Global existence and energy decay of solutions for Kirchhoff-Carrier equations with weakly nonlinear dissipation

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Abstract

In this paper we prove the global existence and study decay property of the solutions to the initial boundary value problem for the solutions to the quasilinear wave equation of Kirchhoff-Carrier type with a general weakly nonlinear dissipative term by constructing a stable set in $H^2 \bigcap H_0^1$.

1 Introduction

We consider the problem

$$(P) \qquad \begin{cases} u'' - \Phi(\|\nabla_x u\|_2^2)\Delta_x u + \rho(t, u') + f(u) = 0 \text{ in } \Omega \times [0, +\infty[, \\ u = 0 \text{ on } \Gamma \times [0, +\infty[, \\ u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x) \text{ on } \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma$, $\Phi(s)$ is a C^1- class function on $[0, +\infty[$ satisfying $\Phi(s) \geq m_0 > 0$ for $s \geq 0$ with m_0 constant, $\rho(t, v)$ and $f(u)$ are functions like $\sigma(t)g(v)$ with a positively nonincreasing function $\sigma(t)$ on $\mathbb{R}_+$ and an increasing odd function $g(v)$, and $-|u|^\alpha u$, $\alpha \geq 0$.

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For the problem (P), when $\Phi(s) \equiv 1$ and $\rho(t,x) = \delta x$ $(\delta > 0)$, Ikehata and Suzuki [**9**] investigated the dynamics, they have shown that for sufficiently small initial data (u_0, u_1), the trajectory $(u(t), u'(t))$ tends to $(0,0)$ in $H_0^1(\Omega) \times L^2(\Omega)$ as $t \to +\infty$. When $\rho(t,x) = \delta|x|^{m-1}x$ $(m \geq 1)$ and $f(y) = -\beta|y|^{p-1}y$ $(\beta > 0, p \geq 1)$, Georgiev and Todorova [**5**] have shown that if the damping term dominates over the source, then a global solution exists for any initial data. Quite recently, Ikehata [**7**] proved that a global solution exists with no relation between p and m,

In [**2**], Aassila proved the existence of a global decaying H^2 solution when $\rho(t,x) = h(x)$ has not necessarily a polynomial growth near zero and a source term of the form $\beta|y|^{p-1}y$, but with small parameter β. The decay rate of the global solution depends on the polynomial growth near zero of $h(x)$ as it was proved in [**15**], [**16**],[**12**] and [**7**].

When $\Phi(s)$ is not a constant function, the equation with $\rho(t,x) \equiv 0$ and $f(y) \equiv 0$ is often called the wave equation of Kirchhoff type which has been introduced in order to study the nonlinear vibrations of an elastic string by Kirchhoff [**11**] and the existence of global solutions was investigated by many authors (see [**17**], [**10**],[**6**] , [**3**]...). In [**3**], the first author studied the existence of a global decaying solution for mildly degenerate Kirchhoff-Carrier equation ($\Phi(s) = s^\alpha, \alpha \geq 1$ *and* $\|\nabla_x u_0\|_2 \geq 0$) with two dissipatives terms of polynomial form, we proved a polynomial decay of the energy of the solution using a general method on the energy decay introduced by Nakao[**15**]. Unfortunately this method does not seem to be applicable to the case of more general functions ρ.

In [**8**], the authors discussed the existence of a global decaying solution in the case $\Phi(s) = m_0 + s^{\frac{(\gamma+2)}{2}}$, $\gamma \geq 0$, $\rho(t,v) = |v|^r v$, $0 \leq r \leq \frac{2}{(n-2)}$ $(0 \leq r \leq \infty$ if $n = 1,2)$, $f(u) = -|u|^\alpha u$, $0 < \alpha \leq \frac{4}{(n-2)}$ $(0 < \alpha < \infty$ if $n = 1,2)$ by use of a stable set method due to Sattinger [**18**]. But, then, the method in [**8**] cannot be applied to the case $\alpha > \frac{4}{(n-2)}$, which is caused by the construction of stable set in H_0^1. In [**16**](see also [**1**]) Nakao has constructed a stable set in $H_0^1 \bigcap H^2$ to obtain a global decaying solution to the initial boundary value problem for non-linear dissipative wave equations.

Our purpose in this paper is to give a global solvability in the class $H_0^1 \cap H^2$ and energy decay estimates of the solutions to problem (P) for a general non-linear damping ρ and a polynomial non-linear source term. We use some new techniques introduced in [**2**] to derive a decay rate of the solution. So we use the argument combining the method in [**2**] with the concept of stable set in $H_0^1 \cap H^2$. We also use some ideas from [**13**] introduced in the study of the decay rates of solutions to the wave equation $u_{tt} - \Delta u + h(u_t) = 0$ in $\Omega \times \mathbb{R}^+$.

We conclude this section by stating our plan and giving some notations. In section 2 we shall prepare some lemmas needed for our arguments. Section 3 is devoted to the proof of the global existence and decay estimates to the problem (P). Section 4 is devoted to the proof of the global existence and decay estimates to the problem (P) in the case $\alpha = 0$, i.e., $f(u) = -u$. In this case the smallness of $|\Omega|$ (the volume of Ω) will play an essential role in our argument. In the last section we shall treat the case $\Phi \equiv 1$, we prove only the global decaying H_0^1 solution, but we obtain more results than the case when $\Phi \not\equiv 1$. The condition that β (k_1 in our paper) is small is removed here, also we extend some results obtained by Ikehata, Matsuyama [**8**], Aassila [**2**] and Martinez [**13**].

Throughout this paper the functions considered are all real valued. We erase the space variable x of $u(t,x)$, $u_t(t,x)$ and simply denote $u(t,x)$, $u_t(t,x)$ by $u(t)$, $u'(t)$, respectively, when no confusion arises. Let l be a number with $2 \le l \le \infty$. We denote by $\| \,.\, \|_l$ the L^l norm over Ω. In particular, L^2 norm $\| \,.\, \|_2$ is simply denoted $\| \,.\, \|_2^2$. $(\,.\,)$ denotes the usual L^2 inner product. We use familiar function spaces H_0^1, H^2.

2 Preliminaries

Let us state the precise hypotheses on Φ, ρ and f.
(H.1) Φ is a C^1-class function on $\mathbb{R}^+$ and satisfies

$$\Phi(s) \ge m_0 \text{ and } |\Phi'(s)| \le m_1 s^{\frac{\gamma}{2}} \text{ for } 0 \le s < \infty \tag{1}$$

with some $m_0 > 0$, $m_1 \ge 0$ and $\gamma \ge 0$.
(H.2) $\rho(t,v)$ satisfies the following hypotheses: There exists a nonincreasing function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ of class C^1 on $\mathbb{R}_+$ satisfying $\displaystyle\int_0^{+\infty} \sigma(t)\,dt = +\infty$ and a strictly increasing and odd function g of class C^1 on $[-1,1]$ such that $g(v) = v$ for all $|v| \ge 1$ and

$$\forall t \ge 0, \forall v \in \mathbb{R}, \quad \sigma(t)g(|v|) \le |\rho(t,v)| \le g^{-1}\left(\frac{|v|}{\sigma(t)}\right), \tag{2}$$

where g^{-1} denotes the inverse function of g. In particular, this implies that $v \mapsto \rho(t,v)$ has a linear growth at infinity, and that $\sigma(0) \le 1$.

Define

$$H(y) = \frac{g(y)}{y} \tag{3}$$

Note that $H(0) = g'(0)$.

We will study the following cases:
Hyp.1 We assume that (2) is satisfied, and that $g(v) = v$ for all $v \in \mathbb{R}$.
Hyp.2 We assume that (2) is satisfied, and that there exists some $p > 1$ such that $g(v) = v^p$ on $[0,1]$.
Hyp.3 We assume that (2) is satisfied, and that $g'(0) = 0$ and the function H is nondecreasing on $[0,\eta]$ for some $\eta > 0$. (Note $H(0) = 0$.)
(H.3) $f(.)$ belongs to $C^1(\mathbb{R})$ and satisfies (for typical example, we can take $f(u) = -|u|^\alpha u$):

$$|f(u)| \le k_2|u|^{\alpha+1} \text{ and } |f'(u)| \le k_2|u|^\alpha \text{ for } u \in \mathbb{R} \tag{4}$$

with some $k_2 > 0$ and

$$0 < \alpha < \frac{2}{(N-4)^+}, \tag{5}$$

where $(N-4)^+ = \max\{N-4, 0\}$.

We first state three well known lemmas, and then we recall and give the proof of three other lemmas that will be needed later.

Lemma 2.1 (Sobolev-Poincaré inequality). *Let q be a number with $2 \leq q < +\infty$ $(n = 1, 2)$ or $2 \leq q \leq 2n/(n-2)$ $(n \geq 3)$, then there is a constant $c_* = c(\Omega, q)$ such that*

$$\|u\|_q \leq c_* \|\nabla u\|_2 \quad \text{for} \quad u \in H_0^1(\Omega).$$

Lemma 2.2 (Gagliardo-Nirenberg). *Let $1 \leq r < q \leq +\infty$ and $p \leq q$. Then, the inequality*

$$\|u\|_{W^{m,q}} \leq C \|u\|_{W^{m,p}}^{\theta} \|u\|_r^{1-\theta} \quad \text{for} \quad u \in W^{m,p} \bigcap L^r$$

holds with some $C > 0$ and

$$\theta = \left(\frac{k}{n} + \frac{1}{r} - \frac{1}{q}\right)\left(\frac{m}{n} + \frac{1}{r} - \frac{1}{p}\right)^{-1}$$

provided that $0 < \theta \leq 1$ (we assume $0 < \theta < 1$ if $q = +\infty$).

