A proof of the Great Picard Theorem

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Abstract

In this paper we present a simple and self contained proof of the great Picard Theorem based on certain Harnack-type inequalities due to J. Lewis.

1 Introduction.

The classical "**Little Picard Theorem**" asserts that a nonconstant entire function omits at most one complex value. This result was first proved by Picard and subsequently a number of different proofs have been given, the most recent being due to J. Lewis [6].

Rickman [7] obtained an analogue of Picard's theorem for entire quasiregular mappings. He proved that a nonconstant entire quasiregular mapping in $\mathbb{R}^n$ can omit only finitely many values. Rickman's original proof of this result was obtained via the "modulus method". In [4], Eremenko and Lewis studied uniform limits of certain $\mathcal{A}$-harmonic functions in a ball of $\mathbb{R}^n$ where $\mathcal{A}$ satisfies certain elliptic structure conditions and, as an application of their work, they obtained a completely P.D.E. proof of Rickman's theorem. Finally, J. Lewis proved in [6] that both Picard's theorem and Rickman's theorem are rather easy consequences of a Harnack-type inequality.

There is a stronger version of Picard's theorem: "An entire function which is not a polynomial takes every complex value, with at most one exception, infinitely many times".

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Actually, the latter theorem is true in a local situation:

"If z_0 is a point of the Riemann sphere $\mathbb{C}^*$ and f is a function which is holomorphic in a punctured neighborhood of z_0 and has an essential singularity at z_0 then in every neighborhood of z_0 the function f takes every complex value, with at most one exception, infinitely many times".

This is the so called "**Great Picard Theorem**" which is a remarkable strengthening of the Theorem of Casorati-Weierstrass (see e.g. [1, p. 129] or [3, p. 109]) which simply asserts that the image of every neighborhood of z_0 is dense in $\mathbb{C}$ if f has an essential singularity at z_0.

Many proofs of the Great Picard Theorem are known. In this paper we shall show that Lewis' ideas can be used to obtain a new elementary one. Section 2 will be devoted to recall the Harnack-type results which are the key ingredients in Lewis' method. These results will be used in Section 3 to give a proof of the Great Picard Theorem.

We finish this section fixing some notation. If $R > 0$ and $w \in \mathbb{C}$ then $\Delta(w, R)$ will denote the open disc of center w and radius R, i. e., $\Delta(w, R) = \{z \in \mathbb{C} : |z-w| < R\}$. The unit disc will be denoted by Δ, hence, $\Delta = \Delta(0, 1)$. If E is a subset of $\mathbb{C}$ then $\overline{E}$ will denote the closure of E. Finally, if h is a real valued function defined in the disc $\Delta(w, R)$ then we shall set

$$M(r, h, w) = \sup_{|z-w|<r} h(z), \quad 0 < r < R.$$

2 Harnack-type results.

In this section we shall start recalling Harnack's inequality for positive harmonic functions. We must remark that we could state more general forms of this inequality, however, we have preferred to state it in the simplest form which will be enough for our purposes. The same commentary can be applied to some other results which will be stated below. We cite chapter 3 of [2] and chapter 11 of [8] as general references for this topic.

The Poisson kernel $P_r(\theta) = \frac{1-r^2}{|1-re^{i\theta}|^2}$ satisfies the simple inequality

$$\frac{1}{3} \le P_r(\theta) \le 3, \quad 0 < r \le \frac{1}{2}, \quad \theta \in \mathbb{R}.$$

This inequality and the representation of harmonic functions in a disc as Poisson integrals easily imply the so called Harnack's inequality.

Harnack's inequality. *Let h be a positive harmonic function in the disc $\Delta(w, R)$. Then*

$$\frac{1}{3}h(w) \le h(z) \le 3h(w), \quad \textit{if} \quad |z-w| \le \frac{R}{2}, \tag{2.1}$$

or, equivalently,

$$\frac{1}{3}h(w) \le M\left(\frac{R}{2}, h, w\right) \le 3h(w). \tag{2.2}$$

Using this result we can prove the following Lemma of Lewis [6] which is a result of Harnack type for (non necessarily positive) harmonic functions.

