Conjugation of standard morphisms and a generalization of singular words

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Abstract

We study the action of right conjugates of a standard morphism on the infinite word (if it exists) generated by this morphism. When g is the i-th right conjugate of a standard morphism generating an infinite word $\mathbf{x}$, $g(\mathbf{x})$ is the i-th conjugate of $\mathbf{x}$. We design an algorithm to obtain a canonical decomposition of all the right conjugates of a standard morphism. As an application we compute the sequence of conjugates of the powers of the Fibonacci morphism and then we generalize, to all the conjugates of $\mathbf{F}$, Wen and Wen's decomposition of the Fibonacci word $\mathbf{F}$ in singular words.

Résumé

Nous étudions l'action des conjugués à droite d'un morphisme standard sur le mot infini (s'il existe) engendré par ce morphisme. Quand g est le i-ème conjugué à droite d'un morphisme standard engendrant un mot infini $\mathbf{x}$, $g(\mathbf{x})$ est le i-ème conjugué de $\mathbf{x}$. Nous décrivons un algorithme pour obtenir une décomposition canonique de tous les conjugués à droite d'un morphisme standard, puis, appliquant ce résultat au calcul de la suite de tous les conjugués à droite du morphisme de Fibonacci, nous généralisons à l'ensemble des conjugués de $\mathbf{F}$ la décomposition de Wen et Wen du mot de Fibonacci $\mathbf{F}$ en mots singuliers.

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1 Introduction

A Sturmian word is an infinite binary word which contains exactly $n+1$ distinct factors of length n for every positive integer n. These words have many characterizations and numerous properties. They are used in various fields of Mathematics such as symbolic dynamics or the study of continued fraction expansion, but also in Physics and, of course, in some domains of Computer Science as infography, formal language theory, algorithms on words, combinatorics on words. For a wide description of various characterizations and properties of Sturmian words, as well as many references and historical remarks, see [1], chapter 2.

The best known example of a Sturmian word is the Fibonacci word which has a lot of beautiful properties. In particular, in [7], the authors establish a decomposition of this word in what they call "singular words".

The Fibonacci word is generated by a morphism which is Sturmian, i.e., a morphism that preserves Sturmian words. It is proved in [3] that these morphisms are obtained by composing in any number and order three particular morphisms, but such a decomposition is in general not unique for a given Sturmian morphism (see [5]). Standard morphisms are a particular subset of Sturmian morphisms, using only two of the three particular morphisms. In [6] and [1], it is proved that any Sturmian morphism can be obtained as a right conjugate of a standard one.

We begin in Section 2 by presenting a few preliminary definitions and a basic result (Lemma 2.2) which allows us to compute the image of the word generated by a standard morphism associated to each right conjugate of this morphism. Then in Section 3 we give a construction of Sturmian morphisms as right conjugates of standard ones which allows to associate canonically to each Sturmian morphism a unique decomposition. Using this, in Section 4 we compute the sequence of conjugates of a power of the Fibonacci morphism (4.1) and then we extend the notion of singular words and of decomposition of the conjugates of the Fibonacci word in such factors (4.2).

2 Preliminaries

The terminology and notations are mainly those of Lothaire, 2002 [1].

Let A be a finite set called *alphabet* and A^* the free monoid generated by A.

The elements of A are called *letters* and those of A^* are called *words*. The *empty word* ε is the neutral element of A^* for the concatenation of words (the *concatenation* of two words u and v is the word uv), and we denote by A^+ the semigroup $A^* \setminus \{\varepsilon\}$.

If $u \in A^*$, then $|u|$ is the *length* of u (in particular $|\varepsilon| = 0$).

Let $w \in A^*$. The *inverse* of w, say w^{-1}, is defined by $ww^{-1} = w^{-1}w = \varepsilon$. Note that this is simply notation, i.e., for $u_1, u_2, w \in A^*$, the words $u_1^{-1}w$ and wu_2^{-1} are defined only if w starts with u_1 and ends with u_2. Remark that if $w = uv$ then $wv^{-1} = u$ and $u^{-1}w = v$.

A word w is called a *factor* (resp. a *prefix*) of u if there exist words x, y such that $u = xwy$ (resp. $u = wy$). The factor (resp. the prefix) is *proper* if $xy \neq \varepsilon$ (resp. $y \neq \varepsilon$).

An *infinite word* (or *sequence*) over A is an application $\mathbf{a} : \mathbb{N} \to A$. It is written $\mathbf{a} = a_0a_1 \dots a_i \dots,\ i \in \mathbb{N}, a_i \in A$.

The notion of factor is extended to infinite words as follows: a (finite) word u is a *factor* (resp. a *prefix*) of an infinite word $\mathbf{a}$ over A if there exist $n \in \mathbb{N}$ (resp. $n = 0$) and $m \in \mathbb{N}$, $m = |u|$, such that $u = a_n \dots a_{n+m-1}$ (by convention $a_n \dots a_{n-1} = \varepsilon$). If $\mathbf{a} = u\mathbf{a}'$, $u \in A^*$, then $u^{-1}\mathbf{a} = \mathbf{a}'$.

In what follows, we will consider morphisms on A.

