Integral representations on equipotential and harmonic sets

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Abstract

The sets we are going to consider here are of the form $\{z \in \mathbb{C};\ |A(z)| = 1\}$ (equipotential) and $\{z \in \mathbb{C};\ \mathfrak{Im}\, A(z) = 0\}$ (harmonic) with A being a polynomial with complex coefficients. There are two themes which we want to focus on and which come out from invariance property of inner products on $\mathbb{C}[Z]$ related to the aforesaid sets. First, we formalize the construction of integral representation of the inner products in question with respect to matrix measure. Then we show that these inner products when represented in a Sobolev way are precisely those with discrete measures in the higher order terms of the representation. In this way we fill up the case already considered in [3] by extending it from the real line to harmonic sets on the complex plane as well as we describe completely what happens in this matter on equipotential sets. As a kind of smooth introduction to the above we are giving an account of standard integral representations on the complex plane in general and of those supported by these two kinds of real algebraic sets.

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1 Introduction

Denote as usually by $\mathbb{C}[Z]$ the algebra of all polynomials in a single variable having complex coefficients. In this paper we are going to consider a *semi-inner product* s on $\mathbb{C}[Z]$, that is a mapping $s : \mathbb{C}[Z] \times \mathbb{C}[Z] \mapsto \mathbb{C}$ with the properties: the mapping $p \mapsto s(p, \cdot)$ is linear, $s(p,q) = \overline{s(q,p)}$ for $p, q \in \mathbb{C}[Z]$ and $s(p,p) \geq 0$ for $p \in \mathbb{C}[Z]$. Generally speaking, we are interested here in s being represented either as

$$s(p,q) = \int_{\mathbb{C}} p\bar{q}\,\mathrm{d}\mu, \quad p, q \in \mathbb{C}[Z], \tag{1}$$

or, more intriguing, as [1]

$$s(p,q) = \sum_{i=0}^{N} \int_{\mathbb{C}} p^{(i)}\overline{q^{(i)}}\,\mathrm{d}\mu_i, \quad p, q \in \mathbb{C}[Z], \tag{2}$$

where μ and μ_i, $i = 0, 1, \ldots, N$, are positive (=non-negative) Borel measures on $\mathbb{C}$ having support on a real algebraic set like lemniscates and harmonic curves.

If we define the *moments* of s by

$$s_{m,n} \stackrel{\mathrm{df}}{=} s(Z^m, Z^n), \quad m, n = 0, 1, \ldots, \tag{3}$$

then (1) says that $\{s_{m,n}\}_{m,n=0}^{\infty}$ is a complex moment bisequence on $\mathbb{C}$ (the references [11], [10], and [12] have to be recommended in this matter); in this context the representation (2) refers to that of Sobolev type (here we refer to [8] and [9]). Moreover, there is a way back, given a bisequence $\{s_{m,n}\}_{m,n=0}^{\infty}$, we can always extend it in a sesquilinear way to a semi-inner product on $\mathbb{C}[Z]$ provided

$$\sum_{m,n=0}^{K} s_{m,n}\xi_m\bar{\xi}_n \geq 0, \quad \xi_0, \ldots, \xi_K \in \mathbb{C},\ K = 0, 1, \ldots, \tag{4}$$

which means positive definiteness of every finite section $\{s_{m,n}\}_{m,n=0}^{K}$ in the matrix sense. Thus, while these two possibilities are equivalent (via (3)), we prefer here to use the semi-inner approach.

In this paper we consider representations of the form (1) (call them *standard*), also with a matrix valued measure replacing the scalar one; this is the first part. Then, in the second part, we deal with representations (2) of Sobolev type. Before doing this we want to say a bit what can be done for representations (1) themselves.

Let us mention that different kinds or integral representations of inner products are discussed in [7].