Lemma 2.3 ([12]). *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and assume that there are two constants $p \geq 1$ and $A > 0$ such that*

$$\int_S^{+\infty} E^{\frac{p+1}{2}}(t)\,dt \leq AE(S), \quad 0 \leq S < +\infty,$$

then we have

$$E(t) \leq cE(0)(1+t)^{\frac{-2}{p-1}} \;\; \forall t \geq 0, \quad \text{if} \;\; p > 1$$

and

$$E(t) \leq cE(0)e^{-\omega t} \;\; \forall t \geq 0, \quad \text{if} \;\; p = 1$$

c and ω are positive constants independent of the initial energy $E(0)$.

Lemma 2.4 ([13]-[2]). *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non increasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing C^2 function such that*

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \to +\infty \quad \text{as} \quad t \to +\infty.$$

Assume that there exist $p \geq 1$ and $A > 0$ such that

$$\int_S^{+\infty} E(t)^{\frac{p+1}{2}}(t)\phi'(t)\,dt \leq AE(S), \quad 0 \leq S < +\infty,$$

then we have

$$E(t) \leq cE(0)(1+\phi(t))^{\frac{-2}{p-1}} \;\; \forall t \geq 0, \quad \text{if} \;\; p > 1$$

and

$$E(t) \leq cE(0)e^{-\omega\phi(t)} \;\; \forall t \geq 0, \quad \text{if} \;\; p = 1$$

c and ω are positive constants independent of the initial energy $E(0)$.

Proof of Lemma 2.4. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $f(x) := E(\phi^{-1}(x))$, (we remark that ϕ^{-1} has a sense by the hypotheses assumed on ϕ). f is non-increasing, $f(0) = E(0)$ and if we set $x := \phi(t)$ we obtain

$$\int_{\phi(S)}^{\phi(T)} f(x)^{\frac{p+1}{2}}\,dx = \int_{\phi(S)}^{\phi(T)} E\left(\phi^{-1}(x)\right)^{\frac{p+1}{2}}\,dx$$

$$= \int_S^T E(t)^{\frac{p+1}{2}} \phi'(t)\, dt \le AE(S) = Af(\phi(S)) \quad 0 \le S < T < +\infty.$$

Setting $s := \phi(S)$ and letting $T \to +\infty$, we deduce that

$$\int_s^{+\infty} f(x)^{\frac{p+1}{2}}\, dx \le Af(s) \quad 0 \le s < +\infty.$$

Thanks to lemma 2.3, we deduce the desired results. ∎

Lemma 2.5. *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non increasing function and assume that there exists $q \ge 0, q' \ge 0, c \ge 0$ and $\omega > 0$ such that*

$$\int_S^{+\infty} E(t)^{q+1}\, dt \le \frac{1}{\omega} E(S)^{1+q} + \frac{c}{(1+S)^{q'}} E(0)^q E(S) \quad \forall S \ge 0. \tag{6}$$

Then we have

$$E(t) \le E(0) e^{1-\omega t} \quad \forall t \ge 0, \quad \text{if } q = 0 = c, \tag{7}$$

and there exists $C > 0$ such that

$$E(t) \le E(0) \frac{C}{(1+t)^{\frac{(1+q')}{q}}} \quad \forall t \ge 0, \quad \text{if } q > 0. \tag{8}$$

Proof of Lemma 2.5. If $E(0) = 0$, then $E \equiv 0$ and there is nothing to prove. Otherwise, replacing the function E by $\frac{E}{E(0)}$ we may assume that $E(0) = 1$. We note by C any constant depending of σ, σ' and c.

First, we deduce from (6) that

$$\int_S^{+\infty} E(\tau)^{1+q} d\tau \le CE(t).$$

Thanks to lemma 2.3, we deduce that

$$E(t) \le \frac{C}{(1+t)^{\frac{1}{q}}}$$

we reinject this estimate in (6) to deduce from it that E verify

$$\int_S^{+\infty} E(\tau)^{1+q} d\tau \le C \frac{E(S)}{1+S} + C \frac{E(S)}{(1+S)^{q'}}$$

Let $\sigma_1 = \inf\{1, q'\}$. Then

$$\int_S^{+\infty} E(\tau)^{1+q} d\tau \le C \frac{E(S)}{(1+S)^{\sigma_1}}$$

Let $g : \mathbb{R}_+ \to \mathbb{R}_+,\ g(t) = \dfrac{E(t)}{(1+t)^{\sigma_1}}$ and applying lemma 2.4 to the function g with $\phi(t) = (1+t)^{\sigma_1(1+\sigma)+1} - 1$, we deduce:

$$E(t) \le \frac{C}{(1+t)^{\frac{(1+\sigma_1)}{q}}}$$

If $\sigma' \le 1$, we obtain (8). If not

$$E(t) \le \frac{C}{(1+t)^{\frac{2}{q}}}$$

and we start again until obtaining the desired exposant: let $n \in \mathbb{N}$ such that $\sigma \in [n, n+1]$. We prove by induction that for each $k \in \mathbb{N}, k \le n, E$ satisfy:

$$E(t) \le \frac{C_k}{(1+t)^{\frac{(1+k)}{q}}} \quad \forall t \ge 0 \tag{9}$$

We have proved (9) for $k = 0$ and for $k = 1$ if $n \ge 1$. Suppose that $n \ge 2$ and that (9) is true for some $k < n$. Then we use (9) to deduce from (6) that E satisfy:

$$\int_S^{+\infty} E(\tau)^{1+q} d\tau \le C \frac{E(S)}{(1+S)^{1+k}} + C \frac{E(S)}{(1+S)^{q'}}.$$

As $1 + k \le n \le \sigma'$, we have

$$\int_S^{+\infty} E(\tau)^{1+q} d\tau \le C \frac{E(S)}{(1+S)^{1+k}} \quad \forall t \ge 0,$$

and we deduce from Lemma 2.3 that

$$E(t) \le \frac{C_k}{(1+t)^{\frac{(1+k)}{q}}} \quad \forall t \ge 0,$$

that shows

$$E(t) \le \frac{C_k}{(1+t)^{\frac{(n+1)}{q}}} \quad \forall t \ge 0,$$

and with the same argument:

$$E(t) \le \frac{C_k}{(1+t)^{\frac{(q'+1)}{q}}} \quad \forall t \ge 0.$$

■

Lemma 2.6 ([13]). *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non increasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing C^2 function such that*

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \to +\infty \quad \text{as} \quad t \to +\infty.$$

Assume that there exists $q \ge 0, q' \ge 0, c \ge 0$ and $\omega > 0$ such that

$$\int_S^{+\infty} E(t)^{q+1} \phi'(t)\, dt \le \frac{1}{\omega} E(S)^{1+q} + \frac{c}{(1+\phi(S))^{q'}} E(0)^q E(S) \quad \forall S \ge 0. \tag{10}$$

Then we have

$$E(t) \le E(0) e^{1-\omega\phi(t)} \quad \forall t \ge 0, \quad \text{if } q = 0 = c, \tag{11}$$

and there exists $C > 0$ such that

$$E(t) \le E(0) \frac{C}{(1+\phi(t))^{\frac{(1+q')}{q}}} \quad \forall t \ge 0, \quad \text{if } q > 0. \tag{12}$$

Proof of Lemma 2.6. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $f(x) := E(\phi^{-1}(x))$, (we remark that ϕ^{-1} has a sense by the hypotheses assumed on ϕ). f is non-increasing, $f(0) = E(0)$ and if we set $x := \phi(t)$ we obtain

$$\int_{\phi(S)}^{+\infty} f(x)^{q+1}\,dx = \int_{\phi(S)}^{+\infty} E\left(\phi^{-1}(x)\right)^{q+1}\,dx = \int_S^{+\infty} E(t)^{q+1}\phi'(t)\,dt \leq$$
$$\frac{1}{\omega} f(\phi(S))^{q+1} + \frac{c}{(1+\phi(S))^{q'}} f(\phi(0))^q f(\phi(S)) \quad 0 \leq S < \infty.$$

Setting $s := \phi(S)$, we deduce that

$$\int_s^{+\infty} f(x)^{\frac{p+1}{2}}\,dx \leq \frac{1}{\omega} f(s)^{q+1} + \frac{c}{(1+s)^{q'}} f(0)^q f(\phi(S)), \quad 0 \leq s < +\infty.$$

Thanks to lemma 2.5, we deduce the desired results. ∎

Remark 2.1. *The use of a 'weight function' $\phi(t)$ to establish the decay rate of solutions to hyperbolic PDE was successfully done by Aassila* [**2**]*, Martinez* [**13**]*, and Mochizuki and Motai* [**14**].

Lemma 2.7. *There exists a function $\phi : [1, +\infty[\to [1, +\infty[$ increasing and such that ϕ is concave and*

$$\phi(t) \to +\infty \text{ as } t \to +\infty, \tag{13}$$

$$\phi'(t) \to 0 \text{ as } t \to +\infty, \tag{14}$$

$$\phi'(t) = \sigma(t) H\left(\frac{1}{\phi(t)}\right), \quad \forall t \geq 1. \tag{15}$$

$$\frac{1}{\phi(t)} \leq g^{-1}\left(\frac{1}{1 + \displaystyle\int_1^t \sigma(\tau)\,d\tau}\right). \tag{16}$$

Proof of Lemma 2.7. We need lemma 2.7 to prove the decay estimate (30) below in the case when the function ρ satisfies Hyp.3.

