The fundamental class of a rational space, the graph coloring problem and other classical decision problems

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Abstract

The problem of k-coloring a graph is equivalent to deciding whether a particular cohomology class of a certain rational space vanishes. Although this problem is NP-hard we are able to construct a fast (polynomial) algorithm to give a representative of this class. We also associate to other classical decision problems rational spaces so that the given problem has a solution if and only if the associated space is not elliptic. As these spaces have null Euler homotopy characteristic we easily characterize when the given problem has a solution in terms of commutative algebra.

1 Introduction

In [14] the authors associate to a given graph G and any integer $k \geq 2$ a rational space $S_{G,k}$ and prove that the graph can be k-colored if and only if the singular cohomology of $S_{G,k}$ with rational coefficients is infinite dimensional. On the other hand, in [15], the second author gives an explicit formula for a cohomology class of the formal dimension of certain spaces (the so called finite pure spaces) in such a way that the non vanishing of this class is equivalent to the finiteness of the rational cohomology of the space. As the spaces $S_{G,k}$ are pure, the problem of k-coloring a graph is equivalent to determining when a particular cohomology class vanishes (let us call it the fundamental class).

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At this point one could search for new algorithms to determine when a given graph is k-colorable by finding procedures which let us know if the fundamental class is null. Then we are facing two complexity problems: one is to compute the cycle representing this class, and the other is to check whether this cycle is a boundary. All of this having in mind, of course, the well known fact that the problem of determining the k-coloring of a graph belongs to the class of NP-complete problems which are believed to require more than polynomial time (with respect to the data) to be solved. Also it is important to note that the formula given in [15] requires an exponential amount of time to be computed.

In this paper we start by describing a slightly different method than the one used in [15] to obtain a representative of the fundamental class (proposition 2). Also, we give a process which lets us compute the fundamental class of any elliptic space, i.e., a space with finite dimensional rational homotopy and cohomology (proposition 6). Part of this is folklore on rational homotopy theory and we include it here for clarity.

Our first objective is then to modify the space $S_{G,k}$ associated to the graph in such a way that: 1) it is still true that the new space is not elliptic if and only if the graph can be k-colored, and 2) we can produce, using the new method, a representative of the fundamental class of this space in polynomial time (see theorems 8 and 9).

As an immediate consequence we deduce that the problem of deciding whether a given cycle of a space (or a model) represents the zero class is NP-hard.

After this, we shall define a new space associated to the 3-coloring of a graph so that it has null homotopy Euler characteristic (note that this was not at all the case of the space associated to a graph in [14] or in the mentioned modification above). From this, we characterize the 3-coloring problem in terms of commutative algebra. (see theorem 11 and corollaries 12, 13 and 14)

To finish we mimic this situation in the context of two other classical decision problems:

1. The "satisfiability problem", i.e., given a propositional formula (a set of clauses), does there exist an assignment of logical values to the variables of the clauses such that the formula is satisfied, i.e., every clause is satisfied? Recall that this was the first problem that was proved to be NP–complete [5].

2. The "subset sum problem" which, having as data a set $B \subset \mathbb{N}$ and a positive integer n_0, consists in deciding whether there exists a subset of B whose elements sum up to n_0. This is also an NP-complete problem [8].

We will be able to associate to any of these problems a new rational space in such a way that the given problem has a solution if and only if the associated space has infinite dimensional rational cohomology (see theorems 16 and 19). A very important property of these spaces is that again, they are "pure" and have null homotopy Euler characteristic. This particular behavior of these spaces will let us, using standard techniques from the theory of minimal models, characterize when a given problem has a solution in terms of pure commutative algebra (see corollaries 17, and 20).

These characterizations can be seen themselves as algorithms for these problems and all the machinery of Gröbner basis (see [3], [4] or [13]) may then be applied to improve the performance of these algorithms.

Our results rely heavily on the understanding of basic facts and tools from rational homotopy theory, and in particular, Sullivan minimal models. Standard and good references are [6],[11] and [18]. With respect to complexity of algorithms the non familiar reader may consult [1] or [19] for basic facts. All spaces and models will be 1-connected of finite type. The ground field $\mathbb{K}$ will be, unless explicitly stated otherwise, of characteristic zero. All graphs considered are non oriented, finite, connected and simple.

2 The fundamental class of spaces

As a very quick synopsis about minimal models we remark that to any space S corresponds a KS-complex, i.e., a commutative differential graded algebra $(\Lambda V, d)$ which algebraically models the rational homotopy type of the space and is called the minimal model of S. By ΛV we mean the free commutative algebra generated by the graded vector space V, i.e., $\Lambda V = TV/I$ where TV denotes the tensor algebra over V and I is the ideal generated by $v \otimes w - (-1)^{|v||w|} w \otimes v$, $v, w \in V$. The differential d of any element of V is a "polynomial" in ΛV with no linear term.

Recall that a KS-complex $(\Lambda V, d)$ is *pure* if $dV^{\text{even}} = 0$ and $dV^{\text{odd}} \subset \Lambda V^{\text{even}}$. Such a KS-complex admits a bigradation $\Lambda V = \sum_{n,j\geq 0} (\Lambda_j V)^n$, where $(\Lambda_j V)^n = (\Lambda V^{\text{even}} \otimes \Lambda^j V^{\text{odd}})^n$, for which d has bidegree $(1, -1)$. Hence the cohomology algebra is also bigraded $H^*(\Lambda V, d) = \sum_{n,j\geq 0} H_j^n(\Lambda V, d)$. Recall also that if $(\Lambda V, d)$ is elliptic, then the *homotopy Euler characteristic* $\chi_\pi = \sum_q (-1)^q \dim V^q$ is non-positive. Moreover if $k = -\chi_\pi$ then $H_k(\Lambda V, d) \neq 0$ and $H_{>k}(\Lambda V, d) = 0$.

