rn, q^n + 1) = 1$ then π arises from a Baer subgeometry partition.

As an further application of this result, now assume that we have a cyclic and regular group on the line at infinity and the kernel contains $GF(q)$. Assume that $n > 1$ and q not 2, then there is a p -primitive divisor of $q^{2n} - 1$ which must be linear; i.e. is in $GL(1, q^{2n})$. The centralizer of an element of this order is $GL(1, q^{2n})$. Since we do have a cyclic group of order $q^{2n} - 1$ which centralizes the indicated element, we have $GL(1, q^{2n}) \supseteq GL(1, q^2)$.

Hence, we have:

Corollary 31. (Baker, Dover, Ebert, Wantz [1] (see Theorem (2.2))

Any flag-transitive translation plane of order q^n and kernel containing $GF(q)$ for n odd > 1 and $(n, q) \neq (3, 2)$ that admits a collineation group inducing a cyclic and regular group on the line at infinity may be obtained by geometric lifting from a Baer subgeometry partition.

It might be noted that any three components of a translation plane of even order automatically form a $GF(2)$ -regulus. Hence,

Theorem 32. *Let π be a translation plane of order 2^n for n odd.*

(1) Then the spread for π is a union of $GF(2)$ -reguli.

(2) π arises from a Baer subgeometry partition if and only if π admits a non-planar collineation of order 3.

Proof. Since 3 divides $2^n + 1$ when n is odd and any three components form a $GF(2)$ -regulus, we have the proof to (1).

Now assume that π admits a non-planar collineation τ of order 3. Suppose that τ leaves invariant some component L . Then it follows that τ must leave invariant at least three components as 3 does not divide 2^n or $2^n - 1$. Assuming that τ is in the translation complement then on each fixed component τ must fix at least two points which implies that τ is planar.

Hence, τ has $(2^n + 1)/(2 + 1)$ orbits of components of length 3. Since each orbit is now a $GF(2)$ -regulus and there are exactly $(2^n - 1)$ Desarguesian subplanes of order 2 on each $GF(2)$ -regulus net, it follows that τ leaves at least one Desarguesian subplane invariant on each orbit. Hence, τ induces a faithful element of $GL(2, 2)$. Thus, $\langle \tau \rangle \cup \{0\}$ is a field isomorphic to $GF(4)$. Since $GF(4)^*$ acts fixed-point-free, π is decomposed as a $GF(4)$ -vector space which admits the scalar group $GF(4)^*$ as a collineation group. Since the orbits of components are all of length $2 + 1$, it follows from the fundamental theorem that π may be retracted from a Baer subgeometry partition of $PG(n - 1, 4)$. The converse now follows from the section on geometric lifting. ■

In [1], the following two questions are asked? Does a spread of order q^n for n odd and > 1 , which is a disjoint union of $GF(q)$ -reguli, arise from a Baer subgeometry partition? And, if there are more than $(q^n + 1)/(q + 1)$ $GF(q)$ -reguli, must the spread be regular (Desarguesian)? It is also pointed out that the Hering spread of order 27 shows that the general answer to these question is no. However, the question was raised as to whether the Hering spread is the only exception.

Note that, in particular, any semifield plane of order 2^n for n odd cannot admit a collineation group of order 3 which does not leave invariant a component and hence is planar. Hence any such spread cannot arise from a Baer subgeometry partition. Furthermore, even when there is a non-planar element of order 3, there are $\binom{2^n + 1}{3}$ $GF(2)$ -reguli.

Hence, we see that the $GF(2)$ -reguli trivially answer these questions negatively. So, these questions more properly need to exclude the case when $q = 2$.

9.1 Transitive and Two-Transitive Baer Subgeometry Partitions.

Some easy consequences of the fundamental theorem regarding transitivity from the other direction are as follows.

Definition 33. A Baer subgeometry partition of $PG(2m, q^2)$ is said to be ‘transitive’ or ‘two-transitive’ if and only if there is a collineation group in $P\Gamma G(2m + 1, q^2)$ which acts transitive or two-transitively respectively on the sets of $PG(2m, q)$'s of the partition.

Theorem 34. A transitive Baer subgeometry partition of $PG(2m, q^2)$ produces a flag-transitive translation plane of order q^{2m+1} with spread in $PG(4m + 1, q)$ admitting $GF(q^2)^*$ as a collineation group.