A *morphism on A* (in short *morphism*) is an application $f : A^* \to A^*$ such that $f(uv) = f(u)f(v)$ for all $u, v \in A^*$. It is uniquely determined by its value on the alphabet A. If w is a (finite) word over A and f a morphism on A then $f(w^{-1})$ is defined as $[f(w)]^{-1}$.

A morphism is *prolongable on* x_0, $x_0 \in A$, if there exists $u \in A^+$ such that $f(x_0) = x_0u$. If, for all $n \in \mathbb{N}$, the word $f^n(x_0)$ is a proper prefix of the word $f^{n+1}(x_0)$ then the sequence $(f^n(x_0))_{n\geq 0}$ converges to a unique infinite word

$$\mathbf{x} = x_0uf(u)f^2(u)\dots f^n(u)\dots$$

which is the limit of the sequence $(f^n(x_0))_{n\geq 0}$. We write $\mathbf{x} = f^\omega(x_0)$ and say that $\mathbf{x}$ is *generated* by f.

From now on, A will be the two-letter alphabet $A = \{a, b\}$.

2.1 Sturmian words and morphisms, standard morphisms

A *Sturmian word* is an infinite word over A which contains exactly $n + 1$ distinct factors of length n for every positive integer n. A morphism $f : A^* \to A^*$ is *Sturmian* if $f(\mathbf{x})$ is a Sturmian word whenever $\mathbf{x}$ is.

We define on A the following three morphisms:

$$\begin{array}{lcll}
\varphi & : & \varphi(a) = ab, & \varphi(b) = a \\
\tilde{\varphi} & : & \tilde{\varphi}(a) = ba, & \tilde{\varphi}(b) = a \\
E & : & E(a) = b, & E(b) = a
\end{array}$$

We denote by St the monoid of Sturm, i.e., the set of all the morphisms obtained by composing φ, $\tilde{\varphi}$ and E in any number and order: $St = \{E, \varphi, \tilde{\varphi}\}^*$.

Theorem 2.1 ([3]). *A morphism f is Sturmian if and only if $f \in St$.*

Example. The morphism φ, called the *Fibonacci morphism* is Sturmian. It generates the *Fibonacci word* $\mathbf{F} = \varphi^\omega(a)$. Moreover, for any $n \in \mathbb{N}$, φ^n is a Sturmian morphism.

A morphism is *non trivial* if it is different from E and Id_A.

A Sturmian morphism f is *standard* if $f \in \{E, \varphi\}^*$.

2.2 Conjugate words and morphisms

Two words u and v over A are *conjugate* if there exists $s \in A^*$ such that $su = vs$.

Let $k \in \mathbb{N}$. The k-th conjugate of an infinite word $\mathbf{x}$ over A is the infinite word $\mathbf{x}'$ such that $\mathbf{x} = u\mathbf{x}'$ where $u \in A^*$, $|u| = k$.

Let f be a morphism on A. A morphism g is a *right conjugate* of f if there is a word w such that

$$f(x)w = wg(x) \quad \text{for all words } x \in A^*.$$

A morphism f has at most $|f(ab)|$ different right conjugates (including itself when $w = \varepsilon$). Indeed, if $|w| \geq |f(ab)|$ then $w = f(ab)w' = w'g(ab)$ for some word w' and, since $f(ab)w = wg(ab) = f(a)wg(b)$, this implies $f(a)w' = w'g(a)$ and $f(b)w' = w'g(b)$. So all the right conjugates of f are given by words w such that $|w| < |f(ab)|$. When it exists, the right conjugate obtained with such a w is called the $|w|$-th conjugate of f and is denoted by $f_{|w|}$. For a general study see, e.g., [4].

We first prove a useful lemma.

Lemma 2.2. *Let f be a morphism on A which generates an infinite word $\mathbf{t}$. Let $i \in \mathbb{N}$, $0 \leq i < |f(ab)|$, and denote by u the prefix of $\mathbf{t}$ of length i. If it exists, f_i is such that $f_i(\mathbf{t}) = u^{-1}\mathbf{t}$ (and $w = u$).*

Proof. Let $i \in \mathbb{N}$, $0 \leq i < |f(ab)|$, and suppose f has a i-th right conjugate f_i, i.e., there exists a word w such that $|w| = i$ and $f(x)w = wf_i(x)$ for all words $x \in A^*$. Then, for any sequence $x_1 \cdots x_p$ of words over A, $wf_i(x_1 \cdots x_p) = f(x_1 \cdots x_p)w$. Expanding to infinity, this implies that if $\mathbf{x}$ is an infinite word over A then $wf_i(\mathbf{x}) = f(\mathbf{x})$. Thus $wf_i(\mathbf{t}) = f(\mathbf{t})$ which implies that w is the prefix of length i of $\mathbf{t}$ ($w = u$) and $f_i(\mathbf{t}) = w^{-1}f(\mathbf{t}) = u^{-1}\mathbf{t}$ ($f(\mathbf{t}) = \mathbf{t}$ because $\mathbf{t}$ is generated by f). ■

Example. It is known [6] that a standard morphism f generates an infinite (Sturmian) word if and only if $f \in \{\varphi, E\varphi, \varphi E, E\varphi E\}^+ \setminus (\{E\varphi\}^+ \cup \{\varphi E\}^+)$.