Lemma 1 (Lewis). *There exists a positive constant A such that if u is any bounded harmonic function in the unit disc Δ with $u(0)=0$ then there exist $w \in \Delta$ and $r>0$ (which depend on u) such that*

$$u(w)=0, \tag{2.3}$$

$$\overline{\Delta(w,2r)} \subset \Delta, \tag{2.4}$$

and

$$A^{-1}M\left(\frac{1}{2},u,0\right) \le M(2r,u,w) \le AM(r,u,w). \tag{2.5}$$

Let us remark that actually Lewis proved in [6] a form of this lemma for the so called Harnack functions in $\mathbb{R}^n$. Hence, we shall include a proof of the lemma in our setting for the sake of completeness.

Proof of Lemma 1. Let u be a bounded harmonic function in Δ with $u(0)=0$. Given $z \in \Delta$, we define

$$\delta(z) = 1-|z| = \operatorname{dist}(z,\partial\Delta). \tag{2.6}$$

Let

$$E = \{z \in \Delta : u(z)=0\},$$

$$F = \overline{\bigcup_{z\in E} \Delta\left(z, \frac{\delta(z)}{4}\right)}.$$

Define

$$\gamma = \sup\{u(z) : z \in F\} = \sup_{z\in E} M\left(\frac{\delta(z)}{4}, u, z\right),$$

since u is bounded, $\gamma<\infty$. Choose $w \in E$ such that

$$M\left(\frac{\delta(w)}{4}, u, w\right) \ge \frac{\gamma}{2}, \tag{2.7}$$

and take

$$r = \frac{\delta(w)}{4}. \tag{2.8}$$

We shall prove that the conclusion of Lemma 1 holds with this choice of w and r.

Since $w \in E$, we have that $u(w)=0$. Also, (2.8) implies that $\overline{\Delta(w,2r)} \subset \Delta$. Hence, it remains to prove (2.5). Notice that (2.7) is equivalent to

$$M(r,u,w) \ge \frac{\gamma}{2}. \tag{2.9}$$

Let $z \in \Delta(0,\frac{1}{2})$ with $u(z)\ge 0$.
-If $z \in F$ then $u(z) \le \gamma \le 2M(r,u,w)$.
-If $z \notin F$ then, since $0 \in F$ and F is closed, we have that (using interval notation to denote the segment connecting two points) there exists $z' \in [z,0] \cap F$ such that $[z,z') \cap F = \emptyset$. Then, it is clear that $u>0$ in $[z,z')$. In fact, we have that

$$u>0, \quad \text{in } \Delta\left(\xi, \tfrac{1}{10}\right), \quad \text{for every } \xi \in [z,z'). \tag{2.10}$$

Proof of 2.10. Let $\xi \in [z, z')$ and $\xi' \in \Delta\left(\xi, \frac{1}{10}\right)$. Since $\xi \in [0, z]$ and $|z| < \frac{1}{2}$, we have that $|\xi| < \frac{1}{2}$ which implies that $\delta(\xi) > \frac{1}{2}$. Now,

$$\delta(\xi') \geq \delta(\xi) - |\xi - \xi'| \geq \frac{1}{2} - \frac{1}{10} = \frac{4}{10} > 4|\xi - \xi'|,$$

hence,

$$|\xi - \xi'| < \frac{\delta(\xi')}{4}$$

which implies that $u(\xi') \neq 0$ (because if $u(\xi') = 0$ then ξ would belong to F) and then $u(\xi') > 0$. This proves (2.10). ■

Using (2.10) and Harnack's inequality (2.2), we deduce that

$$M\left(\frac{1}{20}, u, \xi\right) \leq 3u(\xi), \quad \text{for every } \xi \in [z, z']. \tag{2.11}$$

Define

$$\xi_k = z' + \frac{z - z'}{10}k, \quad k = 0, 1, \ldots, 10.$$

then, it is clear that $\xi_0 = z'$, $z_{10} = z$, and that

$$\xi_k \in [z, z') \quad \text{and} \quad |\xi_k - \xi_{k-1}| < \frac{1}{20} \quad \text{for every } k.$$

Then, using (2.11) with $\xi = \xi_k$, and $k = 9, 8, \ldots, 1, 0$ successively, we obtain

$$u(z) = u(z_{10}) \leq 3u(z_9) \leq \cdots \leq 3^k u(z_{10-k}) \leq \cdots \leq 3^{10} u(z_0) = 3^{10} u(z').$$

Now, since $z' \in F$, having in mind (2.9), we see that $u(z') \leq \gamma < 2M(r, u, w) \leq 2M(2r, u, w)$ and, hence

$$u(z) \leq 2 \times 3^{10} M(2r, u, w).$$

This proves the left hand side inequality of (2.5) with $A = 2 \times 3^{10}$.