Conversely a flag-transitive spread permitting spread-retraction (whose group is in $\Gamma L(2m + 1, q^2)$) produces a transitive Baer subgeometry partition.

Theorem 35. A two-transitive Baer subgeometry partition of $PG(2m, q^2)$, for $(q, 2m + 1) \neq (2, 5)$ produces a Desarguesian spread of order q^{2m+1} with spread in $PG(4m + 1, q)$.

Proof. A transitive Baer subgeometry partition produces a translation plane whose spread is covered by $GF(q)$ -reguli each of which is an orbit of components under a group isomorphic to $GF(q^2)^*$. Since the Baer subgeometry partitions is transitive,

there is a group which is transitive on the line at infinity of the translation plane so that there is a flag-transitive group. The converse is immediate.

A two-transitive Baer subgeometry partition produces a translation plane of order q^{2m+1} with kernel containing $GF(q)$ admitting a group acting transitively on the components and whose stabilizer of a $GF(q)$ -reguli is transitive on the remaining $q(1+q+q^2+\dots+q^{2m-2})$ $GF(q)$ -reguli. By Foulser (see [6] and [7]), either the group is non-solvable or the flag-transitive group is a subgroup of $\Gamma L(1, q^{2(2m+1)})$. Hence, it must be that the group is non-solvable or $(q, 2m+1) = (2, 3), (3, 3), (4, 3),$ or $(2, 5)$. There are no non-Desarguesian planes of orders 8 and 27 with the properties stated (see e.g. Dempwolff [4]). The $(4, 3)$ possibility may be resolved by appealing to the work of Baker et al [1], who discuss possible Baer subgeometry partitions in $PG(2, 16)$, and there are no non-Desarguesian translation planes emerging from such orders that admit doubly transitive groups on the regulus nets. Hence, we may assume that the group is non-solvable. So, the translation planes are the Desarguesian, Lüneburg-Tits of order 2^{2r} , r odd with kernel $GF(2^r)$, Hering of order 27 and Hall or order 9. We note subsequently that the Hering plane does not correspond to a Baer subgeometry partition. If the plane is a Lüneburg-Tits plane and the order of the plane is q^{2m+1} with kernel containing $GF(q)$, and 2^{2r} for r odd and equal to q^{2m+1} for $q = 2^s$ implies that $2r = s(2m+1)$. Thus, s is even. If the kernel contains $GF(2^s)$ and $GF(2^r)$ for r odd, we clearly have a contradiction in this case. Since the order is not 9 by assumption, we must have that the plane is Desarguesian. ■

A open problem of some importance and interest is whether all flag-transitive planes of order q^n with kernel containing $GF(q)$ and n odd correspond to Baer subgeometry partitions. The only known plane that may not be so constructed is the Hering plane of order 27.

9.2 The Hering Plane of order 27

Mathon and Hamilton [14] have shown that the Hering plane of order 27 does not correspond to a Baer subgeometry partition of $PG(2, 9)$ by exhaustive computer search.

We may use the fundamental theorem to give an easy proof that the Hering plane of order 27 does not correspond to a Baer subgeometry partition of $PG(2, 9)$. We need to ask if there is a collineation group isomorphic to $GF(9)^*$ with point orbits of length 8 and component orbits of length 4. However, the full collineation group (full translation complement) of the Hering planes is $SL(2, 13)$ (see Dempwolff [4]). Moreover, in $SL(2, 13)$, the 2-groups are generalized quaternion of order 8. Hence, no such cyclic group exists.

Hence, by the fundamental theorem:

Theorem 36. *The Hering plane of order 27 does not arise from a Baer subgeometry partition of $PG(2, 9)$.*

10 Maximal Partial Projective Partitions.

Suppose we have a maximal partial spread in $PG(4n - 1, q)$ that admits $GF(q^2)^*$ as a collineation group which has components of lengths 1 or $q + 1$. Then clearly, we may form a ‘retraction’ of the partial spread.

Theorem 37. (1) Let \mathcal{P} be a translation net with partial spread in $PG(4n - 1, q)$ such that the associated vector space is a $GF(q^2)$ -space for some quadratic extension field of the underlying field isomorphic to $GF(q)$ and $GF(q^2)^*$ acts as a collineation group of \mathcal{P} . Assume that the partial spread components are in orbits of lengths 1 or $q + 1$ under $GF(q^2)^*$ and that there are β orbits of length 1 and α orbits of length $q + 1$.