Assume $\dim V < \infty$, call $X = V^{\text{even}}$, $Y = V^{\text{odd}}$, choose homogeneous basis $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_m\}$ of X and Y respectively, and write

$$dy_j = a_j^1 x_1 + a_j^2 x_2 + \cdots + a_j^{n-1} x_{n-1} + a_j^n x_n, \qquad j = 1, \ldots, m,$$

where each a_j^i is a polynomial in the variables $x_i, x_{i+1}, \ldots, x_n$. For any $1 \leq j_1 < \ldots < j_n \leq m$, denote by $P_{j_1 \ldots j_n}$ the determinant of the matrix of order n:

$$\begin{pmatrix} a_{j_1}^1 & \ldots & a_{j_1}^n \\ \vdots & \ddots & \vdots \\ a_{j_n}^1 & \ldots & a_{j_n}^n \end{pmatrix}$$

Then (see [15]) the element $P \in \Lambda V$

$$P = \sum_{1\leq j_1 < \ldots < j_n \leq m} (-1)^{j_1+\cdots+j_n} P_{j_1\ldots j_n}\, y_1 \ldots \hat{y}_{j_1} \ldots \hat{y}_{j_n} \ldots y_m,$$

is a cycle representing a basis of the image of the evaluation map

$$\mathrm{ev}_{(\Lambda V, d)} \colon \mathrm{Ext}_{\Lambda V}(\mathbb{K}, \Lambda V) \to H^*(\Lambda V, d).$$

This is a $\mathbb{K}$-linear map which associates to a class $[f]$ represented by $f \colon P \to (\Lambda V, d)$ (P a $(\Lambda V, d)$-semifree resolution of $\mathbb{K}$) the class $[f(1)]$. $\mathrm{Ext}(\mathbb{K}, \Lambda V)$ turns out to have dimension 1 [7] and its image lies in the formal dimension N of the cohomology of $(\Lambda V, d)$ which is given by the classical formula

$$N = \sum_{i=1}^{n} (1 - |x_i|) + \sum_{j=1}^{m} |y_j|.$$

Moreover, in [16] is proven that this class, i.e., the image of this map, is non zero if and only if $\dim H^*(\Lambda V, d) < \infty$. In this case, the mentioned class is precisely the fundamental class of the cohomology algebra which is a Poincaré duality algebra. Hence, even if $(\Lambda V, d)$ is not elliptic, we shall still call it the fundamental (or top) class.

We shall now give another procedure to describe this class.

First note (here we just require $dV^{\text{even}} = dX = 0$ and no assumption about the derivative of the odd part) that the projection $\rho\colon (\Lambda V \otimes \Lambda \overline{X}, d) \xrightarrow{\simeq} (\Lambda Y, \overline{d})$, with $\overline{X} = sX$ the suspension of X and $d\overline{x} = x$, is a quasi-isomorphism. Since $\Pi_{j=1}^m y_j$ is a cycle representing the fundamental class of $(\Lambda Y, \overline{d})$, the top class of $(\Lambda V \otimes \Lambda \overline{X}, d)$ is represented by an element $\gamma = \Pi_{j=1}^m y_j + \Omega$ with $\Omega \in \Lambda^+(X \oplus \overline{X}) \otimes \Lambda Y$. Write $\Omega = \phi(\Pi_{i=1}^n \overline{x}_i) + \psi$ with $\phi \in \Lambda V$ and $\psi \in \Lambda V \otimes \Lambda^{<n}\overline{X}$. Then we have

Lemma 1. *ϕ is a cycle representing the fundamental class of ΛV.*

Proof: The result follows by an obvious induction on $n = \dim X$ considering the following fact: let $A \otimes \Lambda y$ be an extension of A and y of odd degree. If $\dim H^*(A) < \infty$ and $[\alpha y + \beta]$ $(\alpha, \beta \in A)$ is the fundamental class of the extension then $[\alpha]$ is the fundamental class of A. ∎

Now, if $(\Lambda V, d)$ is an elliptic pure model, choose elements $\Phi_j \in \Lambda X \otimes \Lambda \overline{X}$ for which $d\Phi_j = dy_j$, $j = 1, \ldots, m$.

Proposition 2. *The element*

$$w = \textit{coefficient of } \prod_{i=1}^n \overline{x}_i \textit{ in the development of } \prod_{j=1}^m (y_j - \Phi_j)$$

represents the fundamental class of $(\Lambda V, d)$.

Proof: Apply the lemma above noting that $\prod_{j=1}^m (y_j - \Phi_j)$ is a cycle which is sent to $\Pi_{j=1}^m y_j$ by ρ. ∎

Even if $(\Lambda V, d)$ has not finite dimensional cohomology, the class $[w]$ of the preceding proposition lives in $H^N(\Lambda V, d)$ being N the formal dimension. Moreover, in Lemma 1.1 of [15] it is stated that $[w]$ is a basis of the image of the evaluation map described above. Therefore, applying theorem 3.2 of [16] we get

Corollary 3. *Let $(\Lambda V, d)$ be a pure model with V finite dimensional and let w be as in Proposition 2. Then $[w] \neq 0$ if and only if $(\Lambda V, d)$ is elliptic.*

Remark. Suppose now that our ground field is of any characteristic. In [17] it is proven that, over any field, if $\dim V < \infty$ then the model $(\Lambda V, d)$ is Gorenstein and moreover, $(\Lambda V, d)$ is elliptic if and only if $\mathrm{ev}_{(\Lambda V, d)} \neq 0$. But again one can easily observe that proposition 2 is true for any field. Hence, we conclude that with any coefficients $(\Lambda V, d)$ is elliptic if and only if $[w] \neq 0$, i.e., corollary 3 is true over any field.

Using the remark above together with corollary 3 we find a topological criteria which lets us know when a given system of homogeneous equations has non trivial solutions: let $f_j = f_j(x_1, \dots, x_n) \in \mathbb{K}[x_1, \dots, x_n]$, $j = 1, \dots, m$, be homogeneous polynomials ($\mathbb{K}$ any field) and let $(\Lambda V, d)$ be the pure model defined as follows:

$$V^{\text{even}} = \langle x_1, \dots, x_n \rangle, \quad i = 1, \dots, n, \quad |x_i| = 2, \quad dx_i = 0,$$
$$V^{\text{odd}} = \langle y_1, \dots, y_m \rangle, \quad j = 1, \dots, m, \quad |y_j| = 2|f_j| - 1, \quad dy_j = f_j.$$

Proposition 4. $(\Lambda V, d)$ *is not elliptic if and only if the system*

$$\{f_j(x_1, \dots, x_n) = 0, \quad 1 \le j \le m\}$$

has non trivial solutions in a finite algebraic extension of the ground field $\mathbb{K}$.

Proof: If $H^*(\Lambda V, d)$ is finite dimensional then the class $[x_i]$ is nilpotent for any i. That is to say, there are integers n_i and elements $\alpha^i_j \in (\Lambda V, d)$ such that $x_i^{n_i} = \sum_{j=1}^m \alpha^i_j f_j$, $1 \le i \le n$. Hence if $x^0_1, \dots, x^0_n$ is a solution of the system, the above equality clearly implies that $x^0_1 = \dots = x^0_n = 0$.