[1] the superscript $^{(i)}$ stands for the i-th derivative with respect to the independent variable

2 Positivity conditions and standard representations

In the sequel we distinguish the conditions

$$s(p,p) \geq 0, \quad p \in \mathbb{C}[Z], \tag{5}$$

which is included in the definition of semi-inner product, by referring to it as to *positivity* of s and this corresponds to *positivity* of $\{s_{m,n}\}_{m,n=0}^{\infty}$ understood as in (4). There is another condition, much stronger, which requires s to satisfy

$$\sum_{m,n=0}^{K} s(Z^m p_n, Z^n p_m) \geq 0, \quad p_0, \dots, p_K \in \mathbb{C}[Z],\ K = 0, 1, \dots \tag{6}$$

We call it *positive definiteness* of the semi-inner product s and this corresponds in the moment version of (3) to

$$\sum_{k,l,m,n=0}^{K} s_{k+n,l+m} \xi_{k,l} \bar{\xi}_{m,n} \geq 0, \quad \{\xi_{i,j}\}_{i,j=0}^{K} \subset \mathbb{C},\ K = 0, 1, \dots, \tag{7}$$

which is recognized as *positive definiteness* of the bisequence $\{s_{m,n}\}_{m,n=0}^{\infty}$. Let us mention by the way that condition (6) ensures the possibility of extending the definition

$$\tilde{s}(\bar{Z}^m p, \bar{Z}^n q) \overset{\text{df}}{=} s(Z^n p, Z^m q), \quad p, q \in \mathbb{C}[Z]$$

in a sesquilinear way to an semi-inner product (thus positive) on $\mathbb{C}[Z, \bar{Z}]$. As a consequence we get, cf. [13],

$$\tilde{s}(p,q) = L(p\bar{q}),$$

for some positive linear functional L on $\mathbb{C}[Z, \bar{Z}]$. This shows how strong positive definiteness is.

Positive definiteness is equivalent for s to be of the form $s(p,q) = L(p\bar{q})$, where L is a linear functional on $\mathbb{C}[Z]$ (cf. [13]) and it is *not* sufficient to guarantee existence of the representation (1); still some stronger conditions are needed (see [12] for recent results in this matter). On the other hand, for the very special cases when $s(Zp, q) = s(p, Zq)$ or $s(Zp, Zq) = s(p,q)$, $p, q \in \mathbb{C}[Z]$ (which determines the Hankel or the Toeplitz structure of the infinite matrix $\{s_{m,n}\}_{m,n=0}^{\infty}$) even positivity of s implies the representation (1). Because these two cases correspond to the situation when the support of the measure μ is either the real line $\{z;\ \mathfrak{Im}\, z = 0\}$ or the unit circle $\{z;\ |z| = 1\}$ the question is whether the same happens for other real algebraic sets on the complex plane $\mathbb{C}$. Though this is not so in general (see [11] for some discussion) we try to say a word on the circumstances under which positivity of the semi-inner product together with its invariance (like this mentioned above in the real line or unit circle case) lead to representation of the form (1) with measure supported on a real algebraic set: the *equipotential* sets, that is sets of the form $\{z \in \mathbb{C};\ |A(z)| = 1\}$ and the *harmonic* ones, that is of the form $\{z \in \mathbb{C};\ \mathfrak{Im}\, A = 0\}$, for some $A \in \mathbb{C}[Z]$.

Fortunately, for the equipotential sets the answer is affirmative.

Theorem 1 ([11], Theorem 4). *Suppose s is a semi-inner product on $\mathbb{C}[Z]$ such that $s(Ap, Aq) = s(p, q)$, $p, q \in \mathbb{C}[Z]$ for some nonconstant $A \in \mathbb{C}[Z]$. Then the following conditions are equivalent*

(i) *s satisfies* (6)*;*

(ii) *there is a positive measure μ such that*

$$s(p,q) = \int_{\mathcal{E}(A)} p\bar{q}\,\mathrm{d}\mu, \quad p,q \in \mathbb{C}[Z],$$

where $\mathcal{E}(A) \stackrel{\mathrm{df}}{=} \{z \in \mathbb{C};\ |A(z)| = 1\}$.

Unfortunately, for the harmonic case nothing like this exists so far; one has to add some extra conditions, like the Carleman one, to get a sufficiency result.

