Kazhdan's Property T for the Symplectic Group over a Ring

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Abstract

Kazhdan constants for $\mathrm{Sp}(n, \Omega)$ where Ω is a commutative topological ring with dense finitely generated subring with unity are determined. This implies Kazhdan's property T for these groups. As application explicit Kazhdan constants are determined for the loop groups corresponding to $\mathrm{Sp}(n, \mathbf{C})$. These are further examples of groups with property T which are infinite dimensional Lie groups and not locally compact.

1 Introduction

Let G be a Hausdorff topological group, $\varepsilon > 0$, $Q \subset G$ compact, and π a strongly continuous unitary representation of G on a Hilbert space H_π. In this paper, "representation" shall always mean "unitary representation". The representation π has a (Q, ε)-invariant vector if there exists a $\xi \in H_\pi$ such that $\|\pi(g)\xi - \xi\| < \varepsilon \|\xi\|$ for all $g \in Q$. Such a pair (Q, ε) is a Kazhdan pair if every representation π of G which has a (Q, ε)-invariant vector has in fact a nonzero invariant vector. If there exists a Kazhdan pair (Q, ε) for the group G then G has Kazhdan's property T. For an account of this group theoretic property and its remarkable applications see [1] and [6].

The methods used here are due to Y. Shalom for $\mathrm{SL}(n, \Omega)$ and adapted for $\mathrm{Sp}(n, \Omega)$, where Ω is a commutative topological ring with unit 1. The strategy here

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will therefore be similar to the one of Shalom in [4]. The main difference appears in Theorem 1.1. In comparison with [4, Main Theorem] the Kazhdan set has to be enlarged. Namely not only the generating elements of the finitely generated subring have to be taken into account but also all products of different generating elements. The reason for this can be observed in the proof of Proposition 3.2. It is due to the action of elementary matrices of $\mathrm{SL}(2,\Omega[t]) \cong \mathrm{Sp}(1,\Omega[t])$ on the dual group of $S^2\left((\Omega[t])^2\right)$ where $S^2(\Omega^2)$ is the Ω-module of symmetric 2×2-matrices. This action does not only involve linear terms in t as it is the case in [4, Main Theorem] for the action on $(\Omega[t])^2$ but also quadratic terms in t^2.

Let now $I_n \in \Omega^{n\times n}$ be the identity matrix and

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

then the symplectic group is defined by

$$G = \mathrm{Sp}(n,\Omega) = \left\{ g \in \mathrm{GL}(2n,\Omega) : g^T J g = J \right\}.$$

Let $E_{j,k} \in \Omega^{n\times n}$ be the elementary matrix which is zero in every entry except for the entry 1 at (j,k). The elementary symplectic matrices are the following

$$\begin{aligned} a_{j,k}(x) &= \begin{pmatrix} I_n + xE_{j,k} & 0 \\ 0 & I_n - xE_{k,j} \end{pmatrix}, \\ b_{j,k}(x) &= \begin{pmatrix} I_n & x(E_{j,k}+E_{k,j}) \\ 0 & I_n \end{pmatrix}, c_{j,k}(x) = (b_{j,k}(x))^T \end{aligned}$$

for $j,k = 1,\ldots,n$, $j \neq k$ and

$$b_{k,k}(x) = \begin{pmatrix} I_n & xE_{k,k} \\ 0 & I_n \end{pmatrix}, c_{k,k}(x) = (b_{k,k}(x))^T.$$

The group G is called boundedly elementary generated if there exists an integer ν such that every element $g \in G$ is a product of at most ν elementary symplectic matrices. The smallest such ν will be denoted by $\nu_n(\Omega)$. If Ω is a field, G is always boundedly elementary generated. This can be observed by using a "Gauss algorithm" which is adapted to the symplectic case by using only elementary symplectic matrices. For further details see also the proof of Theorem 6.2 which shows bounded elementary generation for G with Ω the ring of continuous functions $f : S^1 \to \mathbf{C}$ where $S^1 = \{x \in \mathbf{C} : |x| = 1\}$.