Conversely if the system has just the trivial solution over any algebraic extension of $\mathbb{K}$ then the variety Ω defined by $\{f_j(x_1, \dots, x_n),\ j = 1, \dots, m\}$ is a point. By the Hilbert Nullstellensatz $x_i \in \operatorname{Rad} I$, with I the ideal of $\overline{\mathbb{K}}[x_1, \dots, x_n]$ generated by $\{f_i\}$ and $\overline{\mathbb{K}}$ the algebraic closure of $\mathbb{K}$. Hence, the cohomology class $[x_i]$ is nilpotent for any i, and therefore by [10], $H^*(\Lambda V, d)$ (with coefficients in $\overline{\mathbb{K}}$, and therefore over $\mathbb{K}$ too!) is finite dimensional. ■

In the same setting, if w is the representative given in proposition 2 of $(\Lambda V, d)$, the model associated to the given system, we get:

Corollary 5. $[w] = 0$ *if and only if the system has non trivial solutions in a finite algebraic extension of* $\mathbb{K}$.

Examples. (1) Over $\mathbb{Z}_5$ the system

$$\left.\begin{aligned} x^2 - 3y^2 &= 0 \\ x^4 - 9y^4 &= 0 \end{aligned}\right\}$$

has only the trivial solution. Otherwise $y \ne 0$ and $(x/y)^2 = 3$ which is a contradiction since there is no square root of 3 in $\mathbb{Z}_5$. On the other hand, proceeding as in proposition 2 and choosing $\Phi_1 = x\overline{x} - 3y\overline{y}$, $\Phi_2 = x^3\overline{x} - 9y^3\overline{y}$, the fundamental class of the model associated to this system is 0. Indeed the system has a non trivial solution over the extension $\frac{\mathbb{Z}_5[u]}{u^2-3}$.

(2) Consider the system

$$\left.\begin{aligned} 6x^3 + y^3 &= 0 \\ y^2 + 4x^2 + 2xy &= 0 \end{aligned}\right\}$$

and choose $\Phi_1 = 6x^2\overline{x} + y^2\overline{y}$, $\Phi_2 = y\overline{y} + (4x + 2y)\overline{x}$ to obtain $w = 6x^2y - 4xy^2 - 2y^3$. To see whether this cycle is a boundary we solve:

$$\lambda(6x^3 + y^3) + (\beta_1 x + \beta_2 y)(4x^2 + 2xy + y^2) = 6x^2y - 4xy^2 - 2y^3$$

and we get

$$\left.\begin{aligned} 6\lambda + 4\beta_1 &= 0 \\ \lambda + \beta_2 &= -2 \\ 2\beta_1 + 4\beta_2 &= 6 \\ \beta_1 + 2\beta_2 &= -4 \end{aligned}\right\}$$

The last two equations imply that $\overline{14} = \overline{0}$, and the system only have non trivial solutions in fields of characteristic different from 2 and 7. For instance, $x = 1, y = 0$; $x = y \neq 0$ and $x = 2, y = 2$ are solutions in $\mathbb{Z}_2$ and $\mathbb{Z}_7$.

Next we shall describe a process for the computation of the fundamental class of any elliptic space. Let $(\Lambda V, d)$ be an elliptic minimal model and $(\Lambda V, d_\sigma)$ its associated pure model [10], i.e., $dv = d_\sigma v + \varphi$, $\varphi \in \Lambda^+ V^{\text{odd}} \otimes \Lambda V^{\text{even}}$. Observe that w_0, obtaining as in proposition 2 and representing the fundamental class of $(\Lambda V, d_\sigma)$, lives in $(\Lambda V^{\text{even}} \otimes \Lambda^{m-n} V^{\text{odd}})^N$ in which: $m = \dim V^{\text{odd}}$, $n = \dim V^{\text{even}}$ and N is the formal dimension.

Write $M_j^i = (\Lambda V^{\text{even}} \otimes \Lambda^j V^{\text{odd}})^i$, $p = m - n$, and observe that

$$dw_0 = \alpha_1^0 + \alpha_3^0 + \cdots + \alpha_k^0, \quad \alpha_i^0 \in M_{p+i}^{N+1}, \quad k \leq N/3 - p.$$

Since $d^2 w_0 = 0$ it follows that $d_\sigma \alpha_1^0 = 0$ (indeed this is the only summand of $d^2 w_0$ in M_{p+1}^*). Hence α_1^0 is a d_σ-boundary: $d_\sigma \beta_1 = \alpha_1^0$, $\beta_1 \in M_{p+2}^N$. Consider $w_1 = w_0 - \beta_1$ and note that

$$dw_1 = \alpha_3^1 + \alpha_5^1 + \cdots + \alpha_k^1, \quad \alpha_i^1 \in M_{p+i}^{N+1}, \quad k \leq N/3 - p.$$

Again, for the same reason, $d_\sigma \alpha_3^1 = 0$, so there exists $\beta_2 \in M_{p+4}^N$ such that $d_\sigma \beta_2 = \alpha_3^1$. Hence we define inductively elements $w_j, \beta_j \in (\Lambda V)^N$ satisfying $w_j = w_{j-1} - \beta_j$ and $dw_j \in \sum_{i=2j+1}^k M_{p+i}^{N+1}$. Hence, for the first j_o such that $2j_o + 1 > k$ this process stops and w_{j_o} is a d-cycle which we denote by w. Then we have:

Proposition 6. *w represents the fundamental class of* $(\Lambda V, d)$.

Proof: Recall [10] that the *odd spectral sequence* is obtained from a filtration of $(\Lambda V, d)$ by $F^{p,q} = \sum_{j+q \geq 0} M_j^{p+q}$. This defines a spectral sequence of algebras of the first and second quadrant whose (E_0, d_0)-term is precisely $(\Lambda V, d_\sigma)$ and which converges to $H^*(\Lambda V, d)$. Since $(\Lambda V, d)$ is elliptic then each of the terms of the odd spectral sequence is a Poincaré duality algebra (see [10]) with the same formal dimension. Hence, w_0 representing the fundamental class of (E_0, d_0) survives until E_∞. Finally it is a straightforward calculation to show that the element w_j represents the fundamental class of E_{2j} and therefore the theorem follows. ∎

Example. Let $(\Lambda V, d)$ be the model defined by: $V^{\text{even}} =< x_1, x_2, x_3 >$, $V^{\text{odd}} =< y_1, y_2, y_3, y_4, y_5 >$, $|x_i| = 2$, $dx_i = 0$, $|y_1| = |y_2| = 3$, $|y_3| = |y_4| = 5$, $|y_5| = 15$, $dy_i = x_i^2$, $1 \leq i \leq 3$, $dy_4 = x_1 x_2 x_3$, $dy_5 = -x_1^2 x_2 x_3^2 y_1 y_2 + x_1 x_2^2 x_3 y_1 y_4 - x_1^3 x_3 y_2 y_4$.