Let us define $\widetilde{\psi}(t)$ by

$$\widetilde{\psi}(t) := 1 + \int_1^t \frac{1}{H\left(\frac{1}{\tau}\right)}\,d\tau, \quad t \geq 1. \tag{17}$$

Then $\widetilde{\psi} : [1, +\infty[\to [1, +\infty[$ is a strictly increasing and convex function of class C^2 and

$$\widetilde{\psi}'(t) = \frac{1}{H\left(\frac{1}{t}\right)} \to +\infty \text{ as } t \to +\infty,$$

Hence

$$\widetilde{\psi}(t) \to +\infty \text{ as } t \to +\infty,$$

Now define

$$\widetilde{\phi}(t) = \widetilde{\psi}^{-1}(t) \quad \forall t \geq 1. \tag{18}$$

and

$$\phi(t) = \widetilde{\phi}\left(1 + \int_1^t \sigma(\tau)\, d\tau\right) \quad \forall t \geq 1. \tag{19}$$

We see that $\phi(t)$ verify all the hypotheses of lemma 2.7, indeed define

$$S(t) = 1 + \int_1^t \sigma(\tau)\, d\tau.$$

Since σ is a positive function of class C^1, ϕ is a strictly increasing function of class C^2 and

$$\phi'(t) = \sigma(t)\widetilde{\phi}'(S(t)).$$

The decrease of σ implies that ϕ is concave. Moreover, we note that

$$\phi(t) \to +\infty \text{ as } t \to +\infty \text{ because } \int_1^\infty \sigma(\tau)\, d\tau = +\infty,$$
$$\phi'(t) \to 0 \text{ as } t \to +\infty, \text{ because } \widetilde{\phi}'(t) \to 0.$$

Next we remark that ϕ satisfies (15): indeed

$$\phi'(t) = \sigma(t)\widetilde{\phi}'(S(t)) = \sigma(t)\frac{1}{\widetilde{\psi}'(\widetilde{\phi}(S(t)))} = \sigma(t)\frac{1}{\widetilde{\psi}'(\phi(t))} = \sigma(t)H\left(\frac{1}{\phi(t)}\right).$$

At last we verify that ϕ satisfies (16): for t large enough we have

$$\widetilde{\psi}(t) \leq 1 + \frac{t-1}{H\left(\frac{1}{t}\right)} \leq \frac{t}{H\left(\frac{1}{t}\right)} = \frac{1}{g\left(\frac{1}{t}\right)},$$

provided that $H\left(\frac{1}{t}\right) \leq 1$. Therefore

$$t \leq \widetilde{\phi}\left(\frac{1}{g\left(\frac{1}{t}\right)}\right).$$

Thus we get that for t large enough

$$\frac{1}{\phi(t)} = \frac{1}{\widetilde{\phi}\left(1 + \int_1^t \sigma(\tau)\, d\tau\right)} \leq g^{-1}\left(\frac{1}{1 + \int_1^t \sigma(\tau)\, d\tau}\right)$$

Note that as a consequence of (15), we see that there exist $k > 0$ such that

$$\phi'(t) \leq k\sigma(t), \quad \forall t \geq 0. \tag{20}$$

∎

Now, we shall construct a stable set in $H_0^1 \bigcap H^2$. For this, we need define some functionals defined on H_0^1. We set

$$J(u) \equiv \frac{1}{2}\int_0^{\|\nabla_x u\|_2^2} \Phi(s)\,ds + \int_\Omega \int_0^u f(\eta)\,d\eta\,dx \qquad \text{for } u \in H_0^1,$$

$$\tilde{J}(u) \equiv \Phi(\|\nabla_x u\|_2^2)\|\nabla_x u\|_2^2 + \int_\Omega f(u)u\,dx \qquad \text{for } u \in H_0^1$$

and

$$E(u,v) \equiv \frac{1}{2}\|v\|_2^2 + J(u) \qquad \text{for } (u,v) \in H_0^1 \times L^2.$$

Lemma 2.8. *Let* $0 < \alpha < \dfrac{4}{(N-4)^+}$. *Then, for any* $K > 0$, *there exists a number* $\varepsilon_0 \equiv \varepsilon_0(K) > 0$ *such that if* $\|\Delta_x u\| \le K$ *and* $\|\nabla_x u\| \le \varepsilon_0$, *we have*

$$J(u) \ge \frac{m_0}{4}\|\nabla_x u\|_2^2 \quad \text{and} \quad \tilde{J}(u) \ge \frac{m_0}{2}\|\nabla_x u\|_2^2. \tag{21}$$

Proof: We see from the Gagliardo-Nirenberg inequality that

$$\begin{aligned} \|u\|_{\alpha+2}^{\alpha+2} &\le C\|u\|_{\frac{2N}{(N-2)}}^{(\alpha+2)(1-\theta)}\|\Delta_x u\|_2^{(\alpha+2)\theta} \\ &\le C\|\nabla_x u\|_2^{(\alpha+2)(1-\theta)}\|\Delta_x u\|_2^{(\alpha+2)\theta} \end{aligned} \tag{22}$$

with

$$\theta = \left(\frac{N-2}{2N} - \frac{1}{\alpha+2}\right)^+ \left(\frac{2}{N} + \frac{N-2}{2N} - \frac{1}{2}\right)^{-1} = \frac{((N-2)\alpha-4)^+}{2(\alpha+2)} (\le 1). \tag{23}$$

Here, we note that
(24)

$$(\alpha+2)(1-\theta)-2 = \begin{cases} \alpha > 0 & \text{if } 0 < \alpha \le \frac{4}{N-2}\ (0 < \alpha < \infty \text{ if } N = 1,2), \\ \frac{(4-N)\alpha+4}{2} > 0 & \text{if } \frac{4}{N-2} < \alpha < \frac{4}{N-4}\ (\frac{4}{N-2} < \alpha < \infty \text{ if } N = 3,4). \end{cases}$$

Hence, if $\|\Delta_x u\|_2 \le K$, we have

$$\begin{aligned} J(u) &\ge \frac{m_0}{2}\|\nabla_x u\|_2^2 - \frac{k_2}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2} \\ &\ge \frac{m_0}{2}\|\nabla_x u\|_2^2 - C\|\nabla_x u\|_2^{(\alpha+2)(1-\theta)}\|\Delta_x u\|_2^{(\alpha+2)\theta} \\ &\ge \left\{\frac{m_0}{2} - CK^{(\alpha+2)\theta}\|\nabla_x u\|_2^{(\alpha+2)(1-\theta)-2}\right\}\|\nabla_x u\|_2^2. \end{aligned} \tag{25}$$

Using (24), we define $\varepsilon_0 \equiv \varepsilon_0(K)$ by

$$CK^{(\alpha+2)\theta}\varepsilon_0^{(\alpha+2)(1-\theta)-2} = \frac{m_0}{4}.$$

Thus, we obtain

$$J(u) \ge \frac{m_0}{4}\|\nabla_x u\|_2^2 \tag{26}$$

if $\|\nabla_x u\|_2 \le \varepsilon_0$. It is clear that (26) is valid for $\tilde{J}(u)$. ∎

Let us define a stable set defined in $H_0^1 \bigcap H^2$ as follows:

$$\mathcal{W}_K \equiv \left\{(u,v) \in (H_0^1 \cap H^2) \times H_0^1 |\ \|\Delta_x u\|_2 < K, \|\nabla_x v\|_2 < K \text{ and } \sqrt{4m_0^{-1}E(u,v)} < \varepsilon_0\right\}$$

for $K > 0$.

Remark 2.2. *If $f(u)u \geq 0$, we need not take $\varepsilon_0(K)$, and $\mathcal{W}_K$ is replaced by*

$$\tilde{\mathcal{W}}_K \equiv \{(u,v) \in (H_0^1 \cap H^2) \times H_0^1 |\ \|\Delta_x u\|_2 < K, \|\nabla_x v\|_2 < K\}$$

3 Global Existence and Asymptotic Behavior

A simple computation shows that

$$E'(t) = -\int_\Omega u' g(u')\, dx \leq 0,$$

hence the energy is non-increasing and we have in particular $E(t) \leq E(0)$ for all $t \geq 0$.

Lemma 3.1. *Let $u(t)$ be a strong solution satisfying $(u(t), u'(t)) \in \mathcal{W}_K$ on $[0, T[$ for some $K > 0$. Assume that the function σ satisfies*

$$\int_0^\infty \sigma(t)\, dt = +\infty. \tag{27}$$

Then we have
1. Under Hyp.1, there exists a positive constant ω such that the energy of the solution u of (P) decays as:

$$E(t) \leq E(0) \exp\left(1 - \omega \int_0^t \sigma(\tau)\, d\tau\right) \qquad on\ [0, T[. \tag{28}$$

2. Under Hyp.2, there exists a positive constant $C(E(0))$ $(C(0) = 0)$ depending on $E(0)$ in a continuous way such that the energy of the solution u of (P) decays as:

$$E(t) \leq \left(\frac{C(E(0))}{\displaystyle\int_0^t \sigma(\tau)\, d\tau}\right)^{\frac{2}{(p-1)}} \qquad on\ [0, T[. \tag{29}$$

3. Under Hyp.3, there exists a positive constant $C(E(0))$ $(C(0) = 0)$ depending on $E(0)$ in a continuous way such that the energy of the solution u of (P) decays as:

$$E(t) \leq C(E(0)) \left[g^{-1}\left(\frac{1}{1 + \displaystyle\int_1^t \sigma(\tau)\, d\tau}\right)\right]^2 \qquad \forall t \geq 1. \tag{30}$$