Theorem 1.1. *Let $n \geq 2$ and G boundedly elementary generated. Suppose for $1 \leq m < \infty$ that there are elements $\alpha_1,\ldots,\alpha_m \in \Omega$ generating a dense subring. Let $Q_1 \subset G$ be the (finite) set of elementary symplectic matrices with $x = 1$ and Q_2 the set of $a_{k,k+1}(\alpha_r)$, $1 \leq k \leq n-1$ and $1 \leq r \leq m$, and $b_{j,k}\left(\alpha_{r_1}\cdots\alpha_{r_p}\right)$, $c_{j,k}\left(\alpha_{r_1}\cdots\alpha_{r_p}\right)$ with $1 \leq j,k \leq n$, $0 \leq p \leq m$, and $1 \leq r_1 < \ldots < r_p \leq m$. Let $Q = Q_1 \cup Q_2$ and $\varepsilon = 3 \times 14^{-2m-2}(\nu_n(\Omega))^{-1}$ then (Q,ε) is a Kazhdan pair for G. If every neighborhood of $0 \in \Omega$ contains such $\alpha_1,\ldots,\alpha_m \in \Omega$ for fixed m, then (Q_1,ε) is a Kazhdan pair of G.*

This will be applied to the loop group $\mathfrak{L}(\mathrm{Sp}(n, \mathbf{C}))$, the group of all continuous mappings $g : S^1 \to \mathrm{Sp}(n, \mathbf{C})$ with pointwise multiplication and the topology of uniform convergence. The loop group is isomorphic to $\mathrm{Sp}(n, \Omega)$ where Ω is the ring of continuous functions $\alpha : S^1 \to \mathbf{C}$. A dense subgroup of this ring is generated by the functions

$$\alpha_0(x) = \sqrt{2} + i, \alpha_1(x) = x, \alpha_2(x) = \overline{x}$$

by the Stone–Weierstrass theorem. So the theorem above can be applied.

Theorem 1.2. *For $n \geq 2$, $\varepsilon = 3 \times 14^{-8}(3n^2 + 4n)^{-1}$, and Q_1 the set of elementary symplectic matrices with off-diagonal entry* 1 *the group $\mathfrak{L}(\mathrm{Sp}(n, \mathbf{C}))$ has the Kazhdan pair (Q_1, ε).*

Note that there are plenty of unitary representations of $\mathfrak{L}(\mathrm{Sp}(n, \mathbf{C}))$. For example every unitary representation of $\mathrm{Sp}(n, \mathbf{C})$ gives rise to a unitary representation of $\mathfrak{L}(\mathrm{Sp}(n, \mathbf{C}))$ by evaluating the elements of $\mathfrak{L}(\mathrm{Sp}(n, \mathbf{C}))$ at a fixed point and then applying the representation of $\mathrm{Sp}(n, \mathbf{C})$.

These loop groups give further examples of groups with Kazhdan's property T which are not finite dimensional and not locally compact. The first examples of such groups, namely the loop groups associated with $\mathrm{SL}(n, \mathbf{C})$ were shown in [4] to have Kazhdan's property T.

In contrast to this result $\mathfrak{L}(\mathrm{Sp}(n, \mathbf{R}))$ does not have property T as the fundamental group $\pi_1(\mathrm{Sp}(n, \mathbf{R})) \cong \mathbf{Z}$, see for example [2, page 173], and the fundamental group is the factor group of $\mathfrak{L}(\mathrm{Sp}(n, \mathbf{R}))$ by the connected component of the identity.

In [5] it is shown that $\mathrm{Sp}(n, \mathfrak{o})$, $n \geq 2$, is boundedly elementary generated where $\mathfrak{o}$ is the ring of integers of an algebraic number field. More generally, this result is proved for rings of algebraic S-integers and general Chevalley groups of normal and twisted type having rank ≥ 2. But an explicit upper bound for $\nu_n(\mathfrak{o})$ is only provided for the rank 2 case in which the estimate $\nu_2(\mathfrak{o}) \leq 180\Delta + 27$ is determined where Δ denotes the number of distinct prime divisors of the discriminant of the number field. For $n \geq 3$ the estimate $\nu_n(\mathfrak{o}) \leq 3n^2 + 4n + 68\Delta + 16$ can be deduced from [7]. Hence the following holds.