To prove the other inequality we argue in a similar way. Take $w' \in \Delta(w, 2r)$ with $u(w') \geq 0$.

-If $w' \in F$, then $u(w') \leq \gamma \leq 2M(r, u, w)$.

-If $w' \notin F$, then, since $w \in F$, there exists $w_1 \in (w', w) \cap F$ such that $[w', w_1) \cap F = \emptyset$. Then, arguing as before, we can prove that

$$u > 0, \quad \text{in } \Delta\left(\xi, \tfrac{r}{5}\right), \quad \text{for every } \xi \in [w', w_1),$$

which, using Harnack's inequality as above, gives us

$$u(w') \leq 2 \times 3^{10} M(r, u, w).$$

Hence, we have the right hand side inequality of (2.5) also with $A = 2 \times 3^{10}$. This finishes the proof of Lemma 1. ■

Harnack's inequalities lead to important convergence theorems for harmonic functions. Among them we shall mention the following which is basic in Lewis' method.

Proposition 1. *Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of positive harmonic functions in the unit disc Δ. Then there are only two possibilities: Either $u_n(z) \to \infty$ for every $z \in \Delta$, or there exist a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ of $\{u_n\}$ and u, a function which is harmonic in Δ, such that $u_{n_k} \to u$, as $k \to \infty$, uniformly on every compact subset of Δ.*

Proposition 1 can be deduced from Harnack's inequalities (see [2, p. 57]). Alternatively, it can also be deduced from Montel's theorem on normal families. Indeed, suppose that $\{u_n\}_{n=1}^{\infty}$ is a sequence of positive harmonic functions in the unit disc Δ and that there exists a point $z_0 \in \Delta$ such that $u_n(z_0)$ does not tend to ∞. Then there exist a constant $M > 0$ and a subsequence $\{u_{n_j}\}_{j=1}^{\infty}$ of $\{u_n\}$ such that $u_{n_j}(z_0) \leq M$ for every j. For every j, let v_{n_j} denote the conjugate harmonic function of u_{n_j}, normalized so that $v_{n_j}(z_0) = 0$, $f_{n_j} = u_{n_j} + iv_{n_j}$ and $g_{n_j} = f_{n_j} - u_{n_j}(z_0)$. Then, for every j, g_{n_j} is a function which is holomorphic in Δ with $g_{n_j}(z_0) = 0$ and the image of g_{n_j} is contained in the half-plane $\{w \in \mathbb{C} : \operatorname{Re} w > -M\}$. Schwarz's lemma easily implies that the sequence $\{g_{n_j}\}$ is uniformly bounded in every compact subset of Δ. Since $u_{n_j}(z_0) \leq M$ for every j, the sequence $\{f_{n_j}\}$ is also uniformly bounded in every compact subset of Δ and then, using Montel's theorem [3, p. 153], we deduce that $\{f_{n_j}\}$ contains a subsequence which converges uniformly in every compact subset of Δ. Then the conclusion of Proposition 1 follows.

3 A proof of the Great Picard Theorem.

As we have already mentioned, the Great Picard Theorem can be proved in a number of different ways. One of them is based on the so called Schottky's theorem (see e.g. [3, p. 298-301]) which asserts that for any $\alpha > 0$ the family of those functions f which are analytic in the unit disc Δ, omit the values 0 and 1 and such that $|f(0)| \leq \alpha$ is uniformly bounded in every compact subset of Δ. Using the results stated in section 2 we shall prove the following proposition which is inspired by [4, Th. 1] and implies a result which is slightly weaker than Schottky's theorem but which is enough to give a proof of the Great Picard Theorem.