Proof of lemma 3.1. From now on, we denote by c various positive constants which may be different at different occurrences. We multiply the first equation of (P) by $E\phi' u$, where ϕ is a function which will be chosen later, we obtain

$$\begin{aligned} 0 &= \int_S^T E^q\phi' \int_\Omega u(u'' - \Phi(\|\nabla_x u\|_2^2)\Delta u + \rho(t,u') + f(u))\,dx\,dt \\ &= \left[E^q\phi' \int_\Omega uu'\,dx\right]_S^T - \int_S^T (qE'E^{q-1}\phi' + E^q\phi'')\int_\Omega uu'\,dxdt - 2\int_S^T E^q\phi' \int_\Omega u'^2\,dxdt \\ &+ \int_S^T E^q\phi' \int_\Omega \left(u'^2 + \Phi(\|\nabla_x u\|_2^2)|\nabla u|^2 + f(u)u\right)\,dxdt + \int_S^T E^q\phi'\int_\Omega u\rho(t,u')\,dxdt \\ &. \end{aligned}$$

under the assumption $(u(t), u'(t)) \in \mathcal{W}_K$, the functionals $J(u(t))$ and $\tilde{J}(u(t))$ are both equivalent to $\|\nabla_x u(t)\|_2^2$ by lemma 2.8. So we deduce that

$$\begin{aligned} \int_S^T E^{q+1}\phi'\,dt &\le -\left[E^q\phi'\int_\Omega uu'\,dx\right]_S^T + \int_S^T (qE'E^{q-1}\phi' + E^q\phi'')\int_\Omega uu'\,dxdt \\ &+2\int_S^T E^q\phi'\int_\Omega u'^2\,dxdt - \int_S^T E^q\phi'\int_\Omega u\rho(t,u')\,dxdt \\ &\le -\left[E^q\phi'\int_\Omega uu'\,dx\right]_S^T + \int_S^T (qE'E^{q-1}\phi' + E^q\phi'')\int_\Omega uu'\,dxdt \\ +2\int_S^T E^q\phi'\int_\Omega u'^2\,dxdt &+ c(\varepsilon)\int_S^T E^q\phi'\int_\Omega \rho(t,u')^2\,dxdt + \varepsilon\int_S^T E^q\phi'\int_\Omega u^2\,dxdt \end{aligned}$$

for every $\varepsilon > 0$. Choosing ε small enough, we deduce that

$$\begin{aligned} \int_S^T E^{q+1}\phi'\,dt &\le -\left[E^q\phi'\int_\Omega uu'\,dx\right]_S^T + \int_S^T (qE'E^{q-1}\phi' + E^q\phi'')\int_\Omega uu'\,dxdt \\ &+ c\int_S^T E^q\phi'\int_\Omega u'^2 + \rho(t,u')^2\,dxdt \end{aligned} \tag{31}$$

Since E is nonincreasing and ϕ' is a bounded nonnegative function on $\mathbb{R}_+$ (and we denote by μ its maximum), we easily estimate the right-hand side terms of (31)

$$|E(t)^q\phi'\int_\Omega u'u| \le \frac{c\mu}{q+1}E(t)^{q+1}$$

$$\begin{aligned} \int_S^T (qE'E^{q-1}\phi' + E^q\phi'')\int_\Omega uu'\,dxdt &\le c\mu\int_S^T -E'(t)E(t)^q\,dt \\ &\le +c\int_S^T E(t)^{q+1}(-\phi''(t))\,dt \\ &\le c\mu E(S)^{1+q} \end{aligned}$$

Then, we obtain the estimate

$$\int_S^T E(t)^{1+q}\phi'(t)\,dt \le cE(S)^{1+q} + c\int_S^T E(t)^q\phi'(t)\int_\Omega u'^2 + \rho(t,u')^2\,dx\,dt. \tag{32}$$

Proof of (28). We consider the case

$$\sigma(t)|v| \le |\rho(t,v)| \le \frac{1}{\sigma(t)}|v| \quad \forall t\in\mathbb{R},\ \forall v\in\mathbb{R}.$$

Then we have

$$u'^2 + \rho(t,u')^2 \leq \frac{2}{\sigma(t)} u'\rho(t,u') \quad \forall t \in \mathbb{R},\ \forall x \in \Omega. \tag{33}$$

Therefore we deduce from (32) (applied with q=0) that

$$\int_S^T E(t)\phi'(t)\,dt \leq CE(S) + 2C\int_S^T \phi'(t)\int_\Omega \frac{1}{\sigma(t)} u'\rho(t,u')\,dx\,dt \tag{34}$$

Define

$$\phi(t) = \int_0^t \sigma(\tau)\,d\tau. \tag{35}$$

It is clear that ϕ is a concave nondecreasing function of class C^2 on $\mathbb{R}_+$. The hypothesis (27) ensures that

$$\phi(t) \to +\infty \text{ as } t \to +\infty. \tag{36}$$

Then we deduce from (34) that

$$\int_S^T E(t)\phi'(t)\,dt \leq CE(S) + 2C\int_S^T \int_\Omega u'\rho(t,u')\,dx\,dt \leq 3CE(S), \tag{37}$$

and thanks to lemma 2.6 we obtain

$$E(t) \leq E(0)e^{(1-\phi(t))/(3C)} \tag{38}$$

Proof of (29). Now we assume that there exists $p > 1$ such that (2) is satisfied with $g(v) = v^p$ on $[0,1]$. Define ϕ by (35). We apply Lemma 2.6 with $q = \dfrac{(p-1)}{2}$.

We need to estimate

$$\int_S^T E^q\phi' \int_\Omega u'^2 + \rho(t,u')^2\,dx\,dt$$

For $t \geq 0$, consider

$$\Omega^t_{1,v} = \{x \in \Omega, |u'| \leq 1\} \text{ and } \Omega^t_{2,v} = \{x \in \Omega, |u'| > 1\}$$
$$\Omega^t_{1,\rho} = \{x \in \Omega, |u'| \leq \sigma(t)\} \text{ and } \Omega^t_{2,\rho} = \{x \in \Omega, |u'| > \sigma(t)\}.$$

First we note that for every $t \geq 0$,

$$\Omega^t_{1,v} \cup \Omega^t_{2,v} = \Omega = \Omega^t_{1,\rho} \cup \Omega^t_{2,\rho}.$$

Next we deduce from Hyp.2 that for every $t \geq 0$,

$$\text{if } x \in \Omega^t_{1,v},\ \text{then } u'^2 \leq \left(\frac{1}{\sigma(t)}u'\rho(t,u')\right)^{\frac{2}{(p+2)}}$$

$$\text{if } x \in \Omega^t_{2,v},\ \text{then } u'^2 \leq \frac{1}{\sigma(t)}u'\rho(t,u')$$

$$\text{if } x \in \Omega^t_{1,\rho},\ \text{then } \rho(t,u')^2 \leq \left(\frac{1}{\sigma(t)}u'\rho(t,u')\right)^{\frac{2}{(p+2)}}$$

$$\text{if } x \in \Omega^t_{2,\rho},\ \text{then } \rho(t,u')^2 \leq \frac{1}{\sigma(t)}u'\rho(t,u')$$

Hence, using Hölder's inequality, we get that

(39)

$$\begin{aligned}
&\int_S^T E^q\phi' \int_\Omega u'^2 + \rho(t,u')^2\,dx\,dt \\
&\le 2\int_S^T E^q\phi' \int_\Omega \frac{1}{\sigma(t)} u'\rho(t,u')\,dx\,dt + 2\int_S^T E^q\phi' \int_\Omega \left(\frac{1}{\sigma(t)} u'\rho(t,u')\right)^{\frac{2}{(p+1)}}\,dx\,dt \\
&\le 2\int_S^T E^q\phi' \int_\Omega \frac{1}{\sigma(t)} u'\rho(t,u')\,dx\,dt + 2c(\Omega)\int_S^T E^q\phi' \left(\int_\Omega \frac{1}{\sigma(t)} u'\rho(t,u')\,dx\right)^{\frac{2}{(p+1)}}\,dt \\
&\le cE(S)^{1+q} + 2c(\Omega)\int_S^T E^q\phi'^{\frac{(p-1)}{(p+1)}} \left(\frac{-E'\phi'}{\sigma(t)}\right)^{\frac{2}{(p+1)}}\,dt.
\end{aligned}$$

Set $\varepsilon > 0$; thanks to Young's inequality and to our definitions of p and ϕ, we obtain

$$\begin{aligned}
&\int_S^T E^q\phi' \int_\Omega u'^2 + \rho(t,u')^2\,dx\,dt \\
&\le cE(S)^{1+q} + 2\frac{p-1}{p+1}\varepsilon^{\frac{(p+1)}{(p-1)}} \int_S^T E^{1+q}\phi'\,dt + \frac{4}{p+1}\frac{1}{\varepsilon^{\frac{(p+1)}{2}}} E(S).
\end{aligned} \tag{40}$$

$$\int_S^T E^{1+q}\phi'\,dt \le 2CE(S),$$

and thanks to lemma 2.6 (applied with $c = 0$) we obtain

$$E(t) \le \frac{C}{\phi(t)^{\frac{2}{(p-1)}}}.$$

Now we assume that Hyp.3 is satisfied with some strictly increasing odd function g of class C^1.

The key point is to construct a suitable weight function ϕ and convenient partitions of Ω. In the following, we assume that the function H is nondecreasing on $[0,1]$ (if H is nondecreasing only on $[0,\eta]$ for some $\eta > 0$, it is easy to adapt the proof, see **[2]**).