Corollary 1.3. *Let $n \geq 2$, $\varepsilon = 14^{-2m-2}(60\Delta + 9)$ in the case $n = 2$ and $\varepsilon = 3 \times 14^{-2m-2}(3n^2 + 4n + 68\Delta + 16)^{-1}$ in the case $n \geq 3$. For $\mathrm{Sp}(n, \mathfrak{o})$ the elementary symplectic matrices together with ε form a Kazhdan pair if there exist $1, \alpha_1, \ldots, \alpha_m \in \mathfrak{o}$ which generate $\mathfrak{o}$ as a ring.*

For $\mathrm{Sp}(n, \Omega)$ subgroups isomorphic to $\mathrm{SL}(2, \Omega) \ltimes S^2(\Omega^2)$ are considered. For such semi-direct products relative Kazhdan pairs, defined as follows, are computed, see Corollary 4.2.

Let U be a closed subgroup of G, $Q \subset G$ compact and $\varepsilon > 0$, then (Q, ε) is a relative Kazhdan pair of (G, U) if every representation π of G which has a (Q, ε)-invariant vector has in fact a U-invariant vector.

Sections 2 and 3 contain the induction argument to transfer relative Kazhdan pairs with respect to a ring to those with respect to its polynomial ring. In Section 4 invariant vectors of the semi-direct product are considered. Section 5 contains the proof of Theorem 1.1. In the last section the results are applied to the loop group.

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2 The ring of integers

Now $\Omega = \mathbf{Z}$ and a relative Kazhdan pair of $(\mathrm{SL}(2,\mathbf{Z}) \ltimes S^2(\mathbf{Z}^2), S^2(\mathbf{Z}^2))$ will be determined in the next theorem. By the isomorphism $S^2(\mathbf{Z}^2) \cong \mathbf{Z}^3$, in the first part $\mathrm{SL}(2,\mathbf{Z})$-invariant means on $(S^1)^3 \cong \widehat{\mathbf{Z}^3}$ will be considered. For the proof of the theorem only $\mathrm{SL}(2,\mathbf{Z})$-invariant means on $\mathbf{R}^3$ are needed.

Lemma 2.1. *Let μ be a mean defined on the Borel sets of $\mathbf{R}^3 \setminus \{0\}$, then there exists a Borel set $W \subset \mathbf{R}^3 \setminus \{0\}$ and an element $g \in \left\{u_{\pm 1}, u_{\pm 1}^T\right\}$, where $u_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, such that $|\mu(gW) - \mu(W)| \geq \frac{1}{12}$.*

Proof. Let

$$\begin{aligned} A &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{R}^3 : 0 < y \leq 2x \right\}, \\ \hat{A} &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{R}^3 : 0 \leq 2x < y \right\}, \end{aligned}$$

then $u_{-1} \cdot \left(A \cup \hat{A}\right) \subset \hat{A}$. Indeed,

$$g \begin{pmatrix} x' & y' \\ y' & z' \end{pmatrix} g^T = \begin{pmatrix} a^2x' + 2aby' + b^2z' & acx' + (ad+bc)\,y' + bdz' \\ acx' + (ad+bc)\,y' + bdz' & c^2x' + 2cdy' + d^2z' \end{pmatrix}$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and hence the action of u_{-1} on the dual $\widehat{\mathbf{R}^3} \cong \mathbf{R}^3$ can be identified with the inverse transpose action given in dual coordinates by $u_{-1} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 2x + y \\ x + y + z \end{pmatrix}$. So u_{-1} maps the elements of $A \cup \hat{A}$ into $\hat{A}$ as $2x + y > 2x \geq 0$. If the inequality would not hold

$$\begin{aligned} \frac{1}{12} &> \mu\left(A \cup \hat{A}\right) - \mu\left(u_{-1} \cdot \left(A \cup \hat{A}\right)\right) \geq \mu\left(A \cup \hat{A}\right) - \mu\left(\hat{A}\right) \\ &= \mu(A). \end{aligned}$$

Similarly $u_{-1} \cdot \left(-A \cup -\hat{A}\right) \subset -\hat{A}$ and $\mu(-A) < \frac{1}{12}$.

It holds that $u_1 \cdot \left(B \cup \hat{B}\right) \subset \hat{B}$ for

$$\begin{aligned} B &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{R}^3 : 0 \le -y < 2x \right\}, \\ \hat{B} &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{R}^3 : 0 < 2x \le -y \right\}, \end{aligned}$$

as $u_1 \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -2x + y \\ x - y + z \end{pmatrix}$ and $2x - y \ge 2x > 0$. So $\mu(B) < \frac{1}{12}$. Analogously $u_1 \cdot \left(-B \cup -\hat{B}\right) \subset -\hat{B}$ and so $\mu(-B) < \frac{1}{12}$.