Let us consider the morphism $f = \varphi\varphi E\varphi : a \mapsto ababa,\ b \mapsto ab$. The morphism f is standard, and it generates the infinite word $f^\omega(a) = ababaababababaabababaabab\ldots$

The right conjugates of f (except itself) are the following five morphisms (see, e.g., [6]).

$f_1 = \tilde{\varphi}\varphi E\varphi : a \mapsto babaa,\ b \mapsto ba \quad (u = w = a)$
$f_2 = \varphi\varphi E\tilde{\varphi} : a \mapsto abaab,\ b \mapsto ab \quad (u = w = ab)$
$f_3 = \tilde{\varphi}\varphi E\tilde{\varphi} : a \mapsto baaba,\ b \mapsto ba \quad (u = w = aba)$
$f_4 = \varphi\tilde{\varphi} E\tilde{\varphi} : a \mapsto aabab,\ b \mapsto ab \quad (u = w = abab)$
$f_5 = \tilde{\varphi}\tilde{\varphi} E\tilde{\varphi} : a \mapsto ababa,\ b \mapsto ba \quad (u = w = ababa).$

For $1 \leq i \leq 5$, $f_i(f^\omega(a)) = u^{-1}f^\omega(a)$.

From [1], chapter 2, we know the following: the number of distinct right conjugates of a standard morphism f is $|f(ab)| - 1$ (they are the morphisms f_0 to $f_{|f(ab)|-2}$). This means that, for all $0 \leq i \leq |f(ab)| - 2$, the right conjugates f_i of f exist and are pairwise different. Conversely f_i is a Sturmian morphism whenever the morphism f is a standard morphism. In this case f_i is obtained by replacing some occurrences of φ by $\tilde{\varphi}$ in the decomposition of f over $\{E, \varphi\}$.

In this particular case (illustrated in the previous example), Lemma 2.2 is very interesting. It means that if f is a standard morphism which generates an infinite word $\mathbf{x}$ then the result of applying the i-th right conjugate f_i of f to $\mathbf{x}$ only consists in deleting the first i letters of $\mathbf{x}$: it is not useful to know explicitly f_i.

3 Computing the right conjugates of a standard morphism

Our aim in this section is to realize a recursive construction of all the right conjugates of any given non trivial standard morphism.

Convention. Associated to each decomposition of a Sturmian morphism g as $g = g_1 \circ g_2 \circ \cdots \circ g_n$ with $g_i \in \{E, \varphi, \tilde{\varphi}\}$ is a word, also denoted by g, written over the alphabet $\{E, \varphi, \tilde{\varphi}\}$ (here $E, \varphi, \tilde{\varphi}$ are considered as letters): the word g is written as $g_1 g_2 \cdots g_n$. In what follows, since there is no ambiguity, we will consider such a g either as the morphism, or as the associated word, and speak of the *writing* of a morphism.

We know from [5] that, in St, the only non trivial generating relations are those of the set

$$\{\tilde{\varphi}(\tilde{\varphi}E)^k E\varphi = \varphi(\varphi E)^k E\tilde{\varphi} \ \ , \ \ k \in \mathbb{N}\}.$$

This implies that a given Sturmian morphism can have many different decompositions over St, thus many different writings. Our first aim is to realize a construction which allows to associate to each morphism of St a *unique* writing.

To do that we will make a recursive construction of the set of all the right conjugates of a standard morphism in order that two writings of the same morphism never occur together (in fact, it is not difficult to verify that a factor of the form $\tilde{\varphi}(\tilde{\varphi}E)^k E\varphi$ never appears in the writing that we obtain for the morphisms).

We recursively construct a (ordered) sequence X_f associated to a non trivial standard morphism f as follows:

- $X_\varphi = (\varphi, \tilde{\varphi})$;
- if $X_f = (f_0, \ldots, f_{|f(ab)|-2})$ then

  $X_{fE} = (f_0 E, \ldots, f_{|f(ab)|-2} E)$,

  $X_{f\varphi} = (f_0\varphi, \ldots, f_{|f(a)|-1}\varphi, f_0\tilde{\varphi}, \ldots, f_{|f(ab)|-2}\tilde{\varphi})$.

We will prove that, for any non trivial standard morphism f, the elements of X_f are, in this order, exactly the right conjugates of f.
(Remark that the construction chosen for $X_{f\varphi}$ was not the only possible one: $X_{f\varphi} = (f_0\varphi, \ldots, f_{|f(ab)|-2}\varphi, f_{|f(b)|-1}\tilde{\varphi}, \ldots, f_{|f(ab)|-2}\tilde{\varphi})$ was another possibility.)

Proposition 3.1. *Let $R_f = (f_0, \ldots, f_{|f(ab)|-2})$ be the sequence of all the right conjugates of a non trivial standard morphism f. Then $X_f = R_f$.*

Proof. The property is true for φ by construction because $\varphi_0 = \varphi$ and $\varphi_1 = \tilde{\varphi}$.

Now, suppose f is a standard morphism and $X_f = R_f$. This implies that, for each $f_i \in X_f$, $0 \le i \le |f(ab)| - 2$, there exists a word w, $|w| = i$, such that $f(a)w = wf_i(a)$ and $f(b)w = wf_i(b)$.