Proof of (30). We estimate the terms of the right-hand side of (32) in order to apply the results of Lemma 2.6: we choose $q = 1$ and study first

$$\int_S^T E\phi' \int_\Omega u'^2\,dx\,dt.$$

We have the following estimate:

Lemma 3.2. *There exists $C > 0$ such that*

$$\int_S^T E\phi' \int_\Omega u'^2\,dx\,dt \le CE(S)^2 + C\frac{E(S)}{\phi(S)} \quad \forall 1 \le S < T. \tag{41}$$

Proof of Lemma 3.2. Introduce

$$h(t) = \frac{1}{\phi(t)}, \quad \forall t \ge 1. \tag{42}$$

h is a decreasing positive function and satisfies

$$h(1) = 1 \text{ and } h(t) \to 0 \text{ as } t \to +\infty.$$

Define for every $t \geq 1$

$$\Omega^t_{3,v} = \{x \in \Omega : |u'| \leq h(t)\}, \tag{43}$$

$$\Omega^t_{4,v} = \{x \in \Omega : h(t) < |u'| \leq h(1)\}, \tag{44}$$

$$\Omega^t_{5,v} = \{x \in \Omega : |u'| > h(1)\}. \tag{45}$$

Fix $S \geq 1$; first we look at the part on $\Omega^t_{5,v}$. We deduce from (2) that

$$v^2 \leq \frac{1}{\sigma} v\rho(t, v), \forall t \geq 1, \forall |v| \geq 1.$$

Thus we have

$$\begin{aligned}\int_S^T E\phi' \int_{\Omega^t_{5,v}} u'^2 \, dx \, dt. &\leq \int_S^T E \frac{\phi'}{\sigma(t)} \int_{\Omega^t_{5,v}} u'\rho(t, u') \, dx \, dt. \\ &\leq \int_S^T E(-E') \, dt \leq kE(S)^2.\end{aligned} \tag{46}$$

Next we look at the part on $\Omega^t_{4,v}$. Set $t \geq 1$ and $x \in \Omega^t_{4,v}$: then $|u'(t, x)| \leq 1$. Thanks to the definition of h, to (15) and to Hyp.3, we have

$$\phi'(t)u'^2 = \sigma(t)H(h(t))u'^2 \leq \sigma(t)H(u')u'^2 \leq u'\rho(t, u').$$

Therefore

$$\int_S^T E\phi' \int_{\Omega^t_{4,v}} u'^2 \, dx \, dt \leq \int_S^T E \int_{\Omega^t_{4,v}} u'\rho(t, u') \, dx \, dt \leq E(S)^2. \tag{47}$$

At last we look at the part on $\Omega^t_{3,v}$:

$$\begin{aligned}&\int_S^T E\phi' \int_{\Omega^t_{4,v}} u'^2 \, dx \, dt \leq \int_S^T E\phi' \left(\int_{\Omega^t_{4,v}} h^2 \, dx \right) dt \\ &\leq |\Omega|E(S) \int_S^T \phi' h(t)^2 \, dt = |\Omega|E(S) \int_S^T \phi' \frac{1}{\phi(t)^2} \, dt \leq |\Omega| \frac{E(S)}{\phi(S)}.\end{aligned} \tag{48}$$

We add (46)-(48) to conclude.

Next we prove in the same way the following

Lemma 3.3. *There exists $C > 0$ such that*

$$\int_S^T E\phi' \int_{\Omega^t_{4,v}} \rho(t, u')^2 \, dx \, dt \leq CE(S)^2 + C\frac{E(S)}{\phi(S)} \quad \forall 1 \leq S < T. \tag{49}$$

Proof of Lemma 3.3. We use the same strategy: define for every $t \geq T_0$

$$\Omega^t_{3,\rho} = \left\{ x \in \Omega : g^{-1}\left(\frac{|u'|}{\sigma(t)}\right) \leq h(t) \right\}, \tag{50}$$

$$\Omega^t_{4,\rho} = \left\{ x \in \Omega : h(t) < g^{-1}\left(\frac{|u'|}{\sigma(t)}\right) \leq 1 \right\}, \tag{51}$$

$$\Omega^t_{5,\rho} = \left\{ x \in \Omega : g^{-1}\left(\frac{|u'|}{\sigma(t)}\right) > 1 \right\}, \tag{52}$$

Then it is easy to verify that

if $x \in \Omega^t_{5,\rho}$, then $\rho(t,u')^2 \leq g^{-1}\left(\frac{|u'|}{\sigma(t)}\right)|\rho(t,u')| = \frac{|u'|}{\sigma(t)}|\rho(t,u')|$;

if $x \in \Omega^t_{3,\rho}$, then $\rho(t,u')^2 \leq h(t)^2$.

At last we see that if $x \in \Omega^t_{4,\rho}$, then

$$\frac{\phi'(t)}{\sigma(t)} = H(h(t)) \leq H\left(g^{-1}\left(\frac{|u'|}{\sigma(t)}\right)\right) = \frac{\dfrac{|u'|}{\sigma(t)}}{g^{-1}\left(\dfrac{|u'|}{\sigma(t)}\right)}$$

thus

$$\phi'(t)|\rho(t,u')| \leq \phi'(t) g^{-1}\left(\frac{|u'|}{\sigma(t)}\right) \leq |u'|.$$

The proof of Lemma 3.3 follows from these three estimates. ∎

Using (32), Lemma 3.2 and Lemma 3.3, we get that

$$\int_S^T E^2\phi\,dt \leq CE(S)^2 + C\frac{E(S)}{\phi(S)} \quad \forall S \geq 1. \tag{53}$$

Then we use Lemma 2.6 and the estimate (16) to conclude that there exists C' and $T_1 \geq 1$ such that

$$E(t) \leq \frac{C'}{\phi(t)^2} \leq C'\left[g^{-1}\left(\frac{1}{1+\int_1^t \sigma(\tau)\,d\tau}\right)\right]^2 \quad \forall t \geq T_1.$$

Thus the proof of Lemma 3.1 is achieved.

Lemma 3.4. *Let $u(t)$ be a strong solution satisfying $(u(t),u'(t)) \in \mathcal{W}_K$ on $[0,T[$ for some $K > 0$. Assume that*

- *Under Hyp.1*

$$\int_0^{+\infty}\left(\exp\left(1-\omega\int_0^t \sigma(\tau)\,d\tau\right)\right)^{\min\left\{\frac{\gamma+1}{2},\frac{\alpha(1-\theta_0)}{2}\right\}} dt < +\infty$$

- *Under Hyp.2*

$$\int_0^{+\infty}\left(\int_0^t \sigma(\tau)\,d\tau\right)^{-\dfrac{\min\{\gamma+1,\alpha(1-\theta_0)\}}{p-1}} dt < +\infty$$

- *Under Hyp.3*

$$\int_0^{+\infty}\left(g^{-1}\left(\frac{1}{1+\int_1^t \sigma(\tau)\,d\tau}\right)\right)^{\min\{\gamma+1,\alpha(1-\theta_0)\}} dt < +\infty.$$

Then we have

$$\|\nabla u'(t)\|_2^2 + \|\Delta u(t)\|_2^2 \leq Q_1^2(I_0, I_1, K),$$

with

$$\lim_{I_0\to 0} Q_1^2(I_0, I_1, K) = I_1^2$$

and where we set

$$I_0^2 = E(0) = \frac{1}{2}\|u_1\|_2^2 + J(u_0)$$
$$I_1^2 = \|\nabla u_1\|_2^2 + \Phi(\|\nabla_x u_0\|_2^2)\|\Delta u_0\|_2^2$$

Proof of lemma 3.4. Multiplying the first equation of (P) by $-\Delta u'(t)$ and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\Big[\|\nabla u'(t)\|_2^2 + \Phi(\|\nabla_x u\|_2^2)\|\Delta u(t)\|_2^2\Big] + \Big(\nabla\rho(t, u'(t)), \nabla u'(t)\Big)$$
$$= -\int_\Omega f'(u)\nabla u.\nabla u'(t)\,dx\Big) + \Phi'(\|\nabla_x u\|_2^2)(\nabla u'(t), \nabla u(t))\|\Delta_x u\|_2^2$$

We set

$$E_1(t) \equiv \|\nabla_x u'\|_2^2 + \Phi(\|\nabla_x u\|_2^2)\|\Delta_x u\|_2^2$$

Using the assumption on Φ, g et f, we have

$$\begin{aligned}\frac{d}{dt}E_1(t) &\leq C\|\nabla_x u\|_2^{\gamma+1}\|\nabla_x u'\|_2\|\Delta_x u\|_2^2 + 2k_2\int_\Omega |u|^\alpha|\nabla_x u||\nabla_x u'|\,dx \\ &\leq C\left\{E(t)^{\frac{(\gamma+1)}{2}}K^3 + \left(\int_\Omega |u|^{2\alpha}|\nabla_x u|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla_x u'|\,dx\right)^{\frac{1}{2}}\right\}\end{aligned}\tag{54}$$

Here, we see from the Gagliardo-Nirenberg inequality that

$$\begin{aligned}\left(\int_\Omega |u|^{2\alpha}|\nabla_x u|^2\,dx\right)^{\frac{1}{2}} &\leq \|u(t)\|_{N\alpha}^\alpha\|\nabla_x u(t)\|_{\frac{2N}{(N-2)}} \\ &\leq C\|u(t)\|_{\frac{2N}{(N-2)}}^{\alpha(1-\theta_0)}\|\Delta_x u(t)\|_2^{\alpha\theta_0}\|\Delta_x u(t)\|_2 \\ &\leq C\|\nabla_x u(t)\|_2^{\alpha(1-\theta_0)}\|\Delta_x u(t)\|_2^{\alpha\theta_0+1} \\ &\leq CE(t)^{\alpha(1-\theta_0)}K^{\alpha\theta_0+1}\end{aligned}\tag{55}$$

with

$$\theta_0 = \left(\frac{N-2}{2} - \frac{1}{\alpha}\right)^+ = \frac{((N-2)\alpha - 2)^+}{2\alpha} \quad (\leq 1).$$

Hence, it follows from (54) and (55) that

$$\frac{d}{dt} E_1(t) \leq C \left\{ E(t)^{\frac{(\gamma+1)}{2}} K^3 + E(t)^{\frac{\alpha(1-\theta_0)}{2}} K^{\alpha\theta_0+2} \right\}. \tag{56}$$

we conclude that

$$\|\Delta_x u(t)\|_2^2 + \|\nabla_x u'(t)\|_2^2 \leq \frac{1}{\min\{1, m_0\}} \left\{ I_1^2 + CK^3 \int_0^\infty E(t)^{\frac{(\gamma+1)}{2}} \, dt \right.$$
$$\left. + CK^{\alpha\theta_0+2} \int_0^\infty E(t)^{\frac{\alpha(1-\theta_0)}{2}} \, dt \right\}$$