3 The matrix representation of s

We start with the algebraic observation (already exploited in this direction in [4], [5] and [6]) that the following direct sum decomposition holds

$$\mathbb{C}[Z] = \mathfrak{P}_A \dotplus Z\mathfrak{P}_A \dotplus Z^2\mathfrak{P}_A \dotplus \cdots \dotplus Z^{d-1}\mathfrak{P}_A \tag{8}$$

where $\mathfrak{P}_A \stackrel{\mathrm{df}}{=} \mathrm{lin}\{1, A, A^2, \dots\}$ [2] and $d = \deg A > 0$. So, if p is in $\mathbb{C}[Z]$ we get a uniquely determined collection

$$\pi(p) \stackrel{\mathrm{df}}{=} (\pi_0(p), \dots, \pi_{d-1}(p)) \tag{9}$$

of polynomials in $\mathbb{C}[Z]$ such that

$$p = \pi_0(p) \circ A + Z(\pi_1(p) \circ A) + \cdots + Z^{d-1}(\pi_{d-1}(p) \circ A). \tag{10}$$

The mappings $p \mapsto \pi_k(p)$ are apparently linear and satisfy

$$\pi_k((r \circ A)p) = r\pi_k(p), \quad p, r \in \mathbb{C}[Z]. \tag{11}$$

Given a semi-inner product s, we can associate with it a Hilbert space $\mathcal{H}$ together with a mapping $\mathbb{C}[Z] \ni p \mapsto \tilde{p} \in \mathcal{H}$ such that [3]

$$s(p,q) = \langle \tilde{p}, \tilde{q} \rangle_{\mathcal{H}}.$$

[2] we prefer to use "lin" for the linear span just to distinguish it from other algebraic spans

[3] $\langle \cdot, \text{-} \rangle$ stands always for an inner product in a Hilbert space, even if it is finite dimensional; in the latter case as a consequence we can drop the usual linear algebra notation and replace it by the more convenient shape free one used in operator theory

3.1 The equipotential case

This case means we are going to work under the assumption of invariance of s of the form

$$s(Ap, Aq) = s(p, q), \quad p, q \in \mathbb{C}[Z] \tag{12}$$

for some fixed nonconstant $A \in \mathbb{C}[Z]$. Then in the Hilbert space $\mathcal{H}$ we can consider [4] an operator $V : \tilde{p} \mapsto \widetilde{Ap}$ which, by (12) is well defined and extends to an isometry on the whole $\mathcal{H}$. Furthermore, we have

$$s(A^m p, A^n q) = \langle V^m \tilde{p}, V^n \tilde{q} \rangle_{\mathcal{H}}. \tag{13}$$

Now, by classical means, we can imbed $\mathcal{H}$ isometrically in another Hilbert space $\mathcal{K}$ into such a way that V extends to a unitary operator U which leaves $\mathcal{H}$ invariant. Let E be the spectral measure of U. Then we can write (13) as [5]

$$s(A^m p, A^n q) = \int_{\mathbb{T}} z^m \bar{z}^n \langle E(\mathrm{d}z) p, q \rangle_{\mathcal{K}}. \tag{14}$$

Decomposing p and q according to (10) we can write furthermore

$$\begin{aligned} s(p, q) &= \sum_{i,j=0}^{d-1} s(Z^i(\pi_i(p) \circ A), Z^j(\pi_j(q) \circ A)) \\ &= \sum_{i,j=0}^{d-1} \int_{\mathbb{T}} \pi_i(p)(z) \overline{\pi_j(q)(z)} \langle E(\mathrm{d}z) Z^i, Z^j \rangle_{\mathcal{K}}. \end{aligned}$$

All this brings us to [6]

Theorem 2. *A semi-inner product s satisfies* (12) *if and only if*

$$s(p, q) = \int_{\mathbb{T}} \langle M(\mathrm{d}z) \pi(p)(z), \pi(q)(z) \rangle_{\mathbb{C}^N} \tag{15}$$

(*with $\pi(p)$ and $\pi(q)$ are defined by* (9)) *whereas the positive definite $d \times d$-matrix measure M is defined as*

$$M \stackrel{\text{df}}{=} \Big\langle E(\cdot)(1, Z, \ldots, Z^{d-1}), (1, Z, \ldots, Z^{d-1}) \Big\rangle_{\mathcal{K}}.$$

Proof. The only thing which still would need some comment is that if s is represented by (14), then it satisfies (12); but this is straightforward. ∎

Using (15) and (11) we come to the following observation.