The associated pure model satisfies the same equations except that $dy_5 = 0$. The fundamental class of this space is represented by

$$w_0 = x_1^2 x_2^2 y_3 y_5 - x_1 x_2 x_3^2 y_4 y_5$$

and $dw_0 = x_1^4 x_2^3 x_3^2 y_1 y_2 y_3 - x_1^3 x_2^2 x_3^4 y_1 y_2 y_4 + x_1^3 x_2^4 x_3 y_1 y_3 y_4 - x_1^5 x_2^2 x_3 y_2 y_3 y_4$. As explained in the procedure above we calculate $\beta_1 = -x_1^3 x_2^2 x_3 y_1 y_2 y_3 y_4$. Since $w_1 = w_0 - \beta_1$ is already a cocycle the process stops and

$$w_1 = x_1^2 x_2^2 y_3 y_5 - x_1 x_2 x_3^2 y_4 y_5 + x_1^3 x_2^2 x_3 y_1 y_2 y_3 y_4$$

represents the fundamental class of $(\Lambda V, d)$.

3 The graph coloring problem

Let G be a graph with vertices $V(G) = \{v_1, \dots, v_n\}$ and edges $E(G) = \{(v_r, v_s)\}_{(r,s)\in J}$. In [14] we associate to G and any integer $k \geq 2$ the rational space $S_{G,k}$ whose Sullivan model (which is minimal if $k \geq 3$) $(\Lambda V_{G,k}, d)$ is defined as follows:

$$V_{G,k}^{\text{even}} = \langle x_1, \dots, x_n \rangle, \quad i = 1, \dots, n, \quad |x_i| = 2, \quad dx_i = 0,$$

$$V_{G,k}^{\text{odd}} = \langle y_{(r,s)} \rangle, \quad (r,s) \in J, \quad |y_{(r,s)}| = 2k-3, \quad dy_{(r,s)} = \sum_{l=1}^{k} x_r^{k-l} x_s^{l-1}.$$

Then we prove (see [14]) that G is k-colorable if and only if $S_{G,k}$ is not elliptic. As the space $V_{G,k}$ is finite dimensional we may apply corollary 3 to obtain:

Corollary 7. *G is k-colorable if and only if $[w] = 0$ (with w, as in proposition 2, representing the fundamental class of $H^*(\Lambda V, d)$).*

Example. This easy example shows how this method works: Let $V(G) = \{v_1, \dots, v_n\}$, $E(G) = \{(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n), (v_n, v_1)\}$ be the vertices and edges of the cycle G. The space $S_{G,2}$ has the model $(\Lambda V_{G,2}, d)$ with $V_{G,2}^{\text{even}} = \langle x_1, \dots, x_n \rangle$, $V_{G,2}^{\text{odd}} = \langle y_1, \dots, y_n \rangle$ $|x_i| = 2$, $|y_i| = 1$, $1 \leq i \leq n$, $dy_i = x_i + x_{i+1}$, $i < n$, $dy_n = x_n + x_1$.

Following any of the procedures given in the former section we obtain that a representative of the fundamental class is $w = 1 + (-1)^{n-1}$, and therefore G is 2-colorable if and only if n is even.

Now it is a good moment to very briefly recall some basic terminology in complexity: A decision problem may be seen as a function $f\colon \Pi \to \{0,1\}$ in which $\Pi = \{I_\alpha\}_{\alpha\in\Gamma}$ is a family of finite subsets of non negative integers with each I_α representing the codification of an instance of the problem. The language of a decision problem is the set of instances I for which the answer is yes, i.e., $f(I) = 1$. A decision problem $f\colon \Pi \to \{0,1\}$ (or simply Π for convenience) belongs to the class P (polynomial) if there is an algorithm $\mathcal{A}$ and a polynomial p such that for each instance $I \in \Pi$ of length n, $\mathcal{A}$ produces $f(I)$ in a number of steps bounded by $p(n)$. A problem Π belongs to the class NP (nondeterministic polynomial) if there is an algorithm $\mathcal{A}$ and a polynomial p such that: given an instance $I \in \Pi$ of length n and a certificate (a possible solution of the problem for the instance I) $C \in \mathbb{N}$, the algorithm $\mathcal{A}$ determines whether C is in fact a solution for I in a number of steps bounded above by $p(n)$. That is to say, P corresponds to the class of problems which are 'easy' to solve while NP is the class of problems for which it is 'easy' to validate a given solution. P$\subset$NP and it is widely accepted in complexity the conjecture that this inclusion is strict.

A map between problems $T\colon \Pi \to \Pi'$ is a Turing or polynomial reduction if: (i) $T(I)$ belongs to the language of Π' if and only if I is an instance of the language of Π; (ii) there exists a polynomial p such that for each $I \in \Pi$ the length of $T(I)$ is bounded by p evaluated in the length of I. If two problems can be Turing reduced one into the other we say that these two problems are Turing or polynomially equivalent.

A problem $\Pi \in$NP is NP-complete if any other problem in NP can be polynomially reduced to Π. Hence, an algorithm that solves an NP-complete problem would also solve any other problem in NP in the same range of time. If any problem in NP may be polynomially reduced to a problem Π, then Π is said to be NP-hard. Note that NP-hard problems do not necessarily lie in NP but, obviously, any NP-complete problem is NP-hard. Again it is not known whether this inclusion is strict.

The problem of k-coloring graphs is known to be NP-complete (see for example [8]) with respect to the usual codifications of a graph, all of them of length polynomially bounded by n the number of vertices.

Returning to our study, if we see the above corollary as an algorithm to decide whether a given graph can be k-colored, we are facing two complexity problems: one is to calculate w itself and the other is to determine whether this cycle is a boundary or not. It is convenient then to give a codification of a given model in order to study the complexity of these problems [14]: Denote by Γ_k the class of minimal models $(\Lambda V, d)$ for which $dV \subset \Lambda^{<k} V$ (It contains in particular $(\Lambda V_{G,k}, d)$ for any graph G). Any space $(\Lambda V, d) \in \Gamma_k$ is encoded as follows: choose $\{v_1, \ldots, v_m\}$ a homogeneous basis of V and write $dv_j = \sum_{p<k} \lambda^j_{i_1,\ldots,i_p} v_{i_1} \cdots v_{i_p}$, with $1 \le i_1 \le \cdots \le i_p \le m$. The codification string of $(\Lambda V, d)$ consists of m, the degree of each v_j, and the list $\lambda^j_{i_1,\ldots,i_p}$ of the coefficients of the differential. It is important to note that the number of these coefficients is at most $m \sum_{p=2}^{k-1} \frac{(m+p-1)!}{(p-1)!m!}$, clearly bounded by a polynomial in m. In particular the codification of $(\Lambda V_{G,k}, d)$, the model associated to G, has a length bounded polynomially by $n = \dim V^{\text{even}}$ =number of vertices of G.