Proposition 2. *There exists a constant $B > 1$ such that if λ is any positive real number and u_1, u_2 are two harmonic functions in Δ satisfying*

$$\{z \in \Delta : u_1(z) < -\lambda\} \bigcap \{z \in \Delta : u_2(z) < -\lambda\} = \emptyset, \tag{3.1}$$

$$|u_1^+ - u_2^+| \leq \lambda, \tag{3.2}$$

and

$$|u_j(0)| \leq \lambda, \quad j = 1, 2, \tag{3.3}$$

then

$$M\left(\frac{1}{2}, u_j, 0\right) \leq B\lambda, \quad j = 1, 2. \tag{3.4}$$

To prove Proposition 2, it is clear that it suffices to consider functions u_1 and u_2 which are bounded. It is easy to see that if $\lambda > 0$ and u_1, u_2 are bounded harmonic in Δ which satisfy (3.1), (3.2) and (3.3) and

$$u(z) = \frac{u_1(z) - u_1(0)}{3\lambda}, \quad v(z) = \frac{u_2(z)}{3\lambda}, \quad z \in \Delta,$$

then u and v are bounded and harmonic in Δ and satisfy

$$\{z \in \Delta : u(z) < -1\} \bigcap \{z \in \Delta : v(z) < -1\} = \emptyset, \tag{3.5}$$

$$|u^+ - v^+| \leq 1, \tag{3.6}$$

$$u(0) = 0, \tag{3.7}$$

and

$$|v(0)| \leq 1. \tag{3.8}$$

Furthermore,

$$M\left(\frac{1}{2}, u_1, 0\right) \leq 3\lambda M\left(\frac{1}{2}, u, 0\right) + \lambda$$

and

$$M\left(\frac{1}{2}, u_2, 0\right) \leq 3\lambda M\left(\frac{1}{2}, v, 0\right).$$

Hence, it is clear that Proposition 2 follows from the following.

Proposition 3. *There exists a constant $B > 1$ such that if u and v are two bounded harmonic functions in Δ which satisfy (3.5), (3.6), (3.7) and (3.8) then*

$$M\left(\frac{1}{2}, u, 0\right) \leq B \tag{3.9}$$

and

$$M\left(\frac{1}{2}, v, 0\right) \leq B.$$

Proof of Proposition 3. First of all, let us notice that, using (3.6) and (3.7), we have that

$$M\left(\frac{1}{2}, v, 0\right) \leq M\left(\frac{1}{2}, v^+, 0\right) \leq M\left(\frac{1}{2}, u^+, 0\right) + 1 = M\left(\frac{1}{2}, u, 0\right) + 1.$$

Hence, it suffices to prove (3.9). We shall do this arguing by contradiction. Hence, let us suppose that there exist two sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ of bounded harmonic functions in Δ satisfying

$$\{z \in \Delta : u_n(z) < -1\} \bigcap \{z \in \Delta : v_n(z) < -1\} = \emptyset, \tag{3.11}$$

$$|u_n^+ - v_n^+| \leq 1, \tag{3.12}$$

$$u_n(0) = 0, \tag{3.13}$$

and

$$|v_n(0)| \leq 1, \tag{3.14}$$

for every n, and such that

$$m_n = M\left(\frac{1}{2}, u_n, 0\right) \to \infty, \quad \text{as } n \to \infty. \tag{3.15}$$

Using Lemma 1, we deduce that for every n there exist $z_n \in \Delta$ and $r_n > 0$ with $\overline{\Delta(z_n, 2r_n)} \subset \Delta$ such that

$$u_n(z_n) = 0 \tag{3.16}$$

and

$$A^{-1} m_n \leq M_n \leq A M_n' \tag{3.17}$$

where, $M_n = M(2r_n, u_n, z_n)$ and $M_n' = M(r_n, u_n, z_n)$. Notice that (3.15) and (3.17) imply that

$$M_n \to \infty, \quad \text{as } n \to \infty.$$

Define

$$U_n(z) = \frac{u_n(z_n + 2r_n z)}{M_n}, \quad V_n(z) = \frac{v_n(z_n + 2r_n z)}{M_n}, \quad z \in \Delta.$$