Corollary 3. *Suppose the semi-inner product s satisfies* (12). *Then positivity of s is equivalent to*

$$\sum_{m,n=0}^{K} q((r_n \circ A) p_m, (q_m \circ A) p_n) \geq 0, \quad p_1, \ldots, p_K, q_1, \ldots, q_K \in \mathbb{C}[Z]. \tag{16}$$

[4] a rough way of doing this is to pass to the quotient space, complete it afterwards and claim that everything goes well; the other is to quote something, for someone who prefers the latter we refer to [14] where some more information on the technique can be found

[5] this requires some identifications on the way, as dropping the tilde ˜ for instance, which we omit

[6] a unconventional approach to matrix integration can be found in [15]

3.2 The harmonic case

By this we mean s to satisfy

$$s(Ap, q) = s(p, Aq), \quad p, q \in \mathbb{C}[Z] \tag{17}$$

for some fixed $A \in \mathbb{C}[Z]$. The formula $T : \tilde{p} \mapsto \widetilde{Ap}$ makes T a well defined linear operator provided

$$s(Ap, Ap) = 0 \quad \text{implies} \quad s(p, p) = 0. \tag{18}$$

T is apparently a symmetric operator in $\mathcal{H}$ and it has a selfadjoint extension therein (the latter fact finds its argument in [15]). If E is the spectral measure of an extension of T in $\mathcal{H}$, then as above we come to the following

Theorem 4. *Suppose a semi-inner product s is such that* (18) *holds. Then s satisfies* (17) *if and only if*

$$s(p, q) = \int_{\mathbb{R}} \langle M(\mathrm{d}x)\pi(p)(x), \pi(q)(x)\rangle_{\mathbb{C}^N} \tag{19}$$

(*with $\pi(p)$ and $\pi(q)$ are defined by* (9)) *whereas the positive definite $d \times d$-matrix measure M is defined as*

$$M \stackrel{\text{df}}{=} \Big\langle E(\cdot)(1, Z, \dots, Z^{d-1}), (1, Z, \dots, Z^{d-1})\Big\rangle_{\mathcal{K}}.$$

Remark 5. One should mention here that some higher order recurrence relations (considered in [1], [2] and, in a sense also in [3]) are in fact immediately related to the kind of invariance we consider here; their relation to other properties, like matrix representation, is less direct. More precisely, we have the following.

Suppose A is a nonconstant polynomial in $\mathbb{C}[Z]$. Then the following conditions are equivalent

- (α) *s satisfies* (17);
- (β) *the sequence $\{p_n\}_{n=0}^{\infty}$ of polynomials orthonormal with respect to s satisfies the following recurrence relation*

$$Ap_n = \sum_{i=n-d}^{n+d} a_i^{(n)} p_i, \quad n = 0, 1, \dots, \quad a_i^{(n)} = 0, \quad \textit{if} \quad n - d < 0,$$

with $a_i^{(n)}$ such that $a_i^{(n)} = \overline{a_n^{(i)}}$, $n - d \le i \le n + d$ and $a_n^{(n+d)}$ always non-zero.

The proof $\alpha \Rightarrow \beta$ goes the same way as that for orthonormal polynomials on the real line. For the other implication insert the recurrence relation into both sides of (17) and compare.

4 The Sobolev type representation of s

Suppose we know s is already of the form

$$s(p,q)=\sum_{i=0}^{N}\int_{\mathcal{S}_i} p^{(i)}\overline{q^{(i)}}\,\mathrm{d}\mu_i,\quad p,q\in\mathbb{C}[Z], \tag{20}$$

with $N\in\{0,1,\dots\}\cup\{+\infty\}$. For fixed p and q this sum is finite though its length may increase with the degree of those polynomials. We do not fix N *a priori* but we try to *determine* it from the degree of A and the invariance property (12) or (17) as well as to *localize* the supports of μ_i's.