Since f is a standard morphism, fE and $f\varphi$ are also standard morphisms. Thus they have respectively $|fE(ab)| - 1$ and $|f\varphi(ab)| - 1$ different right conjugates (see 2.2). Since f has $|f(ab)| - 1$ different right conjugates (f_0 to $f_{|f(ab)|-2}$), fE also has $|f(ab)| - 1$ different right conjugates, $(fE)_0$ to $(fE)_{|f(ab)|-2}$, and $f\varphi$ has

$|f(aba)|-1$ different right conjugates, namely $(f\varphi)_0$ to $(f\varphi)_{|f(a)|-1}$ and $(f\varphi)_{|f(a)|}$ to $(f\varphi)_{|f(a)|+|f(ab)|-2}$.

The proposition will be proved if we state the following:

1. $f_iE = (fE)_i$, $0 \le i \le |f(ab)|-2$,
2. $f_i\varphi = (f\varphi)_i$, $0 \le i \le |f(a)|-1$,
3. $f_i\tilde{\varphi} = (f\varphi)_{|f(a)|+i}$, $0 \le i \le |f(ab)|-2$.

1. For $x \in \{a,b\}$ and $0 \le i \le |f(ab)|-2$, $fE(x)w = f(y)w = wf_i(y) = wf_iE(x)$ where $y \in \{a,b\}$, $y \ne x$. Thus there exists a word of length i, namely w, such that for any $u \in A^*$, $fE(u)w = wf_iE(u)$, which means that $f_iE = (fE)_i$.

2. For $0 \le i \le |f(ab)|-2$, one has $f\varphi(a)w = f(ab)w = f(a)f(b)w = f(a)wf_i(b) = wf_i(a)f_i(b) = wf_i(ab) = wf_i\varphi(a)$ and $f\varphi(b)w = f(a)w = wf_i(a) = wf_i\varphi(b)$. This implies $f\varphi(x)w = wf_i\varphi(x)$ for every word x.

   Since $|w| = i$, this means $f_i\varphi = (f\varphi)_i$, $0 \le i \le |f(ab)|-2$, so this is true in particular for $0 \le i \le |f(a)|-1$ (because $|f(b)| \ge 1$).

3. For $0 \le i \le |f(ab)|-2$, one has $f\tilde{\varphi}(a)w = f(ba)w = f(b)f(a)w = f(b)wf_i(a) = wf_i(b)f_i(a) = wf_i(ba) = wf_i\tilde{\varphi}(a)$ and $f\tilde{\varphi}(b)w = f(a)w = wf_i(a) = wf_i\tilde{\varphi}(b)$.

   Since $|w| = i$, this means $f_i\tilde{\varphi} = (f\tilde{\varphi})_i$, $0 \le i \le |f(ab)|-2$.

   But $f\varphi(a)f(a) = f(ab)f(a) = f(a)f(b)f(a) = f(a)f(ba) = f(a)f\tilde{\varphi}(a)$ and $f\varphi(b)f(a) = f(a)f(a) = f(a)f\tilde{\varphi}(b)$.

   Thus $f\tilde{\varphi} = (f\varphi)_{|f(a)|}$, which implies $(f\tilde{\varphi})_i = (f\varphi)_{|f(a)|+i}$.

   Consequently $f_i\tilde{\varphi} = (f\varphi)_{|f(a)|+i}$, $0 \le i \le |f(ab)|-2$. ■

Thus, given a standard morphism f (over $\{E,\varphi\}$), we obtain in a unique way a decomposition of each right conjugate of f, i.e., of each Sturmian morphism f_i (over $\{E,\varphi,\tilde{\varphi}\}$), $0 \le i \le |f(ab)|-2$.

The converse, that is given a Sturmian morphism g to determine the integer i such that $g = f_i$ for a standard morphism f, is given by a construction due to Richomme [4].

We saw in 2.2 that each Sturmian morphism g is the i-th right conjugate of a standard morphism for some integer i, $0 \le i \le |g(ab)|-2$. Let us denote by $N_bL(g)$ this integer i. Since $E^2 = Id_A$, one has $St = \{E,\varphi,\tilde{\varphi}\}^* = \{E,\varphi E,\tilde{\varphi}E\}^*$. Thus each Sturmian morphism can be decomposed over $\{E,\varphi E,\tilde{\varphi}E\}$. The following result proves that the integer i is related to the decomposition of the morphism f in the last basis.

Lemma 3.2 ([4]). *Let* $g = g_0 \dots g_{n-1}$ *with* $g_j \in \{E,\varphi E,\tilde{\varphi}E\}$, $0 \le j \le n-1$.
Then $N_bL(g) = \sum_{g|0\le j\le n-1|g_j=\tilde{\varphi}E} |g_0 \dots g_{j-1}(a)|$.
(With $|g_0 \dots g_{j-1}(a)| = 1$ *if* $j = 0$.*)*

This means that if $g = f_i$ where f is a standard morphism, then i is obtained from any decomposition of g over $\{E, \varphi E, \tilde{\varphi} E\}$, $g = g_0 \dots g_{n-1}$, by summing the lengths of the words $g_0 \dots g_{j-1}(a)$ for each j, $0 \le j \le n-1$, such that $g_j = \tilde{\varphi} E$.