Then there is a mixed partial partition of $PG(n - 1, q^2)$ of β $PG(n - 1, q^2)$'s and α $PG(2n - 1, q)$'s.

Note that it is possible to have $\beta = 0$ in this case producing a partial Baer subgeometry partition of $PG(2n - 1, q^2)$.

(2) Let \mathcal{P} be a translation net with partial spread in $PG(4m + 1, q)$ such that the associated vector space is a $GF(q^2)$ -space for some quadratic extension field of the underlying field isomorphic to $GF(q)$ and $GF(q^2)^*$ acts as a collineation group of \mathcal{P} . Assume that the partial spread components are in orbits of lengths $q + 1$ under $GF(q^2)^*$ and that there are α orbits of length $q + 1$.

Then there is a partial Baer subgeometry partition of $PG(2m, q^2)$ by α $PG(m - 1, q^2)$'s.

(3) Define the partial spread \mathcal{P} to be ‘ $GF(q^2)$ -maximal’ if and only if there is no extension partial spread in $PG(4n - 1, q^2)$ or respectively $PG(4m + 1, q)$ admitting $GF(q^2)^*$ as a collineation group.

Then \mathcal{P} is $GF(q^2)$ -maximal if and only if there the associated partial mixed or partial Baer subgeometry partition is maximal in $PG(2n - 1, q^2)$ or $PG(2m, q^2)$ respectively.

Proof. The proofs of (1) and (2) follow directly from our previous arguments.

Assume that \mathcal{P} is $GF(q^2)$ -maximal but the associated partial partition of the projective geometry over $GF(q^2)$ is not. Then, our arguments show that the geometric lifting procedure produces a spread extending an isomorphic version of \mathcal{P} admitting $GF(q^2)^*$ as a collineation group which is a contradiction. Similarly, the partial partition is maximal implies the associated partial spread admitting $GF(q^2)^*$ as a collineation group is maximal. ■

To see how this might come above from a spread in $PG(3, q^2)$. Suppose we have two mutually disjoint derivable nets D_1 and D_2 .

Choose a basis properly, assume that D_1 and D_2 are isomorphic to

$$x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u^{\sigma_i} \end{bmatrix} \text{ for all } u \in GF(q^2)$$

where $\sigma_i \neq 1, \sigma_1 \neq 1$ or q and $\sigma_2 = q$.

Then, the set of Baer subplanes of D_i incident with the zero vector has exactly two $GF(q^2)$ -subspaces, the remaining subspaces are all subspaces over $Fix\sigma_i$ and none are subspaces over any superfield of $Fix\sigma_i$.

Now take as a partial spread, the two $GF(q^2)$ -Baer subplanes of D_1 incident with the zero vector and the remaining components of the original spread. This is a maximal partial spread in $PG(3, q^2)$. Moreover, since the remaining Baer subplanes of D_1 are, in fact, not $GF(q)$ -subspaces, this is also a maximal partial spread in $PG(7, q)$ of exactly $q^4 - q^2 + 2$ components of $PG(3, q^2)$.

Now take the two $GF(q^2)$ Baer subplanes of D_2 along with the $q - 1$ orbits of $GF(q)$ -subspaces under the kernel homology group $GF(q^2)^*$ of the original translation plane together with the remaining $q^4 - q^2 + 2 - (q^2 + 1)$ components outside of D_2 of the maximal partial spread. This produces a maximal partial spread in $PG(7, q)$ of $q - 1$ orbits of length $q + 1$ and $q^4 - 2q^2 + 1 + 2 = q^4 - 2q^2 + 3$ remaining components which are $GF(q^2)$ -subspaces.

Hence, we have a partial spread consisting of $q^4 - 2q^2 + 3$ $GF(q^2)$ -spaces and $q^2 - 1$ $GF(q)$ -spaces in orbits of length $q + 1$.

This partial spread is maximal for if not, another $GF(q)$ -component can only lie across the original D_1 and it follows from Jha and Johnson [9] that this must be a Baer subplane of D_1 . Moreover, it is also $GF(q^2)$ -maximal. But, such Baer subplanes are not $GF(q)$ -subspaces. Hence, the partial spread is maximal in $PG(7, q)$.