Let us short the notation by

$$s_k(p,q)\overset{\mathrm{df}}{=}\int_{\mathcal{C}} p^{(k)}\overline{q^{(k)}}\,\mathrm{d}\mu_k,\quad p,q\in\mathbb{C}[Z]\quad,k=0,1,\dots, \tag{21}$$

$$\mathcal{E}_i(A)\overset{\mathrm{df}}{=}\begin{cases}\{z\in\mathbb{C};\ |A(z)|=1\} & \text{if } i=0\\ \{z\in\mathbb{C};\ |A(z)|=1\}\cap\mathcal{Z}(A')\cap\cdots\cap\mathcal{Z}(A^{(i)}) & \text{if } i=1,\dots,d-1\end{cases} \tag{22}$$

and

$$\mathcal{H}_i(A)\overset{\mathrm{df}}{=}\begin{cases}\{z\in\mathbb{C};\ \mathfrak{Im}\,A(z)=0\} & \text{if } i=0\\ \{z\in\mathbb{C};\ \mathfrak{Im}\,A(z)=0\}\cap\mathcal{Z}(A')\cap\cdots\cap\mathcal{Z}(A^{(i)}) & \text{if } i=1,\dots,d-1\end{cases} \tag{23}$$

where

$$\mathcal{Z}(p)\overset{\mathrm{df}}{=}\{z\in\mathbb{C};\ p(z)=0\},\quad p,q\in\mathbb{C}[Z].$$

4.1 The equipotential case

Theorem 6. *Let $A\in\mathbb{C}[Z]$ with $d=\deg A$.*

(i) If s has an integral representation[7]

$$s(p,q)=\sum_{i=0}^{\infty}\int_{\mathcal{S}_i} p^{(i)}\overline{q^{(i)}}\,\mathrm{d}\mu_i,\quad p,q\in\mathbb{C}[Z] \tag{24}$$

with $\operatorname{supp}\mu_i\subset\mathcal{S}_i$ *for* $i=0,1,\dots$ *and has the invariance property* (12) *then* (*cf.* (22))

$$\operatorname{supp}\mu_i\subset\mathcal{S}_i\cap\mathcal{E}_i(A),\quad i=0,1,,\dots,d-1, \tag{25}$$

$$\operatorname{supp}\mu_i=\varnothing,\quad i=d,d+1,\dots. \tag{26}$$

provided[8] $|A(z)|\geq 1$ *for* $z\in\mathcal{S}_k$, $k=0,1,\dots$.

[7] the summation on the right hand side of (24) always terminates, its length depends on degree of p and q as well as on multiplicities of the roots of A (the lower the multiplicities are the shorter the summation becomes)

[8] this enforces a *question*: does there exist a representation (24) of s satisfying (12) and such that $\operatorname{supp}\mu_0\cap\{z\in\mathbb{C};\ |A(z)|<1\}\neq\varnothing$; in other words, is this what follows in the main text essential or technical?

(ii) If s is such that

$$s(p,q)=\sum_{i=0}^{d-1}\int_{\mathcal{S}_i} p^{(i)}\overline{q^{(i)}}\,\mathrm{d}\mu_i, \quad p,q\in\mathbb{C}[Z] \tag{27}$$

with $\mathcal{S}_0 \overset{\mathrm{df}}{=} \mathcal{E}_i(A)$, $i=0,1,,\ldots,d-1$ *then* (12) *holds.*

Proof of (i). We prove by induction in i that

$$\int_{\mathcal{S}_i}(|A(z)|^2-1)\mu_i(\mathrm{d}z)=0, \quad \int_{\mathcal{S}_{d+i}}\mathrm{d}\mu_{d+i}=0,$$
$$\int_{\mathcal{S}_k}|A^{(k-i)}(z)|^2\mu_k(\mathrm{d}z)=0, \quad k=1,2,\ldots\,. \tag{28}$$