For the first of the problems cited above, if we look at the formula given in [15] (see also the previous section) observe that the "time" needed to compute w is of exponential order with respect to $\dim V^{\text{even}}$ and thus with respect to the length of the codification of the data. However, we shall show how to modify the space associated to the graph to be able, using the procedure given by proposition 2, to compute w in polynomial time with respect to the model, and therefore to the codification of the graph.

As before, let G be a graph with vertices $V(G) = \{v_1, \ldots, v_n\}$ and edges $E(G) = \{(v_i, v_j)\}_{(i,j)\in I}$, and let T be a spanning tree, i.e., a tree that contains all the vertices. We point out that for our purposes it will be crucial the fact that finding a spanning tree take a polynomial amount of time [8]. Choose v_1 to be the "root" of the tree and, for any $k \ge 2$, define the space $S_{G,T}$ as the rational pure space whose model (minimal if $k \ge 3$) is as follows:

$$\begin{aligned} V^{\text{even}}_{G,T} &= \langle x_1, \ldots, x_n, t\rangle, \quad |x_i| = |t| = 2, \quad i = 1, \ldots, n, \\ V^{\text{odd}}_{G,T} &= \langle y_{i,j}, z_{r,s}, u\rangle, \quad (i,j) \in E(T), \quad (r,s) \in E(G) - E(T), \end{aligned}$$

$$|y_{i,j}| = |u| = 2k-3, \quad |z_{r,s}| = 2k-1,$$
$$dx_i = dt = 0, \quad dy_{i,j} = \sum_{l=1}^{k} x_i^{k-l} x_j^{l-1},$$
$$dz_{r,s} = t\sum_{l=1}^{k} x_r^{k-l} x_s^{l-1}, \quad du = x_1^{k-1} - t^{k-1}.$$

Then, we prove:

Theorem 8. *G is k-colorable if and only if $S_{G,T}$ is not elliptic.*

Proof: Assume G k-colorable. Then, by [14], the model $(\Lambda V_{G,k}, d)$ is not elliptic, which is in turn equivalent (see Proposition 4 or the proof of [14, Theorem 3]) to assert that the system

$$\{\sum_{l=1}^{k} v_i^{k-l} v_j^{l-1} = 0,\ (i,j) \in E(G)\} \qquad (*)$$

have non trivial solutions in $\mathbb{C}$. If $\{s_1, \ldots, s_n\}$ is such a solution then obviously $\{s_1, \ldots, s_n, r\}$, with $r = s_1$, is a solution of the system

$$\begin{cases} \sum_{l=1}^{k} v_i^{k-l} v_{i+1}^{l-1} = 0, & 1 \le i \le n-1, \\ t\sum_{l=1}^{k} v_r^{k-l} v_s^{l-1} = 0, & (r,s) \in E(G) - E(T), \\ v_1^{k-1} - t^{k-1} = 0. \end{cases} \qquad (**)$$

Hence, again by [14], $S_{G,T}$ is not elliptic.

Conversely, assume that $S_{G,T}$ is not elliptic, i.e., the system $(**)$ has a non trivial solution $\{s_1, \ldots, s_n, r\}$. Then $r \neq 0$. Otherwise $s_1 = 0$ and, in view of the first set of equations in $(**)$, the solution would be trivial. But, if $r \neq 0$, then $\{s_1, \ldots, s_n\}$ is a non trivial solution of $(*)$ and therefore G is k-colorable. ∎

Now we are able to state a formula which gives, in polynomial time, a representative of the fundamental class of $S_{G,T}$:

Theorem 9. *The fundamental class of $S_{G,T}$ is represented by the cycle*

$$w = \frac{1}{t}\prod_{i=1}^{n} x_i^{k-2} \cdot d\Big(\prod_{(r,s)\in E(G)-E(T)} z_{r,s}\Big).$$

In particular, G is colorable if and only if $[w] = 0$

Proof: To follow the procedure established in proposition 2 we need to choose elements $\phi_{i,j}, \psi_{r,s}, \varphi \in (\Lambda V_{G,T}^{\text{even}} \otimes \Lambda \overline{V_{G,T}^{\text{even}}}, d)$, such that $d\phi_{i,j} = dy_{i,j}$, $d\psi_{r,s} = dz_{r,s}$, and $d\varphi = du$. Let us do it wisely: For each leaf or terminal vertex of the tree v_j let v_i be the only vertex for which $(v_i, v_j) \in E(T)$ and define

$$\phi_{i,j} = \overline{x}_i \sum_{l=2}^{k} x_i^{k-l} x_j^{l-2} + \overline{x}_j x_j^{k-2}.$$

Next, we eliminate from T the terminal leaves and we apply the same procedure to the new tree T'. At the end of this process all the elements $\phi_{i,j}$, $(i,j) \in E(T)$, are defined. Finally set:

$$\psi_{r,s} = \overline{t} \sum_{l=1}^{k} x_r^{k-l} x_s^{l-1}, \qquad (r,s) \in E(G) - E(T),$$
$$\varphi = x_1^{k-2}\overline{x}_1 - t^{k-2}\overline{t}.$$

Then, in view of proposition 2 a representative of the fundamental class of $S_{G,T}$ is the coefficient of $\overline{t}\overline{x}_1 \dots \overline{x}_n$ in:

$$(u-\varphi) \prod_{(i,j)\in E(T)} (y_{i,j} - \phi_{i,j}) \prod_{(r,s)\in E(G)-E(T)} (z_{r,s} - \psi_{r,s}).$$

The factor $\overline{x}_j$ corresponding to the leaves of the tree appears in $\phi_{(i,j)}$ with coefficient x_j^{k-2} (being (v_i, v_j) the only edge of the tree containing v_j).

Again, the factor $\overline{x}_q$ corresponding to the leaves of T' only appears in $\phi_{p,q}$, (v_p, v_q) the only edge of T' containing v_q, with coefficient x_q^{k-2}.