Examples
Consider

$$\rho(t, v) = \frac{1}{t^\theta} g(v)$$

with $\theta \in [0, 1]$ then we have the following estimate (we take in consideration the conditions in lemma 3.4).
a) under Hyp.1:

$$E(t) \leq E(0) e^{1-\omega t^{1-\theta}} \text{ if } \theta \in [0, 1[, \gamma \geq 0, \alpha > 0.$$

b) under Hyp.2:

$$E(t) \leq C(E(0))\ t^{-2\frac{(1-\theta)}{(p-1)}} \text{ if } \theta \in [0, 1[,$$
$$(1-\theta)(1+\gamma) > p-1 \text{ and } \alpha(1-\theta)(1-\theta_0) > p-1.$$

c) Under Hyp. 3:
if $g(x)$ is the inverse function of

$$M(0) = 0 \text{ and } M(x) = \frac{x^\sigma}{(\log(-\log x))^\beta} \text{ for } 0 < x < x_0, (\beta, \sigma > 0).$$

The function g exists and verifies the hypothesis $(H.2)$, when $0 < \sigma < 1$ (see Appendix). So

$$g^{-1}\left(\frac{1}{1 + \int_1^t \sigma(\tau)\, d\tau}\right) = \frac{1}{t^{\sigma(1-\theta)} (\log((1-\theta) \log t))^\beta}$$

the conditions in the lemma 3.4 gives

$$B1 \qquad \int_{t_0}^\infty \frac{1}{t^{\sigma(\gamma+1)(1-\theta)} (\log(\log t))^{\beta(\gamma+1)}} \, dt < \infty$$

and

$$B2 \qquad \int_{t_0}^\infty \frac{1}{t^{\sigma\alpha(1-\theta_0)(1-\theta)} (\log(\log t))^{\beta\alpha(1-\theta_0)}} \, dt < \infty,$$

which are similar to Bertrand integrals. So, when $\gamma = 0$, the first integral $(B1)$ is not finite, we obtain the following cases: if $\sigma(\gamma+1)(1-\theta) > 1$, the integral is finite, if $\sigma(\gamma+1)(1-\theta) = 1$, and $\beta(\gamma+1) > 1$, also the integral is finite. The second integral $(B2)$ is finite under the following conditions:

$$\sigma^{-1}(1-\theta)^{-1} < \alpha \leq \frac{2}{(N-2)^+} \quad \text{for } N = 1, 2, 3$$

or

$$\alpha > \frac{2(1-\sigma(1-\theta))}{\sigma(1-\theta)} \quad \text{for } N = 3$$

or

$$\alpha = \sigma^{-1}(1-\theta) \quad \text{and} \quad \beta^{-1} < \alpha \leq \frac{2}{(N-2)^+} \quad \text{for } N = 1, 2, 3$$

or

$$\alpha = \frac{2(1-\sigma(1-\theta))}{\sigma(1-\theta)} \quad \text{and} \quad \alpha > \frac{2(1-\beta)}{\beta} \quad \text{for } N = 3.$$

Hence, we must restrict ourselves to $1 \leq N \leq 3$.

Theorem 3.1. *Under the hypotheses of lemma 3.1 and 3.4 there exists an open set* $S_1 \subset (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$*, which includes* $(0,0)$ *such that if* $(u_0, u_1) \in S_1$*, the problem* (P) *has a unique global solution* u *satisfying*

$$u \in L^\infty([0,\infty[; H^2(\Omega) \cap H_0^1(\Omega)) \cap W^{1,\infty}([0,\infty[; H_0^1(\Omega)) \cap W^{2,\infty}([0,\infty[; L^2(\Omega)),$$

furthermore we have the decay estimate
Under Hyp.1:

$$E(t) \leq E(0) \exp\left(1 - \omega \int_0^t \sigma(\tau)\, d\tau\right) \quad \forall t > 0. \tag{57}$$

Under Hyp.2:

$$E(t) \leq \left(\frac{C(E(0))}{\displaystyle\int_0^t \sigma(\tau)\, d\tau}\right)^{\frac{2}{(p-1)}} \quad \forall t > 0. \tag{58}$$

Under Hyp.3:

$$E(t) \leq C(E(0)) \left[g^{-1}\left(\frac{1}{1 + \displaystyle\int_1^t \sigma(\tau)\, d\tau}\right)\right]^2 \quad \forall t > 0. \tag{59}$$

Proof of theorem 3.1.

Let $K > 0$. Put
$$S_K \equiv \{(u_0, u_1) \in \mathcal{W}_K | \, Q_1(I_0, I_1, K) < K\}$$
and
$$S_1 \equiv \bigcup_{K>0} S_K$$
Note that if E_0, E_1 are sufficiently small, then S_K in not empty.

If $(u_0, u_1) \in S_K$ for some $K > 0$, then an assumed strong solution $u(t)$ exist globally and satisfies $(u(t), u'(t)) \in \mathcal{W}_K$ for all $t \geq 0$. Let $\{w_j\}_{j=1}^{\infty}$ be the basis of H_0^1 consisted by the eigenfunction of $-\Delta$ with Dirichlet condition. We define the approximation solution u_m (m=1, 2, ...) in the form
$$u_m = \sum_{j=1}^{m} g_{jm} w_j$$
where $g_{jm}(t)$ are determined by

(60)
$$(u_m''(t), w_j)+\Phi(\|\nabla_x u_m(t)\|_2^2)(\nabla_x u_m(t), \nabla_x w_m)+(\rho(t, u_m'(t)), w_j)+(f(u_m(t)), w_j) = 0$$
for $j \in \{1, 2, \ldots, m\}$ with the initial data where $u_m(0)$ and $u_m'(0)$ are determined in such a way that
$$u_m(0) = u_{0m} = \sum_{j=1}^{m}(u_0, w_j)w_j \rightarrow u_0 \text{ strongly in } H_0^1 \bigcap H^2 \text{ as } m \rightarrow \infty,$$
$$u_m'(0) = u_{1m} = \sum_{j=1}^{m}(u_1, w_j)w_j \rightarrow u_1 \text{ strongly in } H_0^1 \text{ as } m \rightarrow \infty,$$
as $m \rightarrow \infty$.

By the theory of ordinary differential equations, (60) has a unique solution $u_m(t)$. Suppose that $(u_0, u_1) \in S_K$ for $K > 0$. Then, $(u_m(0), u_m'(0)) \in S_K$ for large m. It is clear that all the estimates obtained above are valid for $u_m(t)$ and, in particular, $u_m(t)$ exists on $[0, \infty[$. Thus, we conclude that $(u_m(t), u_m'(t)) \in \mathcal{W}_K$ for all $t \geq 0$ and all the estimates are valid for $u_m(t)$ for all $t \geq 0$.

Thus, $u_m(t)$ converges along a subsequence to $u(t)$ in the following way:
$$u_m(.) \rightarrow u(.) \text{ weakly } * \text{ in } L^\infty_{loc}([0, \infty); H_0^1 \bigcap H^2),$$
$$u_m'(.) \rightarrow u_t(.) \text{ weakly } * \text{ in } L^\infty_{loc}([0, \infty); H_0^1),$$
$$u_m(.) \rightarrow u_{tt}(.) \text{ weakly } * \text{ in } L^\infty_{loc}([0, \infty); L^2),$$
and hence,
$$\Phi(\|\nabla_x u_m(.)\|_2^2)\nabla_x u_m(.) \rightarrow \Phi(\|\nabla_x u(.)\|_2^2)\nabla_x u(.) \text{ weakly } * \text{ in } L^\infty_{loc}([0, \infty); H_0^1),$$
$$\rho(t, u_m(.)) \rightarrow \rho(t, u(.)) \text{ weakly } * \text{ in } L^\infty_{loc}([0, \infty); H_0^1),$$
Therefore, the limit function $u(t)$ is a desired solution belonging to
$$L^\infty([0, \infty[; H_0^1 \cap H^2) \cap W^{1,\infty}([0, \infty[; H_0^1) \cap W^{2,\infty}([0, \infty[; L^2)$$

The uniqueness can be proved by use of the monotonicity of ρ, $n\alpha < \dfrac{2n}{(n-4)}$ and $\sup_{0 \leq t \leq T}(\|u(t)\|_{H^2} + \|u'(t)\|_{H_0^1}) \leq C(T) < \infty$ (see **[2]**). ■

4 The case $\alpha = 0$

In this section we shall discuss the existence of a global solution to the problem (P) with $f(u) \equiv -u$. More precisely, we impose an assumption on $f(u)$ instead of **(H.3)** as follows:
(H.3)' $f(.)$ satisfies

$$f(u) = -k_3 u \quad \text{for } u \in \mathbb{R} \tag{61}$$

with $k_3 C(\Omega) < m_0$, $k_3 > 0$, where $C(\Omega)$ is a quantity such that

$$C(\Omega) = \sup_{u \in H_0^1 \setminus \{0\}} \frac{\|u\|_2}{\|\nabla_x u\|_2} \tag{62}$$

Remark 4.1. *The condition $k_3 C(\Omega) < m_0$ implies that $|\Omega|$ is small in some sense. On the other hand, if $f(u) = u$, we need not take $C(\Omega)$ into consideration.*

Our result reads as follows.

Theorem 4.1. *Under the hypotheses of lemma 3.1 (we replace* **(H.3)** *by* **(H.3)'***) and 3.4 , there exists an open unbounded set S_2 in $(H^2 \cap H_0^1) \times H_0^1$, which includes $(0,0)$, such that if $(u_0, u_1) \in S_2$, the problem (P) has a unique solution u in the sense of theorem 3.1 which satisfies the decay estimate (57) or (58) or (59).*

Proof of theorem 4.1.