Notice first that the condition (12) explicitly means for $p=q$

$$\sum_{k=0}^{\infty}\int_{\mathcal{S}_k}(|(Ap)^{(k)}|^2-|p^{(k)}|^2)\,\mathrm{d}\mu_k=0. \tag{29}$$

Take $p=1$ in (29). This gives

$$\int_{\mathcal{S}_0}(|A|^2-1)\,\mathrm{d}\mu_0+\sum_{k=1}^{\infty}\int_{\mathcal{S}_k}|A^{(k)}|^2\,\mathrm{d}\mu_k=0.$$

This implies directly (28) for $i=0$. Suppose now (28) holds for $i\le n$. Then from (29) with $p=Z^{n+1}$ we get

$$((n+1)!)^2\int_{\mathcal{S}_{n+1}}(|A|^2-1)\,\mathrm{d}\mu_{n+1}-\sum_{k>n+1}\gamma_{n+1,k}^2\binom{k}{n+1}^2\int_{\mathcal{S}_k}|A^{(k-(n+1))}|^2\,\mathrm{d}\mu_k=0,$$

where $\gamma_{m,0}\overset{\mathrm{df}}{=}1$, $\gamma_{m,k}\overset{\mathrm{df}}{=}m(m-1)\cdots(m-k+1)$ if $1\le k\le m$ and $\gamma_{m,k}\overset{\mathrm{df}}{=}0$ otherwise. This is nothing but (28) for $i=n+1$. It is clear that conditions (28) hold for any n lead directly to (25) and (26). ■

Proof of (ii). Perform the indicated differentiation and make use of the specific ranges of integration. ■

4.2 The harmonic case

Here we have the following

Theorem 7. *Let $A\in\mathbb{C}[Z]$ with $d=\deg A$.*

(i) If s has an integral representation

$$s(p,q)=\sum_{i=0}^{\infty}\int_{\mathcal{S}_i} p^{(i)}\overline{q^{(i)}}\,\mathrm{d}\mu_i, \quad p,q\in\mathbb{C}[Z] \tag{30}$$

with $\operatorname{supp}\mu_i\subset\mathcal{S}_i$ *for* $i=0,1,\ldots$ *and it has the invariance property* (17) *then* (*cf.* (23))

$$\operatorname{supp}\mu_i\subset\mathcal{S}_i\cap\mathcal{H}_i(A), \quad i=0,1,,\ldots,d-1, \tag{31}$$
$$\operatorname{supp}\mu_i=\varnothing, \quad i=d,d+1,\ldots\,. \tag{32}$$

(ii) If s is such that

$$s(p,q)=\sum_{i=0}^{d-1}\int_{\mathcal{S}_i} p^{(i)}\overline{q^{(i)}}\,\mathrm{d}\mu_i, \quad p,q\in\mathbb{C}[Z] \tag{33}$$

with $\mathcal{S}_0 \overset{\mathrm{df}}{=} \mathcal{H}(A)$ *and* $\mathcal{S}_i \overset{\mathrm{df}}{=} \mathcal{H}(A)\cap\mathcal{Z}(A')\cap\mathcal{Z}(A'')\cap\cdots\cap\mathcal{Z}(A^{(i)})$, $\quad i=1,2,,\ldots,$ $d-1$ *then* (17) *holds.*

Remark 8. When $A\in\mathbb{R}[Z]$ and all $\mathcal{S}_k=\mathbb{R}$, we are in the situation of [3] (in addition to that the length N is fixed there). Assuming $\mathcal{S}_0=\mathbb{R}$ and letting the remaining $\mathcal{S}_k$'s be arbitrary, we extend the scope of Theorem 1 therein. Because $\mathbb{R}$ is only a subset of $\mathcal{H}(A)$ this still does not exhaust the whole power of the conclusion (i) of Theorem 7.