The functions U_n and V_n are bounded and harmonic in Δ and satisfy

$$U_n(0) = 0, \tag{3.18}$$

$$M\left(\frac{1}{2}, U_n, 0\right) \geq A^{-1}, \tag{3.19}$$

$$|U_n^+ - V_n^+| \leq \frac{1}{M_n}, \tag{3.20}$$

and

$$\{z \in \Delta : U_n(z) < \frac{-1}{M_n}\} \bigcap \{z \in \Delta : V_n(z) < \frac{-1}{M_n}\} = \emptyset. \tag{3.21}$$

Furthermore, it is clear that $U_n \leq 1$ for every n. Hence, applying Proposition 1 to the sequence $\{1 - U_n\}$ and having in mind (3.18), we deduce that a subsequence $\{U_{n_j}\}$ of $\{U_n\}$ converges uniformly in every compact subset of Δ to a function U which is harmonic in Δ. Using (3.20), we see that the sequence $\{V_n\}$ is also uniformly bounded above and, hence, the sequence $\{V_{n_j}\}$ contains a subsequence which is uniformly convergent in every compact subset of Δ to a function V which is either harmonic in Δ or identically equal to $-\infty$ in Δ. In view of (3.18), (3.19), (3.20) and (3.21), we see that

$$U(0) = 0, \tag{3.22}$$

$$M\left(\frac{1}{2}, U, 0\right) \geq A^{-1}, \tag{3.23}$$

$$U^+ = V^+, \tag{3.24}$$

and

$$\max(U, V) \geq 0. \tag{3.25}$$

By (3.23) we see that there exists $\xi \in \Delta$ such that $U(\xi) > 0$. Then (3.24) implies that V is harmonic in Δ and that $U = V$ in a neighborhood of ξ, which implies that $U = V$ in Δ. Then (3.25) implies that $U \geq 0$ in Δ which, with (3.22) and the maximum principle, implies that $U \equiv 0$ in Δ. This is a contradiction with (3.23). Hence, this finishes the proof. ∎

A proof of the Great Picard Theorem. Suppose that the Great Picard Theorem were false. Then there would exist a point $z_0 \in \mathbb{C}^*$, a function f which is holomorphic in a punctured neighborhood of z_0 and has an essential singularity at z_0 and two complex numbers w_1, w_2 with $w_1 \neq w_2$ such that the equations

$$f(z) = w_1, \qquad f(z) = w_2,$$

would have at most a finite number of solutions in some punctured neighborhood of z_0. By replacing this neighborhood by a smaller one, we can assume that

$$f(z) \neq w_1, \quad \text{and} \quad f(z) \neq w_2$$

for every z in a punctured neighborhood G of z_0. Finally, composing f with simple Möbius transformations, we can assume further that

$$z_0 = \infty, \quad w_1 = 0, \quad w_2 = 1, \quad G = \{z \in \mathbb{C} : |z| > R\}.$$

Hence, we have seen that the Great Picard Theorem is equivalent to the following result. ∎

Proposition 4. *Let $R > 0$ and let f be a function which is analytic in $G = \{z \in \mathbb{C} : |z| > R\}$. Suppose that $0, 1 \notin f(G)$. Then ∞ is not an essential singularity of f.*

The proof of Proposition 4 that we are going to present will be based on Proposition 2 and will use arguments which are similar to those used by Fuchs in chapter V of [5] to deduce the Great Picard Theorem directly from Schottky's theorem.

Proof of Proposition 4. Suppose that f satisfies the conditions of Proposition 4 and further that ∞ is an essential singularity of f. Using the Casorati-Weierstrass theorem, we deduce that, for every $r > R$, $f(\{|z| > r\})$ is a dense subset of $\mathbb{C}$. Hence, there exists a sequence of numbers $\{r_n\}_{n=1}^{\infty}$ with

$$2R < r_1 < r_2 < \cdots < r_n < r_{n+1} < \ldots$$

and $r_n \to \infty$, as $n \to \infty$, such that for every n there exists $z_n \in \mathbb{C}$ with $|z_n| = r_n$ and

$$e + 1 < |f(z_n)| < e^2. \tag{3.26}$$

Set

$$u_1(z) = \log|f(z)|, \quad u_2(z) = \log|f(z) - 1|, \quad z \in G. \tag{3.27}$$