Example. Let $g = \tilde{\varphi}\tilde{\varphi}E\tilde{\varphi} = \tilde{\varphi}EE\tilde{\varphi}E\tilde{\varphi}EE$.

Then, $N_bL(g) = 1 + |\tilde{\varphi}EE(a)| + |\tilde{\varphi}EE\tilde{\varphi}E(a)| = 1 + 2 + 2 = 5$.

Indeed, $g = f_5$ with $f = \varphi\varphi E\varphi$.

4 An application to the Fibonacci morphism

4.1 Computing the right conjugates of φ^n

Let us recall that the Fibonacci morphism φ is defined by $\varphi(a) = ab$, $\varphi(b) = a$. It generates the Fibonacci infinite word $\mathbf{F} = \varphi^\omega(a)$.

For any non negative integer n, $|\varphi^n(a)| = f_n$ and $|\varphi^n(b)| = f_{n-1}$ where $(f_i)_{i \ge -1}$ is the sequence of Fibonacci numbers: $f_{-1} = f_0 = 1$, $f_{p+2} = f_{p+1} + f_p, p \ge -1$.

If $n \ge 1$, the morphism φ^n is of course a non trivial standard morphism. In this particular case, the construction realized in Section 3 gives the following: the sequence X_{φ^n} of all the right conjugates of φ^n is recursively defined by $X_\varphi = (\varphi, \tilde{\varphi}) = ((\varphi^1)_0, (\varphi^1)_1)$ and, for any $n \ge 2$,

$X_{\varphi^n} = ((\varphi^{n-1})_0\varphi, (\varphi^{n-1})_1\varphi, \dots, (\varphi^{n-1})_{f_{n-1}-1}\varphi, (\varphi^{n-1})_0\tilde{\varphi}, (\varphi^{n-1})_1\tilde{\varphi}, \dots, (\varphi^{n-1})_{f_n-2}\tilde{\varphi})$
where $(\varphi^{n-1})_i$ is the i-th right conjugate of φ^{n-1} (i.e., the i-th element of the sequence $X_{\varphi^{n-1}}$).

Consequently, one has the following particular case of Proposition 3.1.

Proposition 4.1. *For any $n \ge 1$, X_{φ^n} is the ordered sequence of all the $f_{n+1} - 1$ right conjugates of φ^n: $X_{\varphi^n} = ((\varphi^n)_0, (\varphi^n)_1, \dots, (\varphi^n)_{f_{n+1}-2})$.*

Now, let us recall that the conjugates of φ^n, for all $n \in \mathbb{N}$, are exactly all the elements of the set $\{\varphi, \tilde{\varphi}\}^*$. Thus for the converse, i.e., given a Sturmian morphism g to determine the integer i such that g is the i-th right conjugate of some φ^n, the situation is simplified because there is no E in the decomposition of the standard morphism. Then the analogous of Lemma 3.2 is the following.

Lemma 4.2. *Let $g = g_0 \dots g_{n-1}$ with $g_j \in \{\varphi, \tilde{\varphi}\}$, $0 \le j \le n-1$ (i.e., g is a right conjugate of φ^n).*

Then $N_bL(g) = \sum_{g|0 \le j \le n-1|g_j = \tilde{\varphi}} f_j$.

(*This means that g is the i-th right conjugate of φ^n with $i = \sum_{g|0 \le j \le n-1|g_j = \tilde{\varphi}} f_j$.*)

Proof. Since $\varphi = \varphi EE$ and $\tilde{\varphi} = \tilde{\varphi}EE$, this result can be obtained as a corollary of Lemma 3.2. However, we give here an interesting direct proof which uses only the rank of g in the sequence X_{φ^n}.

The proof is by induction on n.

If $n = 1$ then either $g = \varphi = (\varphi^1)_0$ and $N_bL(g) = 0$, or $g = \tilde{\varphi} = (\varphi^1)_1$ and $N_bL(g) = f_0 = 1$. Consequently, the property is true if $n = 1$.

Suppose now that g is the i-th right conjugate of φ^{n+1} for some $n \ge 1$ and $0 \le i \le f_{n+2} - 2$. Then $g = g_0 \dots g_{n-1}g_n$ with $g_n = \varphi$ or $g_n = \tilde{\varphi}$. Moreover, we suppose by induction that $g' = g_0 \dots g_{n-1}$ is the $N_bL(g')$-th right conjugate of φ^n with $N_bL(g') = \sum_{g'|0\le j\le n-1|g_j=\tilde\varphi} f_j$.

We have to prove that $i = N_bL(g) = \sum_{g|0\le j\le n|g_j=\tilde\varphi} f_j$.

- If $0 \le i \le f_n - 1$ then, by construction of $X_{\varphi^{n+1}}$, $g = g'\varphi$ ($g_n = \varphi$) and g' is the i-th right conjugate of φ^n.

  Thus $N_bL(g) = i = N_bL(g') = \sum_{g'|0\le j\le n-1|g_j=\tilde\varphi} f_j = \sum_{g'\varphi|0\le j\le n|g_j=\tilde\varphi} f_j = \sum_{g|0\le j\le n|g_j=\tilde\varphi} f_j$.