The proof of theorem 4.1 is also given in parallel way to the proof of theorem 3.1 and we sketch the outline.

First, let $k_3 C(\Omega) < m_0$. Then, we see by (62)

$$J(u) = \frac{1}{2}\int_0^{\|\nabla_x u\|_2^2} \Phi(s)\,ds - \frac{k_3}{2}\|u\|_2^2 \geq \frac{1}{2}(m_0 - k_3 C(\Omega))\|\nabla_x u\|_2^2. \tag{63}$$

We may assume $\tilde{J}(u)$ also satisfies (63).

If $u(t)$ is a strong solution satisfying $\|\nabla_x u(t)\|_2 < K$ and $\|\nabla_x u'(t)\|_2 < K$ on $[0, T[$ for some $K > 0$, we can derive the decay estimate (28), (29) and (30) by a similar argument as lemma 3.1.

Multiplying the equation by $-\Delta_x u'$, we see

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}E_1(t) &\leq |\Phi'(\|\nabla_x u(t)\|_2^2)|(\nabla_x u(t), \nabla_x u'(t))\|\Delta_x u(t)\|_2^2 + \frac{k_3}{2}\frac{d}{dt}\|\nabla_x u(t)\|_2^2 \\ &\leq C K^3 E(t)^{\frac{(\gamma+1)}{2}} + \frac{k_3}{2}\frac{d}{dt}\|\nabla_x u(t)\|_2^2 \end{aligned} \tag{64}$$

where we set

$$E_1(t) = \Phi(\|\nabla_x u(t)\|_2^2)\|\Delta_x u(t)\|_2^2 + \|\nabla_x u'(t)\|_2^2.$$

we integrate (64) to obtain

$$\begin{aligned}
&\|\Delta_x u(t)\|_2^2 + \|\nabla_x u(t)\|_2^2 \\
&\le \frac{1}{\min\{1, m_0\}} \left\{ I_1^2 + CK^3 \int_0^\infty E(t)^{\frac{(\gamma+1)}{2}}\, dt + k_3 \|\nabla_x u(t)\|_2^2 - k_3 \|\nabla_x u_0\|_2^2 \right\} \\
&\le \frac{1}{\min\{1, m_0\}} \left\{ I_1^2 + C I_0^2 + C\ I_0^{\gamma+1}\ K^3 \int_0^\infty E(t)^{\frac{(\gamma+1)}{2}}\, dt \right\} \\
&\equiv Q_2^2(I_0, I_1, K) \text{ on } [0, T[.
\end{aligned}$$

In lemma 3.4, We replace $\min\left\{\dfrac{\gamma+1}{2}, \dfrac{\alpha(1-\theta_0)}{2}\right\}$ and $\min\{\gamma+1, \alpha(1-\theta_0)\}$ by $\frac{\gamma+1}{2}$ and $\gamma+1$.
Defining

$$S_K \equiv \{(u_0, u_1) \in H_0^1 \bigcap H^2 |\ Q_2(I_0, I_1, K) < K\}$$

and

$$S_2 \equiv \bigcup_{K>0} S_K$$

we conclude that if $(u_0, u_1) \in S_2$, the corresponding solution to the problem (P) exists globally and satisfies the estimate (28), (29), (30) and $\|\Delta_x u(t)\|_2^2 + \|\nabla_x u'(t)\|_2^2 < K^2$ for all $t > 0$. The proof of theorem 4.1 is complete.

5 The case $\Phi \equiv 1$

It is well known that usually, we study global existence for Kirchhoff equation (i.e. when $\Phi \not\equiv 1$) in the classe $H^2 \cap H_0^1$ (also when $f \equiv \rho \equiv 0$). Thus the condition in lemma 3.4 excludes some functions g which verify $(H.2)$ as for example $\rho(t,x) = e^{-\frac{1}{x}}$ or $\rho(t,x) = e^{-e^{\frac{1}{x}}}$ or the example (1) in some cases in lemma 3.1, so, we consider the case $\Phi \equiv 1$ (or a constant function) and we prove a global decaying H_0^1 solution. Here we do not need the condition of lemma 3.4 and we will take only $\alpha \le \dfrac{4}{(n-2)^+}$ because we work only in $H_0^1(\Omega)$.

Now, we consider the initial boundary value problem

$$(P') \qquad \begin{cases} u'' - \Delta_x u + \rho(t, u') + f(u) = 0 \text{ in } \Omega \times [0, +\infty[, \\ u = 0 \text{ on } \Gamma \times [0, +\infty[, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \text{ on } \Omega, \end{cases}$$

First, we shall construct a stable set in H_0^1. For this, we need define some functionals defined on H_0^1. We set

$$J(u) \equiv \frac{1}{2}\|\nabla_x u\|_2^2 + \int_\Omega \int_0^u f(\eta)\, d\eta\, dx \qquad \text{for } u \in H_0^1,$$

$$\tilde{J}(u) \equiv \|\nabla_x u\|_2^2 + \int_\Omega f(u) u\, dx \qquad \text{for } u \in H_0^1$$

and

$$E(u,v) \equiv \frac{1}{2}\|v\|_2^2 + J(u) \qquad \text{for } (u,v) \in H_0^1 \times L^2.$$

Then we can define the stable set as

$$\mathcal{W} = \{u \in H_0^1(\Omega), \|\nabla_x u\|_2^2 - k_1\|u\|_{\alpha+2}^{\alpha+2} > 0\} \cup \{0\}$$

Lemma 5.1. **(i)** *If* $\alpha < \dfrac{4}{[n-2]^+}$*, then*

(65) $\mathcal{W}$ *is an open neighborhood of* 0 *in* $H_0^1(\Omega)$.

(ii) *If* $u \in \mathcal{W}$*, then*

(66) $$\|\nabla_x u\|_2^2 \leq d_* J(u)$$

with $d_* = \dfrac{2(\alpha+2)}{\alpha}$.

Proof of lemma 5.1.
(**i**) From the Sobolev-Poincaré inequality we have

(67) $$k_1\|u\|_{\alpha+2}^{\alpha+2} \leq Ak_1\|\nabla_x u\|_2^{\alpha}\|\nabla_x u\|_2^2$$

where $A = c_*^{\alpha+2}$. Let

$$U(0) \equiv \left\{u \in H_0^1(\Omega) \Big\backslash \|\nabla_x u\|_2^{\alpha} < \frac{1}{Ak_1}\right\}.$$

Then, for any $u \in U(0)\backslash\{0\}$, we deduce from (67) that

$$k_1\|u\|_{\alpha+2}^{\alpha+2} < \|\nabla_x u\|_2^2,$$

that is, $K(u) > 0$. This implies $U(0) \subset \mathcal{W}$.
(**ii**) By the definition of $K(u)$ and $J(u)$ we have the inequality

$$\begin{aligned} J(u) &\geq \frac{1}{2}\|\nabla_x u\|_2^2 - \frac{k_1}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2} \\ &\geq \frac{\alpha}{2(\alpha+2)}\|\nabla_x u\|_2^2 \end{aligned}$$

■

Lemma 5.2. *Let* $u(t)$ *be a strong solution of* (P')*. Suppose that*

(68) $$u(t) \in \overline{\mathcal{W}} \ \text{ and } \tilde{J}(u(t)) \geq \frac{1}{2}\|\nabla_x u(t)\|_2^2$$

for $0 \leq t < T$*. Then we have (28), (29) and (30) verified on* $[0, T[$.

Examples

1) Consider

$$\rho(t,v) = \frac{1}{t^\theta} g(v)$$

with $\theta \in [0,1]$ then we have the following estimate

a) under Hyp.1:

$$E(t) \leq E(0)e^{1-\omega t^{1-\theta}} \text{ if } \theta \in [0,1[,$$
$$E(t) \leq E(0)\frac{e}{(\log t)^\omega} \text{ if } \theta = 1.$$

b) under Hyp.2:

$$E(t) \leq \frac{C(E(0))}{t^{2\frac{(1-\theta)}{(p-1)}}} \text{ if } \theta \in [0,1[,$$
$$E(t) \leq E(0)\frac{C(E(0))}{(\log t)^{\frac{2}{(p-1)}}} \text{ if } \theta = 1.$$

c) under Hyp.3:

$$\text{If } g(v) = e^{-\frac{1}{v^p}} \text{ for } 0 < v < \frac{1}{2},\ p > 0,$$

then we have

$$E(t) \leq \frac{C(E(0))}{(\ln t)^{2/p}} \text{ if } \theta \in [0,1[.$$
$$E(t) \leq \frac{C(E(0))}{(log(logt))^{2/p}} \text{ if } \theta = 1.$$

2) We can also consider the case where $\rho(t,v) = \sigma(t)g(v)$ with

$$\sigma(t) = \frac{1}{t(logt)(log_2 t)\dots(log_p t)}$$

for t large nought and with some $p \geq 1$. Then there exists $c > 0$ such that

$$\int_0^t \sigma(\tau)\, d\tau = c + log_{p+1}(t).$$

Then, under Hyp. 1:

$$E(t) \leq E(0)e^{1-\omega log_{p+1}(t)} = E(0)\frac{e}{(log_p(t))^\omega}.$$

Proof of lemma 5.2. The functionals $J(u(t))$ and $\tilde{J}(u(t))$ are both equivalent to $\|\nabla_x u(t)\|_2^2$, indeed we have

$$\int_\Omega f(u)u\, dx \leq k_1 \|u\|_{\alpha+2}^{\alpha+2} \leq \|\nabla_x u(t)\|_2^2$$

So, we have
$$\frac{1}{2}\|\nabla_x u\|_2^2 \leq K(u(t)) \leq \frac{3}{2}\|\nabla_x u\|_2^2.$$
Also, we have
$$|J(u(t))| \leq \frac{1}{2}\|\nabla_x u(t)\|_2^2 + \frac{1}{\alpha+2}\|\nabla_x u\|_2^2 \leq \frac{\alpha+4}{2(\alpha+2)}\|\nabla_x u(t)\|_2^2$$
therefore
$$K(u(t)) \geq \frac{1}{2}\|\nabla_x u\|_2^2 \geq \frac{\alpha+2}{\alpha+4}J(u). \tag{69}$$
Now, we can derive the decay estimate (28), (29) and (30) by similar argument as lemma 3.1.