Let $\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then holds $u_{\pm 1}^T = \omega u_{\mp 1} \omega^{-1}$ and the corresponding inequalities $\mu(\omega \cdot (\pm A)) < \frac{1}{12}$ and $\mu(\omega \cdot (\pm B)) < \frac{1}{12}$.

For

$$C = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{R}^3 : y > 0, -y \le 2x < y, -y < 2z \le y \right\}$$

holds $\mathbf{R}^3 \setminus \{0\} = A \cup B \cup (-A) \cup (-B) \cup C \cup \omega \cdot (A \cup B \cup (-A) \cup (-B) \cup C)$.

As $u_{-1}^T \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y + z \\ y + 2z \\ z \end{pmatrix}$, $2x + 2y + 2z \ge y + 2z > 0$, and $2x + y \ge 0$ for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in C$, $u_{-1}^T \cdot C \subset A$. This shows

$$\frac{1}{12} > \mu(C) - \mu\left(u_{-1}^T \cdot C\right) \ge \mu(C) - \mu(A)$$

implying $\mu(C) < \frac{1}{6}$. Similarly $\mu(\omega \cdot C) < \frac{1}{6}$.

From the above follows the contradiction

$$\begin{aligned} 1 &= \mu(A \cup (-A) \cup B \cup (-B) \cup C \cup \omega \cdot (A \cup (-A) \cup B \cup (-B) \cup C)) \\ &< \frac{8}{12} + \frac{2}{6} = 1. \end{aligned}$$

∎

A relative Kazhdan pair for $(\mathrm{SL}(2, \mathbf{Z}) \ltimes S^2(\mathbf{Z}^2), S^2(\mathbf{Z}^2))$ can now be determined with this lemma.

Theorem 2.2. *Let $u_{\pm 1}, u_{\pm 1}^T \in \mathrm{SL}(2, \mathbf{Z})$ be as above,*

$$\alpha^{\pm} = \begin{pmatrix} \pm 1 & 0 \\ 0 & 0 \end{pmatrix}, \beta^{\pm} = \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^{\pm} = \begin{pmatrix} 0 & 0 \\ 0 & \pm 1 \end{pmatrix} \in S^2\left(\mathbf{Z}^2\right)$$

and Q the subset of $\mathrm{SL}(2, \mathbf{Z}) \ltimes S^2(\mathbf{Z}^2)$ consisting of the corresponding 10 elements. Let π be a unitary representation of G on H_π with a $\left(Q, \frac{1}{26}\right)$-invariant vector, then H_π has a nonzero $S^2(\mathbf{Z}^2)$-invariant vector.

Proof. Let E be the spectral measure corresponding to $\pi|_{S^2(\mathbf{Z}^2)}$. Suppose for $\varepsilon = \frac{1}{26}$ there exists a (Q,ε)-invariant unit vector $\xi \in H_\pi$, but no nonzero $S^2(\mathbf{Z}^2)$-invariant vector. Let μ_ξ be the corresponding measure on $(S^1)^3 \cong \widehat{S^2(\mathbf{Z}^2)}$, where $\mu_\xi(B) = \langle E(B)\xi|\xi\rangle$. By assumption $\mu_\xi(\{0\}) = 0$. Now $\widehat{S^2(\mathbf{Z}^2)}$ is identified with $]-1/2,1/2]^3$, where $(x,y,z) \in]-1/2,1/2]^3$ corresponds to the character $\chi(n_1,n_2,n_3) = e^{2\pi i(xn_1+yn_2+zn_3)}$. In the following $W =]-1/6,1/6[^3$ is considered.

It holds, that $\langle \pi(z)\xi|\xi\rangle = \int \chi(z)\, d\mu_\xi(\chi)$ for $z \in S^2(\mathbf{Z}^2)$. With the identification of $(S^1)^3$ and $]-1/2,1/2]^3$, this shows that

$$\begin{aligned}
\int_{]-1/2,1/2]^3} \left|e^{\pm 2\pi i x} - 1\right|^2 d\mu_\xi(x,y,z) &= \left\|\pi\left(\alpha^\pm\right)\xi - \xi\right\|^2 \le \varepsilon^2,\\
\int_{]-1/2,1/2]^3} \left|e^{\pm 2\pi i y} - 1\right|^2 d\mu_\xi(x,y,z) &= \left\|\pi\left(\beta^\pm\right)\xi - \xi\right\|^2 \le \varepsilon^2,\\
\int_{]-1/2,1/2]^3} \left|e^{\pm 2\pi i z} - 1\right|^2 d\mu_\xi(x,y,z) &= \left\|\pi\left(\gamma^\pm\right)\xi - \xi\right\|^2 \le \varepsilon^2.
\end{aligned}$$