- If $f_n \le i \le f_{n+2} - 2$ then, by construction of $X_{\varphi^{n+1}}$, $g = g'\tilde\varphi$ ($g_n = \tilde\varphi$) and g' is the $(i - f_n)$-th right conjugate of φ^n.

  Thus $N_bL(g) = i = f_n + N_bL(g') = f_n + \sum_{g'|0\le j\le n-1|g_j=\tilde\varphi} f_j = \sum_{g'\tilde\varphi|0\le j\le n|g_j=\tilde\varphi} f_j = \sum_{g|0\le j\le n|g_j=\tilde\varphi} f_j$. ∎

Example. $X_{\varphi^3} = (\varphi\varphi\varphi, \tilde\varphi\varphi\varphi, \varphi\tilde\varphi\varphi, \varphi\varphi\tilde\varphi, \tilde\varphi\varphi\tilde\varphi, \varphi\tilde\varphi\tilde\varphi, \tilde\varphi\tilde\varphi\tilde\varphi)$, thus $\tilde\varphi\varphi\tilde\varphi = (\varphi^3)_4$.

From Lemma 4.2, $N_bL(\tilde\varphi\varphi\tilde\varphi) = f_0 + f_2 = 1 + 3 = 4$.

4.2 Singular words

Set $\varphi^{-1}(a) = b$. In [7], Wen and Wen introduced the sequence $(w_j)_{j\ge -1}$ of the *singular words* defined by

$$w_n = \begin{cases} a\varphi^n(a)b^{-1} & \text{if } n \text{ is odd} \\ b\varphi^n(a)a^{-1} & \text{if } n \text{ is even} \end{cases}, \; n \ge -1.$$

Of course, for any $n \ge -1$, $|w_n| = |\varphi^n(a)| = f_n$.

These words have many interesting properties (see [7], and also [2]). In particular, Wen and Wen proved that the Fibonacci word $\mathbf{F}$ is obtained by concatenating all the singular words.

Theorem 4.3 ([7]). $\mathbf{F} = \prod_{j=-1}^{\infty} w_j = a\ b\ aa\ bab\ aabaa\ babaabab \dots$

Our aim in this section is to generalize this construction by showing (Theorem 4.6) that, for each word u, prefix of $\mathbf{F}$, the word $u^{-1}\mathbf{F}$ can be obtained by concatenating the factors of the infinite sequence $[(\varphi^j)_i(a)]_{j\ge p}$ where i and p are fixed integers depending only on $|u|$.

First, we apply Lemma 2.2 to the Fibonacci morphism and word.

Corollary 4.4. *Let $k \in \mathbb{N}$, and let $n \in \mathbb{N}$ be such that $0 \le k \le f_{n+1} - 2$. If u is the prefix of length k of $\mathbf{F}$ $(= (\varphi^n)^\omega(a))$ then $(\varphi^n)_k(\mathbf{F}) = u^{-1}\mathbf{F}$.*

Consequently, to apply the k-th right conjugate of φ^n, $(\varphi^n)_k$, to $\mathbf{F}$ is equivalent to take the k-th conjugate of the word $\varphi^n(\mathbf{F})$.

Considering the decomposition of $\mathbf{F}$ in singular words this gives, for example when $n = 2$ $(0 \le k \le 3)$,

$$\begin{array}{rclcl} \mathbf{F} & = & (\varphi^2)_0(\mathbf{F}) & = & a\ b\ aa\ bab\ aabaa\ babaabab\ aabaababaabaa\dots \\ & & (\varphi^2)_1(\mathbf{F}) & = & b\ aa\ bab\ aabaa\ babaabab\ aabaababaabaa\dots \\ & & (\varphi^2)_2(\mathbf{F}) & = & aa\ bab\ aabaa\ babaabab\ aabaababaabaa\dots \\ & & (\varphi^2)_3(\mathbf{F}) & = & a\ bab\ aabaa\ babaabab\ aabaababaabaa\dots \end{array}$$

Remark that, since φ^n has only $f_{n+1}-1$ different right conjugates (from 0 to $f_{n+1}-2$), if we want to remove a prefix of **F** of length k, it is necessary to consider $(\varphi^n)_k$ with n large enough to ensure that $k \leq f_{n+1}-2$.

Applying Corollary 4.4 to the decomposition of **F** in singular words, we obtain the following result which explains how to remove a prefix to this decomposition.

Proposition 4.5. *Let $k \in \mathbb{N}$, and let $n \in \mathbb{N}$ be such that $k = f_{n+1} - p$ with $2 \leq p \leq f_{n-1}+1$. Then $(\varphi^n)_k(\mathbf{F}) = v^{-1}w_{n-1}\prod_{j=n}^{\infty} w_j$, where v is the prefix of w_{n-1} such that $|v| = f_{n-1}+1-p$.*

Proof. Since $k = f_{n+1}-p$ with $2 \leq p \leq f_{n-1}+1$, one has $f_n - 1 \leq k \leq f_{n+1}-2$. Thus, if $k=0$ then $n=0$ and the result is done.