Theorem 5.1. *Suppose that*
$$\alpha \leq \frac{4}{n-2} \quad (\alpha < \infty \quad \textit{if} \ \ n \leq 2),$$
and suppose that initial data $\{u_0, u_1\}$ *belongs to* $\mathcal{W}$ *and its initial energy* $E(0)$ *is sufficiently small such that*
$$C_4 E(0)^{\frac{\alpha}{2}} < 1, \tag{70}$$
where $C_4 = 2k_1 c_*^{\alpha+2} d_*^{\frac{\alpha}{2}}$.

Then, the problem (P') *has a unique global solution* $u \in \mathcal{W}$ *satisfying*
$$u \in L^\infty([0,\infty[; H_0^1(\Omega)) \cap W^{1,\infty}([0,\infty[; L^2(\Omega)),$$
furthermore we have the decay estimate (28), (29) and (30) for all $t \geq 0$.

Proof of theorem 5.1.

Since $u_0 \in \mathcal{W}$ and $\mathcal{W}$ is an open set, putting
$$T_1 = \sup\{t \in [0,+\infty) : u(s) \in \mathcal{W} \text{ for } 0 \leq s \leq t\},$$
we see that $T_1 > 0$ and $u(t) \in \mathcal{W}$ for $0 \leq t < T_1$. If $T_1 < T_{\max} < \infty$, where $T_{\max}$ is the lifespan of the solution, then $u(T_1) \in \partial\mathcal{W}$; that is
$$K(u(T_1)) = 0 \text{ and } u(T_1) \neq 0. \tag{71}$$
We see from lemma 2.2 and lemma 5.1 that
$$k_1\|u(t)\|_{\alpha+2}^{\alpha+2} \leq \frac{1}{2}B(t)\|\nabla_x u(t)\|_2^2 \tag{72}$$
for $0 \leq t \leq T_1$, where we set
$$B(t) = C_4 E(0)^{\frac{\alpha}{2}} \tag{73}$$

with $C_4 = 2k_1 c_*^{\alpha+2} d_*^{\frac{\alpha}{2}}$.

Next, we put

$$T_2 \equiv \sup\{t \in [0, +\infty) : B(s) < 1 \text{ for } 0 \le s < t\},$$

and then we see that $T_2 > 0$ and $T_2 = T_1$ because $B(t) < 1$ by (70). Then

$$\begin{aligned} K(u(t)) &\geq \|\nabla_x u(t)\|_2^2 - \frac{1}{2}B(t)\|\nabla_x u(t)\|_2^2 \\ &\geq \frac{1}{2}\|\nabla_x u(t)\|_2^2 \end{aligned} \tag{74}$$

for $0 \le t \le T_1$. Moreover, (71) and (74) imply

$$K(u(T_1)) \geq \frac{1}{2}\|\nabla_x u(T_1)\|_2^2 > 0,$$

which is a contradiction, and hence, it might be $T_1 = T_{\max}$. Therefore, (28), (29) and (30) hold true for $0 \le T \le T_{\max}$, and such estimate give the desired a priori estimate; that is, the local solution u can be extended globally (i.e., $T_{\max} = \infty$). The proof of theorem 5.1 is now completed. ■

Remarks:
By a similar argument as the proof of theorem 4.1, we can extend theorem 5.1 to the case $\alpha = 0$.

Appendix

Let $g(x)$ the inverse of the function $M(x)$ defined by

$$M(0) = 0, \quad M(x) = \frac{x^\sigma}{(\log(-\log x))^\beta} \text{ for } 0 < x < x_0, \quad (\sigma, \beta > 0).$$

If we set $x = 1/t (0 < x < x_0)$ we have

$$g^{-1}\left(\frac{1}{t}\right) = \frac{1}{t^\sigma (\log(\log t))^\beta} \quad (t \geq t_0).$$

Now, we prove that the function $g(x)$ exists and verifies the hypothesis $(H.2)$, indeed, we have

$$(M(x))' = \frac{x^\sigma \left[\sigma(\log(-\log x)) - \dfrac{\beta}{\log x}\right]}{(\log(-\log x))^{\beta+1}}, \quad (\sigma, \beta > 0).$$

When x is near 0 $(0 < x < x_0)$, it is clear that $(M(x))' \geq 0$, so $M(x)$ is an increasing continuous function. Thus the function g exists. We have also

$$\frac{x}{M(x)} = \frac{(\log(-\log x))^\beta}{x^{\sigma-1}} \to 0$$

as $x \to 0$ if $0 < \sigma < 1$, so $M(x) \to 0$ (as $x \to 0$) less fast than x (near 0), we deduce that $g(x) \to 0$ as $x \to 0$ more fast than x i.e. $|g(x)| \le c|x|$, we obtain hypothesis $(H.2)$. Now, $\dfrac{M(x)}{x}$ is a decreasing function, indeed,

$$\left(\frac{M(x)}{x}\right)' = \frac{x^{\sigma-2}\left[(\sigma-1)(\log(-\log x)) - \dfrac{\beta}{\log x}\right]}{(\log(-\log x))^{\beta+1}},$$

we take $x = e^{-n}$, n big, we see that $\left(\dfrac{M(x)}{x}\right)' \le 0$. g is a bijective and decreasing function, so for each x and y near 0, such that $x \le y$, we have $\dfrac{M(x)}{x} \ge \dfrac{M(y)}{y}$, also there exist unique x' and y' such that $M(x) = x'$ and $M(y) = y'$(because M is a bijective function), also $M(x)$ is an increasing function, thus, we have

$$x \le y \Longleftrightarrow M(x) = x' \le M(y) = y'$$

So

$$\begin{aligned} x' \le y' &\Longleftrightarrow \frac{x'}{g(x')} \ge \frac{y'}{g(y')} \\ &\Longleftrightarrow \frac{g(x')}{x'} \le \frac{g(y')}{y'} \text{ for } 0 < x < x_0. \end{aligned}$$

References

[1] M. Aassila, *Global existence and energy decay for damped quasi-linear wave equation,* Math. Meth. Appl. Sci. **14** (1998), 1185-1194.

[2] M. Aassila, *Global existence of solutions to a wave equation with damping and source terms,* Differential and Integral Equations, **14** (2001), 1301-1314.

[3] A. Benaissa, *Existence globale et décroissance polynomiale de l'énergie des solutions des équations de Kirchhoff-Carrier moyennement dégénérées avec des termes non linéaires dissipatifs,* Bulletin of the Belgian Math. Society, **47** (2001)-4, 607-622.

[4] A. Benaissa and S A Messaoudi, *Blow up of solutions for the Kirchhoff equation of q Laplacian type with non linear dissipation,* Colloq Math. **94** (2002)-1, 103-109.

[5] V. Georgiev and G. Todorova, *Existence of a solution of the wave equation with nonlinear damping and source terms,* J. Diff. Equat. **109** (1994), 295-308.

[6] D. Gourdin and M Mechab,*Problème de Goursat non linéaire dans les espaces de Gevrey pour les équations de Kirchhoff généralisées.* J. Math. Pures Appl., **IX**, **Sér. 75**, (1996)-6, 569-593.

[7] R. Ikehata, *Some remarks on the wave equations with nonlinear damping and source terms,* Nonlinear Analysis **27** (1996), 1165-1175.

[8] R. Ikehata and T. Matsuyama, *On global solutions and energy decay for the wave equations of Kirchhoff type with nonlinear damping terms,* J. Math Anal. Appl. **204** (1996), 729-753. .

[9] R. Ikehata and T. Suzuki, *Stable and unstable sets for evolution equations of parabolic and hyperbolic type,* Hiroshima Math. J. **26** (1996), 475-491.

[10] K. Kajitani& K. Yamaguti, *On global real analytic solution of the degenerate Kirchhoff equation.* Ann. Sc. Sup. Pisa **4** 1994, 279-297.

[11] G. Kirchhoff, Vorlesungen *ü*ber Mechanik, Leipzig, Teubner, (1883).

[12] V. Komornik, Exact Controllability and Stabilization. The Multiplier Method, Mason-John Wiley, Paris, 1994.

[13] P. Martinez, *A new method to obtain decay rate estimates for dissipative systems,*, ESAIM Control Optim. Calc. Var. **4** (1999), 419-444.

[14] K. Mochizuki & T. Motai, *On energy decay problems for wave equations with nonlinear dissipation term in* $\mathbb{R}^n$, J. Math. Soc. Japan. **47** (1995), 405-421.

[15] M. Nakao, *A difference inequality and its applications to nonlinear evolution equations,* J. Math. Soc. Japan **30** (1978), 747-762.

[16] M. Nakao, *Remarks on the existence and uniqueness of global decaying solutions of the nonlinear dissipative wave equations,* Math. Z. **206** (1991), 265-2276.

[17] S. I. Pohozaev , *On a class of quasilinear hyperbolic equations.* Math. Sbornik **96** (1975), 145-158.

[18] D. H. Sattinger, *On global solution of nonlinear hyperbolic equations,* Arch. Rat. Mech. Anal. **30** (1968), 148-172.

[19] M. Tsutsumi, *Existence and non existence of global solutions for nonlinear parabolic equations,* Publ. RIMS, Kyoto Univ. **8** (1972/73), 211-229.

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