As $|e^{\pm 2\pi i t} - 1| = |e^{\pi i t} - e^{-\pi i t}| = 2|\sin(\pi t)| \ge 1$, for $1/6 \le |t| \le 1/2$, this implies

$$\begin{aligned}
\mu_\xi(\{(x,y,z) : |x| \ge 1/6\}) &\le \varepsilon^2,\\
\mu_\xi(\{(x,y,z) : |y| \ge 1/6\}) &\le \varepsilon^2,\\
\mu_\xi(\{(x,y,z) : |z| \ge 1/6\}) &\le \varepsilon^2.
\end{aligned}$$

So $\mu_\xi(W) \ge 1 - 3\varepsilon^2$.

For all measurable $B \subset]-1/2,1/2]^3$ and $g \in \left\{u_{\pm 1}, u_{\pm 1}^T\right\}$, the following holds

$$|\mu_\xi(gB) - \mu_\xi(B)| \le 2\varepsilon.$$

Indeed,

$$|\mu_\xi(gB) - \mu_\xi(B)| = \left|\left\langle \pi(g) E(B) \pi\left(g^{-1}\right)\xi|\xi\right\rangle - \langle E(B)\xi|\xi\rangle\right|$$

and this is bounded from above by

$$\left|\left\langle \pi(g) E(B)\left(\pi\left(g^{-1}\right)\xi - \xi\right)|\xi\right\rangle\right| + \left|\left\langle E(B)\xi|\pi\left(g^{-1}\right)\xi - \xi\right\rangle\right| \le 2\varepsilon.$$

Now let μ^W be defined on $]-1/2,1/2]^3$ by $\mu^W(B) = \mu_\xi(B \cap W)$, then $0 \le \mu_\xi(B) - \mu^W(B) \le 3\varepsilon^2$.

For the Borel sets B and $g \in \left\{u_{\pm 1}, u_{\pm 1}^T\right\}$ now holds

$$\mu^W(gB) - \mu^W(B) \le 0 + 2\varepsilon + 3\varepsilon^2.$$

Let $\mu = \left(\mu^W(W)\right)^{-1}\mu^W$, then for every measurable B and $g \in \left\{u_{\pm 1}, u_{\pm 1}^T\right\}$,

$$|\mu(gB) - \mu(B)| \le \frac{2\varepsilon + 3\varepsilon^2}{1 - 3\varepsilon^2} = \frac{55}{673} < \frac{1}{12}.$$

As $gW \subset]-1/2,1/2[^3$ for every such g, the measure μ can be considered to be defined on $\mathbf{R}^3$ which yields a contradiction to Lemma 2.1. ∎

3 The polynomial ring

Now Ω is supposed to be discrete. In this section the step from Ω to $\Omega[t]$ will be investigated. For a Kazhdan pair (Q, ε) of $(\mathrm{SL}(2, \Omega) \ltimes S^2(\Omega^2), S^2(\Omega^2))$ a Kazhdan pair (Q_t, δ) of $\left(\mathrm{SL}(2, \Omega[t]) \ltimes S^2\left((\Omega[t])^2\right), S^2\left((\Omega[t])^2\right)\right)$ is determined.

For the following see [4].

The dual group is considered as the group of all characters $\chi : \Omega \to \mathbf{R}/\mathbf{Z}$ with $\chi(r+s) = \chi(r) + \chi(s)$ and denoted by $\widehat{\Omega}$. As Ω is discrete $\widehat{\Omega}$ is compact.

Let X be the group of formal power series $\sum_{k=0}^{\infty} \chi_k t^{-k}$, $\chi_k \in \widehat{\Omega}$, then X is an (additive) abelian group. It is equipped with the product topology via the identification $X \cong \prod_{k=0}^{\infty} \widehat{\Omega}$, which makes X a compact topological group.