Now if $k \geq 1$ then $n \geq 1$ and in this case $f_n - 1 = \sum_{j=-1}^{n-2} f_j$.

From Corollary 4.4, we know that $(\varphi^n)_k(\mathbf{F})$ is the word obtained from **F** by removing its prefix of length k, i.e., of a length at least $\sum_{j=-1}^{n-2} f_j$.

Thus, since for any $j \geq -1$, $|w_j| = f_j$, and from Theorem 4.3, we have that $(\varphi^n)_k(\mathbf{F})$ is obtained from **F** by first removing the prefix $w_{-1}\dots w_{n-2}$ (the remaining is $w_{n-1}\prod_{j=n}^{\infty} w_j$), and then removing a prefix of length $k-(f_n-1) = f_{n+1}-p-f_n+1 = f_{n-1}+1-p$. ∎

The first values are:

$$\begin{aligned}
\mathbf{F} =&\ (\varphi^0)_0(\mathbf{F}) = && w_{-1}w_0w_1w_2w_3\dots\\
a^{-1}\mathbf{F} =&\ (\varphi^1)_1(\mathbf{F}) = && w_0w_1w_2w_3\dots\\
(ab)^{-1}\mathbf{F} =&\ (\varphi^2)_2(\mathbf{F}) = && w_1w_2w_3\dots\\
(aba)^{-1}\mathbf{F} =&\ (\varphi^2)_3(\mathbf{F}) = && a^{-1}w_1w_2w_3\dots\\
(abaa)^{-1}\mathbf{F} =&\ (\varphi^3)_4(\mathbf{F}) = && w_2w_3\dots\\
(abaab)^{-1}\mathbf{F} =&\ (\varphi^3)_5(\mathbf{F}) = && b^{-1}w_2w_3\dots\\
(abaaba)^{-1}\mathbf{F} =&\ (\varphi^3)_6(\mathbf{F}) = && (ba)^{-1}w_2w_3\dots\\
(abaabab)^{-1}\mathbf{F} =&\ (\varphi^4)_7(\mathbf{F}) = && w_3\dots
\end{aligned}$$

Now, we are ready to give for any u prefix of **F** a direct construction of $u^{-1}\mathbf{F}$ as an infinite concatenation of *generalized singular words*.

Set, for any $n \geq -1$, $(\varphi^n)_{-1}(a) = w_n$.

Theorem 4.6. *Let $k \in \mathbb{N}$, and let $n \in \mathbb{N}$ be such that $k = f_{n+1}-p$ with $2 \leq p \leq f_{n-1}+1$. Then $(\varphi^n)_k(\mathbf{F}) = \prod_{j=n-1}^{\infty}(\varphi^j)_{f_{n-1}-p}(a)$.*

Before proving this result, remark that this means that if u is a prefix of **F** then $u^{-1}\mathbf{F}$ can be obtained by concatenating the factors of the infinite sequence $[(\varphi^j)_i(a)]_{j\geq p}$ where i and p are integers depending only on $|u|$. In particular **F** is the special case when $i=p=-1$ (the singular words of Wen and Wen are all the $[(\varphi^j)_{-1}(a)]_{j\geq -1}$): $\mathbf{F} = \prod_{j=-1}^{\infty}(\varphi^j)_{-1}(a)$.

Proof. From Proposition 4.5, $(\varphi^n)_k(\mathbf{F}) = v^{-1}w_{n-1}\prod_{j=n}^{\infty} w_j = v^{-1}\prod_{j=n-1}^{\in fty} w_j$ where v is the prefix of w_{n-1} such that $|v| = f_{n-1}+1-p$.

If $v=\varepsilon$, then $|v| = 0 = f_{n-1}+1-p$, which implies $f_{n-1}-p=-1$. Hence $(\varphi^n)_k(\mathbf{F}) = \prod_{j=n-1}^{\infty} w_j = \prod_{j=n-1}^{\infty}(\varphi^j)_{-1}(a)$, which gives the result.

Now, by definition of the singular words there exists, for any $n \in \mathbb{N}$, $\{\alpha, \beta\} = \{a, b\}$ such that, for any integer $p \geq -1$,

$$w_{n+p} = \begin{cases} \alpha\varphi^{n+p}(a)\beta^{-1} \text{ if } p \text{ is even,} \\ \beta\varphi^{n+p}(a)\alpha^{-1} \text{ if } p \text{ is odd.} \end{cases}$$

Consequently, $\prod_{j=n-1}^{\infty} w_j = \beta\varphi^{n-1}(a)\varphi^{n}(a)\varphi^{n+1}(a)\ldots = \beta\prod_{j=n-1}^{\infty}\varphi^{j}(a)$.

Thus $(\varphi^n)_k(\mathbf{F}) = v^{-1}\beta\prod_{j=n-1}^{\infty}\varphi^j(a)$ and if $v \neq \varepsilon$ then there exists a word v' such that $v^{-1}\beta = v'^{-1}$. This means $v' = \beta^{-1}v$ and one has $|v'| = f_{n-1} - p$.

Since, for any $m \in \mathbb{N}$, $\varphi^m(a)$ is a prefix of $\varphi^{m+1}(a)$ and $|\varphi^m(a)| = f_m$, one has that for any integer $r \geq n-1$, $\varphi^r(a) = v'v_r$, $v_r \in A^*$. So, by definition of the right conjugates, $(\varphi^r)_{f_{n-1}-p}(a) = v_r v'$.