Since f omits the values 0 and 1 in G, we see that u_1 and u_2 are harmonic functions in G. It is easy to see that

$$|u_1^+ - u_2^+| \leq 1, \tag{3.28}$$

and

$$\max(u_1, u_2) \geq \log\frac{1}{2}. \tag{3.29}$$

Take $n \geq 1$. Notice that, since $r_n > 2R$, we have $\overline{\Delta(z, \frac{r_n}{2})} \subset G$ for every z with $|z| = r_n$. Let N be a natural number so that

$$|1 - e^{2\pi i/N}| < \frac{1}{4}. \tag{3.30}$$

Define

$$z_{n,k} = z_n e^{2k\pi i/N}, \quad k = 0, 1, \ldots, N, \tag{3.31}$$

(so $z_{n,0} = z_n$). Notice that

$$z_{n,k} \in \Delta\left(z_{n,k-1}, \frac{r_n}{4}\right), \quad k = 1, 2, \ldots N, \tag{3.32}$$

and that the discs $\Delta\left(z_{n,k}, \frac{r_n}{4}\right)$, $k = 0, 1, \ldots, N-1$ cover the circle $|z| = r_n$.

Define

$$u_{j,k}(z) = u_j\left(z_{n,k} + \frac{r_n}{2}z\right), \quad z \in \Delta, \quad k = 0, 1 \ldots, N-1, \quad j = 1, 2. \tag{3.33}$$

The functions $u_{j,k}$ are bounded and harmonic in Δ. Using (3.26), we see that

$$|u_{j,0}(0)| \leq 3, \quad j = 1, 2, \tag{3.34}$$

and, (3.28) and (3.29) imply

$$|u^+_{1,0} - u^+_{2,0}| \leq 3, \tag{3.35}$$

and

$$\{z \in \Delta : u_{1,0}(z) < -3\} \bigcap \{z \in \Delta : u_{2,0}(z) < -3\} = \emptyset. \tag{3.36}$$

Then, using Proposition 2, we deduce that

$$M\left(\frac{1}{2}, u_{j,0}, 0\right) \leq 3B, \quad j = 1, 2,$$

which is equivalent

$$M\left(\frac{r_n}{4}, u_j, z_{n,0}\right) \leq 3B, \quad j = 1, 2. \tag{3.37}$$

In particular, since $|z_{n,1} - z_{n,0}| < \frac{r_n}{4}$, we have that

$$|u_j(z_{n,1})| \leq 3B, \quad j = 1, 2. \tag{3.38}$$

Using (3.38), (3.28) and (3.29), we obtain that the functions $u_{1,1}$ and $u_{2,1}$ satisfy

$$|u_{j,1}(0)| \leq 3B, \quad j = 1, 2,$$

$$|u^+_{1,1} - u^+_{2,1}| \leq 3B,$$

and

$$\{z \in \Delta : u_{1,1}(z) < -3B\} \bigcap \{z \in \Delta : u_{2,1}(z) < -3B\} = \emptyset.$$

Then, using again Proposition 2, we obtain

$$M\left(\frac{1}{2}, u_{j,1}, 0\right) \leq 3B^2, \quad j = 1, 2,$$

or, equivalently,

$$M\left(\frac{r_n}{4}, u_j, z_{n,1}\right) \leq 3B^2, \quad j = 1, 2.$$

Repeating this process, after N steps, we obtain that setting $M = 3B^N$,

$$M\left(\frac{r_n}{4}, u_j, z_{n,k}\right) \le M, \quad j = 1,2, \quad k = 0,1,\dots N-1. \tag{3.39}$$

Now, since the discs $\Delta\left(z_{n,k}, \frac{r_n}{4}\right)$ cover the circle $|z| = r_n$, (3.39) implies that

$$u_j(z) \le M, \quad \text{if } |z| = r_n, \quad j = 1,2,$$

and, hence,

$$|f(z)| \le e^M, \quad \text{if } |z| = r_n \quad n = 1,2,\dots.$$

Using the maximum principle in each of the rings $r_n \le |z| < r_{n+1}$, we deduce that f is bounded in $|z| > r_1$. This is a contradiction with the assumption that ∞ is an essential singularity of f. Hence, the proof is complete. ■

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