Because of (32) the series (30) terminates, that is s has the representation

$$s(p,q)=\sum_{i=0}^{N}\int_{\mathcal{S}_k} p^{(i)}\overline{q^{(i)}}\,\mathrm{d}\mu_i, \quad p,q\in\mathbb{C}[Z] \tag{34}$$

with $N\le d-1$. It may terminate earlier if a set $\mathcal{S}_k$ is chosen too narrow for some $k<d$; this happens in [3] if some $A^{(k)}$ has exclusively nonreal zeros. Of course, setting $\mathcal{S}_k=\mathbb{C}$, $k=0,1,\ldots$ let us enjoy the full length representation with $N=d-1$. Nevertheless, one has to remember that the condition (17) is a kind of (algebraic) constraint in this game.

Proof of (i). In the sequel we need the following notation

$$P_{m,k}\overset{\mathrm{df}}{=}(Z^mA)^{(k)}\overline{(Z^mA)^{(k)}}, \quad R_{m,k}\overset{\mathrm{df}}{=}(Z^m)^{(k)}\overline{(Z^mA^2)^{(k)}}, \quad m,k=0,1,\ldots.$$

From (17) we get $s(Ap,Aq)=s(p,A^2q)$. Thus, for $p=q=Z^m$ and with the above notation, we have

$$\sum_{k=0}^{\infty}\int_{\mathcal{S}_k}(P_{m,k}-R_{m,k})\,\mathrm{d}\mu_k=0, \quad m=0,1,\ldots \tag{35}$$

We prove by induction in i that

$$\int_{\mathcal{S}_i}(\mathfrak{Im}\,A(z))^2\mu_i(\mathrm{d}z)=0, \quad \int_{\mathcal{S}_{d+i}}\mathrm{d}\mu_{d+i}=0,$$
$$\int_{\mathcal{S}_k}|A^{(k-i)}(z)|^2\mu_k(\mathrm{d}z)=0, \quad k=1,2,\ldots. \tag{36}$$

For $m=0$ we get $P_{0,k}=|A^{(k)}|^2$ for $k=0,1,\ldots$ and $R_{0,k}=0$ for $k=1,2,\ldots,$ $R_{0,0}=\bar{A}^2$. Thus (35) is now

$$\int_{\mathcal{S}_0}(|A(z)|^2-\overline{A(z)^2})\mu_0(\mathrm{d}z)+\sum_{k=1}^{\infty}\int_{\mathcal{S}_k}|A^{(k)}(z)|^2\mu_k(\mathrm{d}z)=0$$

and, consequently,

$$\int_{\mathcal{S}_0}(|A(z)|^2-\mathfrak{Re}\,\overline{A(z)^2})\mu_0(\mathrm{d}z)+\sum_{k=1}^{\infty}\int_{\mathcal{S}_k}|A^{(k)}(z)|^2\mu_k(\mathrm{d}z)=0.$$

Because $|A(z)|^2 - \mathfrak{Re}\,\overline{A(z)^2} = 2(\mathfrak{Im}\,A(z))^2$ we have

$$\int_{\mathcal{S}_0} (\mathfrak{Im}\,A(z))^2 \mu_0(\mathrm{d}z) = 0, \quad \int_{\mathcal{S}_k} |A^{(k)}(z)|^2 \mu_k(\mathrm{d}z) = 0, \quad k = 1, 2, \dots. \tag{37}$$

The first of (37) gives us $\operatorname{supp}\mu_0 \subset \mathcal{H}(A) \cap \mathcal{S}_0$. The second implies

$$\operatorname{supp}\mu_k \subset \mathcal{Z}(A^{(k)}) \cap \mathcal{S}_k, \quad k = 1, 2, \dots, d-1, \quad \operatorname{supp}\mu_d \cap \mathcal{S}_d = 0. \tag{38}$$

These together establish (36) for $i = 0$.