The group X is embedded in $\widehat{\Omega[t]}$ by $\chi(r) = \sum_{k=0}^{\infty} \chi_k(r_k) \in \mathbf{R}/\mathbf{Z}$, where $\chi = \sum_{k=0}^{\infty} \chi_k t^{-k}$ with $\chi_k \in \widehat{\Omega}$ and $r = \sum_{k=0}^{\infty} r_k t^k \in \Omega[t]$. Of course, only finitely many r_k are different from zero.

Lemma 3.1. *The mapping $X \to \widehat{\Omega[t]}$ is a topological group isomorphism.*

Let $\tilde{X}$ be the group of all formal power series $\sum_{k=m}^{\infty} \chi_k t^{-k}$ with $m \in \mathbf{Z}$. The group X can be embedded in $\tilde{X}$. By identifying two elements χ and ψ if $\chi_k = \psi_k$ for all $k \geq 0$, the group $\tilde{X}$ is isomorphic to $\widehat{\Omega[t]}$.

Let $\mathrm{SL}(2, \Omega[t])$ act on $\tilde{X}^3$ by the dual of the action on $S^2\left((\Omega[t])^2\right)$. By definition

$$(g \cdot \chi)(s) = \chi\left(g^{-1} s \left(g^T\right)^{-1}\right)$$

for $g \in \mathrm{SL}(2, \Omega[t])$, $\chi \in \tilde{X}^3$, and $s \in S^2\left((\Omega[t])^2\right)$.

The following is the analogue of Lemma 2.1 for $\Omega[t]$.

Proposition 3.2. *Let Ω be a discrete ring and μ a mean defined on the Borel sets of $\tilde{X}^3 \setminus \{0\}$. Let Q be a finite subset of $\mathrm{SL}(2, \Omega[t])$ that contains the elements $u_{\pm 1}, u_{\pm t}, u_{\pm 1}^T, u_{\pm t}^T$, then exists a $g \in Q$ and a Borel set $W \subset \tilde{X}^3 \setminus \{0\}$ with $|\mu(gW) - \mu(W)| \geq \frac{1}{34}$.*

Proof. For an element $\chi = \sum_{n=m}^{\infty} \chi_n t^{-n}$ define $v(\chi) = 2^{-m}$ if $\chi_m \neq 0$, then $v(t\chi) = 2v(\chi)$ and $v(\chi + \psi) \leq \max\{v(\chi), v(\psi)\}$, where for $v(\chi) \neq v(\psi)$ even equality holds. In case $\chi = 0$, let $v(\chi) = 0$.

Let now

$$A_{s_1,s_2} = \left\{ \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} \in \tilde{X}^3 : \begin{array}{c} \mathrm{sgn}(v(\chi_2) - v(2\chi_1)) = s_1 \\ \mathrm{sgn}(v(2\chi_3) - v(\chi_2)) = s_2 \end{array} \right\}$$

then $u_{-1}^T \cdot (A_{-1,1} \cup A_{0,1} \cup A_{1,1}) \subset A_{-1,0} \cup A_{0,0} \cup A_{1,0}$ as

$$u_{-1}^T \cdot \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} = \begin{pmatrix} \chi_1 + \chi_2 + \chi_3 \\ \chi_2 + 2\chi_3 \\ \chi_3 \end{pmatrix}$$

and $v(\chi_2) < v(2\chi_3)$, so $v(\chi_2 + 2\chi_3) = v(2\chi_3)$. As

$$u_{-t}^T \cdot \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} = \begin{pmatrix} \chi_1 + t\chi_2 + t^2\chi_3 \\ \chi_2 + 2t\chi_3 \\ \chi_3 \end{pmatrix},$$

holds $u_{-t}^T \cdot (A_{0,0} \cup A_{0,1} \cup A_{1,0} \cup A_{1,1}) \subset A_{-1,-1}$. Indeed, $v(\chi_2) < v(2t\chi_3)$ and $v(2\chi_1) < v(t\chi_2)$, so

$$\begin{aligned} v(2\chi_3) &< v(\chi_2 + 2t\chi_3), \\ v(2\chi_1 + t\chi_2) &< v\left(t\chi_2 + 2t^2\chi_3\right) \end{aligned}$$

and hence $v(\chi_2 + 2t\chi_3) < v(2(\chi_1 + t\chi_2 + t^2\chi_3))$. Let $\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then

$$u_{-1}u_1^T u_{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \omega$$

and $\omega \cdot A_{s_1,s_2} = A_{-s_2,-s_1}$, as

$$\omega \cdot \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} = \begin{pmatrix} \chi_3 \\ -\chi_2 \\ \chi_1 \end{pmatrix}.$$

So similarly to the preceding case $u_t \cdot (A_{0,0} \cup A_{-1,0} \cup A_{0,-1} \cup A_{-1,-1}) \subset A_{1,1}$ and $u_1 \cdot (A_{-1,1} \cup A_{-1,0} \cup A_{-1,-1}) \subset A_{0,1} \cup A_{0,0} \cup A_{0,-1}$.