Hence, we obtain a mixed partial partition of $PG(3, q^2)$ of $q - 1$ $PG(3, q)$'s and $q^4 - 2q^2 + 3$ lines.

We note if the mixed partial partition can be extended to a mixed partial partition then, since our arguments show that the geometric lifting process can clearly be extended to partial partitions, there is a partial spread of $PG(7, q)$ extending our maximal partial spread, a contradiction.

Hence, we obtain a maximal mixed partial partition of $PG(3, q^2)$.

Theorem 38. *A translation plane with spread in $PG(3, q^2)$ admitting two mutually disjoint derivable nets which are not regulus nets and exactly one has all transversal Baer subplanes as $GF(q)$ -subspaces produces a maximal mixed partial partition of $PG(3, q^2)$ of $q - 1$ $PG(3, q)$'s and $q^4 - 2q^2 + 3$ lines.*

11 Generalization to Infinite Spreads.

We note that our spread-retraction process generalizes directly to spreads with kernels containing a skewfield F that admit quadratic skewfield extensions K such that K^* , the multiplicative group, is a collineation group of the associated translation plane such that the components are either K^* -invariant or in orbits Λ such that for any component ℓ in Λ , $K_\ell^* = F^*$.

Note that if a space F -isomorphic to V is a K -space then it is also an F -space, and we may use the notation $PG(V, K)$ and $PG(V, F)$, respectively. For example, if the order of the plane is q^{2n} and the dimension of a space $2n$ over $GF(q)$ and n over $GF(q^2)$, then $PG(V, K)$ is $PG(n - 1, q^2)$ and $PG(V, F)$ is $PG(2n - 1, q)$. It will turn out that in order to obtain $PG(V, F)$'s, F and K must be fields, so we assume this from the beginning.

Theorem 39. *Let π be a translation plane with kernel containing the field F and let K be a quadratic field extension of F . Write $\pi = V \oplus V$ where V is an F -subspace.*

Considering π as a K -vector space, assume that K^* , the K -scalar group, acts as a collineation group of π such that each component is either fixed by K^* or in orbits Λ such that for any component ℓ in Λ , $K_\ell^* = F^*$. (We call such orbits ‘long’ orbits.) Then there is a corresponding partition of $PG(\pi, K)$ by $PG(V, K)$'s and $PG(V, F)$'s. If there are zero $PG(V, F)$'s, we obtain merely a projective spread and if there are zero $PG(V, K)$'s, we obtain a Baer subgeometry partition. If there are non-zero projective subspaces of both types, we obtain a mixed partition.

Proof. It remains to show that the ‘long’ orbits produce F -reguli and that these project modulo K into $PG(V, F)$'s. The long orbit components form a net which admit point orbits isomorphic to K^* such that on any component of the orbit, the point orbit of the stabilizer union the zero vector is a 1-dimensional F -subspace. Hence, we have a partition of K as a 2-dimensional F -vector space into 1-dimensional F -subspaces. This makes each point orbit union the zero vector into a Pappian subplane with spread in $PG(3, F)$.

Hence, we have a subplane covered net and by the main theorem of Johnson [10], the net is a pseudo-regulus net coordinatizable by F . But, since F is a field and the subplanes covering the net are Pappian subplanes, it follows that the net is an F -regulus net.

Now when we form the projective space $PG(V, K)$ by taking the lattice of K -subspaces, each F -regulus becomes a $PG(V, F)$ isomorphic to the projective version of any component of the orbit when written over F ; the Pappian subplanes are K -subspaces and hence become ‘points’ in the associated projective geometry. ■

Remark 40. (1) The André type replacements are valid in the infinite case so we obtain a variety of mixed partitions in this case.

(2) The ‘algebraic lifting’ process works for any spread or quasifibration in $PG(3, F)$ where F is a field admitting a quadratic field extension K . Hence, we may construct mixed partitions and/or partial mixed partitions in this case also.

(3) There are infinite versions of the Hughes-Kleinfeld spreads which provide mixed partitions of $PG(3, K)$ containing $PG(3, F)$'s and a regulus.

Remark 41. We note that we do not need finite dimensions for a spread to permit retraction. Also, we may formulate the geometric lifting process without the use of dimension although it is much more cumbersome to do so in the projective case and we omit this construction.

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