In this way we obtain the coefficients of every $\overline{x}_i$, $i = 2, \dots, n$. Finally,$\overline{x}_1$ only appears in φ with coefficient x_1^{k-2}, and $\overline{t}$ is in every $\psi_{r,s}$, $(r,s) \in E(G) - E(T)$, with coefficient $\sum_{l=1}^{k} x_r^{k-l} x_s^{l-1}$. Hence, the theorem follows. ∎

It is a well known fact in complexity that given $k \geq 3$ the problem Π of determining whether a graph can be k-colored is NP-complete. But in view of theorem above we can reduced this problem to the problem Π' of determining whether a given cycle is a boundary. Indeed define a transformation $T\colon \Pi \to \Pi'$ which assigns to each $I \in \Pi$ representing a graph G of n vertices, the instance $T(I)$ given by the cycle w, as in theorem above, representing the fundamental class of $S_{G,T}$. This class is encoded by the formal dimension together with the formula given in theorem 9 whose length is also obviously bounded by a polynomial in n. Hence, by theorem 8, and taking into account that obtaining a spanning tree in a graph requires a polynomial amount of time [8], T is a polynomial reduction and therefore we get the following surprising consequence:

Corollary 10. *Determining whether a cycle of the formal dimension is a boundary is an NP-hard problem.*

Remark. As we pointed out in the first section (see [15]) the class $[w]$ of a pure model $(\Lambda V, d)$ is a basis of the image of the evaluation map [7], $\mathrm{ev}_{(\Lambda V,d)}\colon \mathrm{Ext}_{\Lambda V}(\mathbb{K}, \Lambda V) \to H^*(\Lambda V, d)$ in which $\mathrm{Ext}_{\Lambda V}(\mathbb{K}, \Lambda V)$ is Gorenstein, i.e., it has dimension 1. Therefore, we can see this object carrying the solution of the decision problem of k-coloring graphs.

Next, we shall modify again the space associated to a graph (and the integer 3) so that it has null homotopy Euler characteristic. From this, we characterize the 3-coloring problem in terms of commutative algebra. As complexity is concerned there is no loss of generality in restricting to the 3-coloring problem since it is well known that the k-coloring problem, for any $k \geq 3$, can be polynomially reduced to the case $k = 3$ [5].

As before, Let G be a graph with vertices $V(G) = \{v_1, \dots, v_n\}$ and edges $E(G) = \{(v_i, v_j)\}$. We define the rational space S_G whose Sullivan minimal model $(\Lambda V, d)$ is given by:

$$\begin{aligned}
V_G^{\text{even}} &= \langle x_i \rangle, \quad |x_0| = 4, |x_i| = 2, \quad i = 1, \dots, n, \\
V_G^{\text{odd}} &= \langle y_i \rangle, \quad |y_0| = 11, |y_i| = 5, \quad i = 1, \dots, n, \\
dx_i &= 0, \\
dy_i &= x_i(x_i^2 - x_0), \quad i = 1, \dots, n, \\
dy_0 &= x_0 \sum_{(i,j)\in E(G)} (x_i^2 + x_i x_j + x_j^2 - x_0)^2.
\end{aligned}$$

Again, we can prove:

Theorem 11. *G is 3-colorable if and only if S_G is not elliptic.*

Proof. As in proposition 4 or in the proof of [14, Theorem 3] we first remark that S_G is elliptic if and only if the system

$$\begin{cases} x_i(x_i^2 - x_0) = 0, \quad i = 1, \dots, n, \\ x_0 \sum_{(i,j)\in E(G)} (x_i^2 + x_i x_j + x_j^2 - x_0)^2 = 0, \end{cases}$$

has only the trivial solution over $\mathbb{C}$. Assume the graph G to be 3-colorable and let $f\colon V(G) \to \{1, 0, -1\}$ be a coloring, that is to say, a map for which $f(x_i) \neq f(x_j)$ if $(i, j) \in E(G)$. Then $s_0 = 1$ and $s_i = f(x_i)$, $i = 1, \dots, n$, is a solution of the system: trivially the first set of equations is satisfied. For the last equation observe that if $(i, j) \in E(G)$, then $s_i - s_j \neq 0$ and $s_i^2 + s_i s_j + s_j^2 = (s_i^3 - s_j^3)/(s_i - s_j) = 1$.

Conversely, given $\{s_0, \dots, s_n\}$ a non trivial solution of the system above, then, in view of the first set of equations, $s_0 \neq 0$. Thus we may assume $s_0 = 1$ and therefore $s_i \in \{1, 0, -1\}$ for any i. Hence, the map $f\colon V(G) \to \{1, 0, -1\}$, $f(x_i) = s_i$, is a coloring: Indeed, in view of the last equation $s_i^2 + s_i s_j + s_j^2 = 1$ for any $(i, j) \in E(G)$, and thus $s_i \neq s_j$. ■

As we said the advantage of this space is that it has null homotopy characteristic so its fundamental class lives in $H_0(\Lambda V, d)$. Indeed, consider the adequate election of Φ_i:

$$\begin{aligned}
\Phi_i &= x_i^2(\overline{x}_i - \overline{x}_0) \quad 1 \leq i \leq n, \\
\Phi_0 &= \overline{x}_0 \sum_{(i,j)\in E(G)} (x_i^2 + x_i x_j + x_j^2 - x_0)^2.
\end{aligned}$$

Then, applying the method above for calculating the fundamental class $[w]$ of $H^*(\Lambda V_G, d)$ we obtain (up to a sign!):

$$w = \prod_{i=1}^{n} x_i^2 \sum_{(i,j)\in E(G)} (x_i^2 + x_i x_j + x_j^2 - x_0)^2.$$

Thus, if we consider in $\mathbb{K}[x_0, \dots, x_n]$ the ideal J generated by $x_i(x_i^2 - x_0)$, $i = 1, \dots, n$, and $x_0 \sum_{(i,j)\in E(G)} (x_i^2 + x_i x_j + x_j^2 - x_0)^2$, we deduce the following:

Corollary 12. *G is 3-colorable if and only if $\prod_{i=1}^n x_i^2 \sum_{(i,j)\in E(G)} (x_i^2 + x_i x_j + x_j^2 - x_0)^2 \in J$.*

Proof. By theorem 11 G is 3-colorable if and only if S_G is not elliptic which, in view of observation above, is equivalent to the vanishing of the class $[w]$. Finally, since $[w] \in H_0^*(\Lambda V_G, d)$, the result follows. ■