Suppose (36) holds for $i \le n$. Take $m = n+1$. Notice that, under the induction assumption, we get

$$\int_{\mathcal{S}_k} P_{n+1,k}\,\mathrm{d}\mu_k = \begin{cases} 0 & k < n+1 \\ (k!)^2 \int_{\mathcal{S}_k} |A(z)|^2 \mu_k(\mathrm{d}z) & k = n+1 \\ (k!/(k-n-1)!)^2 \int_{\mathcal{S}_k} |A^{(k-(n+1))}(z)|^2 \mu_k(\mathrm{d}z) & k > n+1 \end{cases}$$

and

$$\int_{\mathcal{S}_k} R_{n+1,k}\,\mathrm{d}\mu_k = \begin{cases} (k!)^2 \int_{\mathcal{S}_k} \overline{A(z)^2} \mu_k(\mathrm{d}z) & k = n+1 \\ 0 & \text{otherwise} \end{cases}$$

Inserting these two formulae into (35) we deduce that (36) holds for $i = n+1$ as well. This completes the proof of (i). ■

Proof of (ii). Straightforward calculation again. ■

Remark 9. Consider a number of examples which exhibit different aspects of Theorems 6 and 7.

(a) Take $A = Z^5 - \sqrt{5}Z^4 + 4Z^5 - 2\sqrt{5}Z$. Then $\mathbb{R} \cup \{\mathrm{i}\} \cup \{-\mathrm{i}\} \subset \mathcal{H}_0(A)$ and $\mathcal{H}_1(A) = \{1/\sqrt{5}\} \cup \{\mathrm{i}\} \cup \{-\mathrm{i}\}$. This show a difference between Theorem 1 of [3] and our Theorem 7.

(b) The polynomial $A = Z^3 + 3Z + 3$ of example 2 of [3], p 459 has $\mathcal{H}_1(A) = \varnothing$ though $\mathbb{R} \subset \mathcal{H}_0(A)$. This is the reason why the Sobolev part of (30) disappears; however the standard part of (30) can be considered for $\mathcal{S}_0 = \mathcal{H}_0(A) = \{z \in \mathbb{C};\ \mathfrak{Re}\,z = 0\} \cup \{z \in \mathbb{C};\ \mathfrak{Im}\,z = 0\} \cup \{z \in \mathbb{C};\ \mathfrak{Re}\,z = \mathfrak{Im}\,z\}$.

(c) For $A = \mathrm{i}\,Z^2 - \mathrm{i}$ one has $\mathcal{H}_0(A) = \mathbb{T} \overset{\mathrm{df}}{=} \{z \in \mathbb{C};\ (\mathfrak{Re}\,z)^2 + (\mathfrak{Im}\,z)^2 = 1\}$. This is the unit circle case though the invariance (17) is *not* the usual one. Nevertheless the Sobolev part may not be trivial because $\mathcal{H}_1(A) = \{\mathrm{i}, -\mathrm{i}\}$

(d) The usual invariance for the unit circle is ceq8 and $A = Z$. Here $\mathcal{E}_0 = \mathbb{T}$ (cf. the case (c) above) and $\mathcal{E}_1(A) = \varnothing$ (no nontrivial Sobolev part on the unit circle under the invariance (12)). For the lemniscate $A = Z^2 - 1$ we have the Sobolev part at $\mathcal{H}_1(A) = \{0\}$.

One always has uniqueness of A in the sense it determines $\mathcal{E}_0(A)$ or $\mathcal{H}_0(A)$ (two different polynomials do not lead to the same set because of harmonicity of the defining functions). This contributes to the question of uniqueness solved in [3] by Theorem 6. That Theorem gives an answer to the question of uniqueness and existence in one: giving a construction expressed by formula (3.25) therein. However if we follow this construction trying to determine a polynomial A for which $\mathcal{Z}(A') = \{-1\} \cup \{\mathrm{i}\}$ and $\mathcal{Z}(A') \cap \mathcal{Z}(A'') = \varnothing$ (and forgetting about $\mathcal{H}_0(A)$ for a while) what we get is $A = \frac{1}{3}Z^3 + (1+\mathrm{i})Z - \mathrm{i}$. Then $\mathcal{H}_1(A)$ becomes empty. Thus the question of uniqueness and existence of A becomes more complex and this is the price we pay extending applicability of results of [3] to much broader context.

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