This implies that
$$\begin{aligned} v^{-1}\beta\prod_{j=n-1}^{\infty}\varphi^j(a) &= v'^{-1}\prod_{j=n-1}^{\infty}\varphi^j(a) \\ &= v'^{-1}\varphi^{n-1}(a)\prod_{j=n}^{\infty}\varphi^j(a) \\ &= v'^{-1}v'v_{n-1}\prod_{j=n}^{\infty}\varphi^j(a) \\ &= v_{n-1}\prod_{j=n}^{\infty}\varphi^j(a) \\ &= v_{n-1}v'v'^{-1}\prod_{j=n}^{\infty}\varphi^j(a) \\ &= (\varphi^{n-1})_{f_{n-1}-p}(a)v'^{-1}\prod_{j=n}^{\infty}\varphi^j(a) \\ &= \ldots \\ &= \prod_{j=n-1}^{\infty}(\varphi^j)_{f_{n-1}-p}(a). \end{aligned}$$

■

Example.

		w_{-1}	w_0	w_1	w_2	w_3	w_4	$\ldots$
$n=0$	$\mathbf{F} =$	$(\varphi^{-1})_{-1}(a)$	$(\varphi^0)_{-1}(a)$	$(\varphi^1)_{-1}(a)$	$(\varphi^2)_{-1}(a)$	$(\varphi^3)_{-1}(a)$	$(\varphi^4)_{-1}(a)$	$\ldots$
$n=1$	$a^{-1}\mathbf{F} =$		$(\varphi^0)_{-1}(a)$	$(\varphi^1)_{-1}(a)$	$(\varphi^2)_{-1}(a)$	$(\varphi^3)_{-1}(a)$	$(\varphi^4)_{-1}(a)$	$\ldots$
$n=2$	$(ab)^{-1}\mathbf{F} =$			$(\varphi^1))_{-1}(a)$	$(\varphi^2)_{-1}(a)$	$(\varphi^3)_{-1}(a)$	$(\varphi^4)_{-1}(a)$	$\ldots$
	$(aba)^{-1}\mathbf{F} =$			$(\varphi^1)_0(a)$	$(\varphi^2)_0(a)$	$(\varphi^3)_0(a)$	$(\varphi^4)_0(a)$	$\ldots$
$n=3$	$(abaa)^{-1}\mathbf{F} =$				$(\varphi^2)_{-1}(a)$	$(\varphi^3)_{-1}(a)$	$(\varphi^4)_{-1}(a)$	$\ldots$
	$(abaab)^{-1}\mathbf{F} =$				$(\varphi^2)_0(a)$	$(\varphi^3)_0(a)$	$(\varphi^4)_0(a)$	$\ldots$
	$(abaaba)^{-1}\mathbf{F} =$				$(\varphi^2)_1(a)$	$(\varphi^3)_1(a)$	$(\varphi^4)_1(a)$	$\ldots$
$n=4$	$(abaabab)^{-1}\mathbf{F} =$					$(\varphi^3)_{-1}(a)$	$(\varphi^4)_{-1}(a)$	$\ldots$
	$(abaababa)^{-1}\mathbf{F} =$					$(\varphi^3)_0(a)$	$(\varphi^4)_0(a)$	$\ldots$
	$(abaababaa)^{-1}\mathbf{F} =$					$(\varphi^3)_1(a)$	$(\varphi^4)_1(a)$	$\ldots$
	$(abaababaab)^{-1}\mathbf{F} =$					$(\varphi^3)_2(a)$	$(\varphi^4)_2(a)$	$\ldots$
	$(abaababaaba)^{-1}\mathbf{F} =$					$(\varphi^3)_3(a)$	$(\varphi^4)_3(a)$	$\ldots$
$n=5$	$(abaababaabaa)^{-1}\mathbf{F} =$						$(\varphi^4)_{-1}(a)$	$\ldots$

5 Concluding remarks

Theorem 4.6 is a generalization of the result of Wen and Wen (Theorem 4.3). Indeed, if we call *generalized singular words* the words $(\varphi^j)_{f_{n-1}-p}(a)$ involved in Theorem 4.6 then we obtain a decomposition in generalized singular words for each conjugate of $\mathbf{F}$. Theorem 4.3 deals with the particular case of $\mathbf{F}$ itself, and the singular words of Wen and Wen are the particular case of the generalized singular words here described when $f_{n-1} - p = -1$.

To end, one can note that other generalizations of singular words have already been considered. For example, Melançon [2] proposed over two letters a generalization of the singular words which gives a decomposition of all the characteristic Sturmian words (an infinite word $\mathbf{x}$ is a *characteristic Sturmian word* if both $a\mathbf{x}$ and $b\mathbf{x}$ are Sturmian words). In the case of alphabets with more than two letters, Wen [8] has defined singular words to decompose the Tribonacci sequence (i.e., the infinite word generated by the three-letter morphism $a \mapsto ab, b \mapsto ac, c \mapsto a$). It could be interesting, in these cases, to search for properties similar to those we proved above.

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