Note that, since $(\Lambda V_G, d)$ has null homotopy Euler characteristic, then $H^*(\Lambda V_G, d) = H_0(\Lambda V_G, d) = \mathbb{K}[x_0, \ldots, x_n]/J$. Hence, by theorem 11, F is satisfiable if and only if $H_0(\Lambda V_G, d)$ is infinite dimensional, which is equivalent to say that J is not generated by a regular sequence. However $\{x_i(x_i^2 - x_0)\}$, $i = 1, \ldots, n$, obviously constitutes a regular sequence in J. Thus, if we consider in $\mathbb{K}[x_0, \ldots, x_n]$ the ideal $I = \langle x_i(x_0 - x_i^2)\rangle$, $1 \le i \le n$, we have:

Corollary 13. *G is 3-colorable if and only if there exists $\alpha \in \mathbb{K}[x_0, \ldots, x_n] - I$ such that*

$$\alpha \sum_{(i,j)\in E(G)} (x_i^2 + x_i x_j + x_j^2 - x_0)^2 \in I.$$

Finally we are able to produce another interesting characterization:

Corollary 14. *G is 3-colorable if and only if*

$$\prod_{(i,j)\in E(G)} (x_i - x_j) \notin I.$$

Proof. Let G be 3-colorable and let $f\colon V(G) \to \{1, 0, -1\}$ be such a coloring. As in the proof of theorem 11 we observe that $s_0 = 1$ and $s_i = f(x_i)$, $i = 1, \ldots, n$, is a solution of the system. Hence any element of the ideal I evaluated in the given solution is zero while $\prod_{(i,j)\in E(G)}(s_i - s_j) \ne 0$. Thus $\prod_{(i,j)\in E(G)}(x_i - x_j) \notin I$.

Conversely, suppose that $\prod_{(i,j)\in E(G)}(x_i - x_j) \notin I$. Observe that

$$\begin{aligned}
&\sum_{(i,j)\in E(G)} (x_i^2 + x_i x_j + x_j^2 - x_0)^2 \prod_{(i,j)\in E(G)} (x_i - x_j) = \\
&\sum_{(i,j)\in E(G)} (x_i - x_j)(x_i^2 + x_i x_j + x_j^2 - x_0)^2 \prod_{(k,l)\ne(i,j)} (x_k - x_l) = \\
&\sum_{(i,j)\in E(G)} \Big((x_i^3 - x_i x_0) - (x_j^3 - x_j x_0)\Big)(x_i^2 + x_i x_j + x_j^2 - x_0) \prod_{(k,l)\ne(i,j)} (x_k - x_l) \in I.
\end{aligned}$$

Hence, by corollary above, G is colorable. ■

Remark. Observe that, in view of proposition 4 and corollary 5, we do not need to assume the ground field $\mathbb{K}$ of characteristic in corollaries 12, 13 and 14.

4 Other decision problems

In this final section we shall associate rational spaces to the satisfiability problem and the subset sum problem so that these spaces are not elliptic if and only if the given problems have solutions. Analogous results to theorem 11 and corollaries 12, 13 are obtained.

We begin by recalling the satisfiability problem: Let $\{l_1, \dots, l_n\}$ be a fixed set of variables. A *literal* is a variable l_i or its negation $\overline{l}_i$. A *clause* is a finite set of literals and a *propositional formula* is a finite set of clauses. Given a propositional formula, the *satisfiability problem* consists in deciding whether it is possible to assign logical (or boolean) values to each of the variables so that the formula is satisfied. This was the first problem that was proved to be NP–complete [5]. Also it is a well known fact [8] that this problem can be reduced in polynomial time to the 3–satisfiability problem, i.e., assuming that in each clause there are at most three literals. Thus, in terms of computability complexity there is no loss of generality in considering the 3–satisfiability problem henceforth.

Let $\{l_1, \dots, l_n\}$ be the set of logical variables and let $F = \{c_1, \dots, c_m\}$ be a propositional formula where each of the clauses $c_j = \{\alpha_{j1}, \alpha_{j2}, \alpha_{j3}\}$ is formed by literals which are either l_i or $\overline{l}_i$. Usually, in conjunctive normal form, F is also written as:

$$F = (\alpha_{11} \vee \alpha_{12} \vee \alpha_{13}) \wedge \dots \wedge (\alpha_{m1} \vee \alpha_{m2} \vee \alpha_{m3}),$$

An *assignment of logical or boolean values* is a map $f\colon \{l_1, \dots, l_n \overline{l}_1, \dots, \overline{l}_n\} \to \{0,1\}$ for which $f(l_i) + f(\overline{l}_i) = 1$. If $f(l_i) = 1$ we say that l_i is true and if $f(l_i) = 0$ we say that l_i is false. Hence l_i is true (respec. false) if and only if $\overline{l}_i$ is false (respec. true). F is *satisfiable* if there exists an assignment f for which F is true, i.e., $\prod_{j=1}^m \Big(\max_{k=1,2,3} f(\alpha_{jk})\Big) = 1$.

Definition 15. *Given a propositional formula F as before, define the rational space S_F whose minimal model $(\Lambda V_F, d)$ is as follows:*

$$\begin{aligned}
V_F^{even} &= \langle x_i \rangle, \quad i = 0, \dots, n, \quad |x_i| = 2,\\
V_F^{odd} &= \langle y_i \rangle, \quad i = 0, \dots, n, \quad |y_i| = 3, \quad i \neq 0, \quad |y_0| = 7.\\
dx_i &= 0,\\
dy_i &= x_i(x_0 - x_i), \quad i = 1, \dots, n,\\
dy_0 &= x_0 \sum_{j=1}^m \phi(\alpha_{j1})\phi(\alpha_{j2})\phi(\alpha_{j3}),
\end{aligned}$$

in which

$$\phi(\alpha) = \begin{cases} x_0 - x_i & \text{if} \quad \alpha = l_i, \\ x_i & \text{if} \quad \alpha = \overline{l}_i. \end{cases}$$

Then we prove:

Theorem 16. *F is satisfiable if and only if the space S_F is not elliptic.*

Proof. Again, As in the proof of [14, Theorem 3] observe that S_F is elliptic if and only if the system

$$\begin{cases} x_i(x_0 - x_i) = 0, \quad i = 1, \dots, n, \\ x_0 \cdot \sum_{j=1}^m \prod_{k=1}^3 \phi(\alpha_{jk}) = 0, \end{cases}$$

has only the trivial solution over $\mathbb{C}$. Let $f\colon \{l_1, \dots, l_n \overline{l}_1, \dots, \overline{l}_n\} \to \{0,1\}$ be a boolean assignment for which F is satisfied. Then $s_i = f(l_i)$, $s_0 = 1$, is a non trivial solution of the above system: trivially $s_i(s_0 - s_i) = 0$, $1 \leq i \leq n$. On the other hand assume that there exists a clause c_j for which (in the values of the solution) $\phi(\alpha_{jk}) \neq 0$ for each k. Hence, either $\alpha_{jk} = l_i$ and $s_i = 0$ so that $f(l_i) = f(\alpha_{jk}) = 0$; or $\alpha_{jk} = \overline{l}_i$ and thus $s_i = 1$ so that $f(\overline{l}_i) = f(\alpha_{jk}) = 0$. Therefore c_j is not satisfied, which is a contradiction.

On the other hand let $\{s_0, \dots, s_n\}$ be a non trivial solution of the above system. Then $s_0 \neq 0$, otherwise, in view of the first set of equations, all s_i would also be null for $i = 1, \dots, n$. Hence $s_i\prime = s_i/s_0, \quad 0 \leq i \leq n$, is another solution for which $s_i\prime \in \{0, 1\}$. Finally, the assignment $f \colon \{l_1, \dots, l_n \, \overline{l}_1, \dots, \overline{l}_n\} \to \{0, 1\}$, $f(l_i) = s_i\prime$ makes F true. Indeed, for each clause c_j there exists at least a literal α for which $\phi(\alpha) = 0$. Then either:

(a) $\phi(\alpha) = s_i\prime = 0$ for some i. Thus, $\alpha = \overline{l}_i$ while $f(l_i) = 0$. Hence, $f(\alpha) = 1$ and the clause is satisfied; or

(b) $\phi(\alpha) = s_0\prime - s_i\prime = 0$ for some i. Then, $\alpha = l_i$ and $f(l_i) = s_i\prime = 1$, i.e., $f(\alpha) = 1$ and again c_j is satisfied. ∎

Next, observe that the space S_F associated to a formula F is pure. Hence choosing $\Phi_i = x_i(\overline{x}_0 - \overline{x}_i)$, $i = 1, \dots, n$, $\Phi_0 = \overline{x}_0 \cdot \sum_{j=1}^{m} \prod_{k=1}^{3} \phi(\alpha_{jk})$, and applying the process above we get (again up to a sign):

$$w = (\prod_{i=1}^{n} x_i) \sum_{j=1}^{m} \prod_{k=1}^{3} \phi(\alpha_{jk}).$$

Then, as in corollary 12, if we denote by J the ideal of $\mathbb{K}[x_0, \dots, x_n]$ generated by $\{dy_i\}$, $i = 0, \dots, n$, we deduce the following:

Corollary 17. *The formula F is satisfiable if and only if $w \in J$.*

Note that, since $(\Lambda V_F, d)$ has null homotopy Euler characteristic, using the same argument as in corollary 13, we get:

Corollary 18. *The formula F is satisfiable if and only if there exists $\alpha \in \mathbb{K}[x_0, \dots, x_n] - I$ such that:*

$$\alpha \cdot \sum_{j=1}^{m} \prod_{k=1}^{3} \phi(\alpha_{jk}) \in I.$$

Finally, analogous results can also be obtained with respect to the *subset sum* problem: Given a subset $B \subset \mathbb{N}$ and an integer $n_0 \in \mathbb{N}$, does there exist $A \subset B$ such that $\sum_{a \in A} a = n_0$? This problem is also known to be NP-complete [8]. Then, we may associate to the data $n_0, B = \{a_1, \dots, a_n\}$ the rational space S_{B,n_0} whose minimal model is the pure KS-complex $(\Lambda V_{B,n_0}, d)$ given by:

$$\begin{aligned} V_{B,n_0}^{\text{even}} &= \langle x_i \rangle, \quad i = 0, \dots, n, \quad |x_i| = 2, \\ V_{B,n_0}^{\text{odd}} &= \langle y_i \rangle, \quad i = 0, \dots, n, \quad |y_i| = 3. \\ dx_i &= 0, \\ dy_i &= x_i(x_0 - x_i), \quad i = 1, \dots, n, \\ dy_0 &= x_0(\sum_{i=1}^{n} a_i x_i - n_0 x_0). \end{aligned}$$

Then, we prove

Theorem 19. *The subset sum problem for B and n_0 has a solution if and only if S_{B,n_0} is not elliptic.*

Proof. Again, S_{B,n_0} is not elliptic is and only if the system

$$\begin{cases} x_i(x_0 - x_i) = 0, \quad i = 1, \dots, n, \\ x_0(\sum_{i=1}^n a_i x_i - n_0 x_0) = 0, \end{cases}$$

has non trivial solutions. Assume that the subset sum problem has a solution and let $A \subset B$ be such that $\sum_{a \in A} a = n_0$. Then

$$\begin{aligned} s_i &= 1, \quad \text{if} \quad i = 0 \quad \text{or} \quad a_i \in A, \\ s_i &= 0 \quad \text{otherwise}, \end{aligned}$$

is trivially a solution of the system. Conversely, given a non trivial solution $\{s_0, \dots, s_n\}$, we see that $s_0 \neq 0$. Thus we may assume that $s_0 = 1$ and $s_i \in \{0, 1\}$, for any i. Then $\sum_{j | s_j = 1} a_j = n_0$. ∎

As before, we now find a representative w of the fundamental class of $H^*(\Lambda V_{B,n_0}, d)$. For that we define

$$\begin{aligned} \Phi_i &= x_i(\overline{x}_i - \overline{x}_0), \quad 1 \leq i \leq n, \\ \Phi_0 &= \overline{x}_0(\sum_{i=1}^n a_i x_i - n_0 x_0), \end{aligned}$$

to obtain (up to a sign)

$$w = \prod_{i=1}^n x_i \, (\sum_{i=1}^n a_i x_i - n_0 x_0).$$

Hence, we deduce that the subset sum problem has a solution if and only if the fundamental class $[w] = 0$, which is in turn equivalent to the following: Denote by J the ideal of $\mathbb{K}[x_0, \dots, x_n]$ generated by $x_i(x_i - x_0)$, $i = 1, \dots, n$, and $x_0 \, (\sum_{i=1}^n a_i x_i - n_0 x_0)$. Then,

Corollary 20. *The subset sum problem for B and n_0 has a solution if and only if $w \in J$.*

Again, let $I \subset \mathbb{K}[x_0, \dots, x_n]$ be the ideal generated by $\{x_i^2 - x_i x_0\}$, $i = 1, \dots, n$. Then we have:

Corollary 21. *The subset sum problem for B and n_0 has a solution if and only if there exists $\alpha \in \mathbb{K}[x_0, \dots, x_n] - I$ such that*

$$\alpha(\sum_{i=1}^n a_i x_i - n_0 x_0) \in I.$$

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