Suppose there is no g and W with $|\mu(W) - \mu(gW)| \geq \frac{1}{34}$, hence

$$\begin{aligned} \mu(A_{0,1} \cup A_{0,0} \cup A_{1,0}) &= \mu(A_{1,1} \cup A_{0,1} \cup A_{0,0} \cup A_{1,0}) - \mu(A_{1,1}) < \frac{2}{17}, \\ \mu(A_{-1,0} \cup A_{0,-1}) &\leq \mu(A_{-1,-1} \cup A_{-1,0} \cup A_{0,0} \cup A_{0,-1}) - \mu(A_{-1,-1}) \\ &< \frac{2}{17}. \end{aligned}$$

This yields

$$\begin{aligned} \mu(A_{-1,1}) &\leq \mu(A_{1,1} \cup A_{0,1} \cup A_{-1,1}) - \mu(A_{1,1}) \\ &< \frac{1}{34} + \mu(A_{-1,0} \cup A_{0,0}) + \mu(A_{1,0}) - \mu(A_{1,1}) \\ &< \frac{2}{17} + \mu(A_{-1,0} \cup A_{0,0} \cup A_{0,-1}) - \mu(A_{1,1}) < \frac{5}{34} \end{aligned}$$

and similarly $\mu(A_{1,-1}) < \frac{5}{34}$. Further

$$\begin{aligned} \mu(A_{1,1}) &< \frac{1}{34} + \mu(A_{-1,0} \cup A_{0,0} \cup A_{1,0}) \\ &< \frac{2}{17} + \mu(A_{0,1}) + \mu(A_{0,0} \cup A_{1,0}) < \frac{4}{17} \end{aligned}$$

and again similarly $\mu(A_{-1,-1}) < \frac{4}{17}$. The inequality

$$\mu\left(\bigcup_{s_1,s_2=-1}^{1} A_{s_1,s_2}\right) < \frac{4}{17} + \frac{5}{17} + \frac{8}{17} = 1$$

yields a contradiction. ∎

With the help of the last lemma the following can be proved.

Theorem 3.3. *Let Ω be a discrete ring. Let $Q \subset \mathrm{SL}(2,\Omega) \ltimes S^2(\Omega^2)$ be a finite set with $Q \supseteq \left\{u_{\pm 1}, u_{\pm 1}^T\right\}$ and $\varepsilon > 0$ such that (Q,ε) is a Kazhdan pair. The ring Ω is canonically embedded in $\Omega[t]$, which yields an embedding of $\mathrm{SL}(2,\Omega) \ltimes S^2(\Omega^2)$ into $\mathrm{SL}(2,\Omega[t]) \ltimes S^2\left((\Omega[t])^2\right)$.*

Let $0 < \delta < \varepsilon/\sqrt{2}$ such that $\left(\delta + 2\sqrt{2}\delta/\varepsilon\right)/\left(1-\sqrt{2}\delta/\varepsilon\right) \leq \frac{1}{68}$, then (Q_t,δ) is a Kazhdan pair of $\left(\mathrm{SL}(2,\Omega[t]) \ltimes S^2\left((\Omega[t])^2\right), S^2\left((\Omega[t])^2\right)\right)$, where Q_t is the union of Q with $\left\{u_{\pm t}, u_{\pm t}^T\right\}$ and all elements in Q where the $S^2(\Omega^2)$-part is replaced by itself multiplied with t.

Proof. Consider the isomorphic semi-direct products $\mathrm{SL}(2,\Omega) \ltimes S^2\left((\Omega t)^2\right)$ and $\mathrm{SL}(2,\Omega) \ltimes S^2(\Omega^2)$. Let Q' be the set corresponding to Q where the $S^2(\Omega^2)$-part is replaced by itself multiplied with t, then (Q',ε) is a Kazhdan pair of $\left(\mathrm{SL}(2,\Omega) \ltimes S^2\left((\Omega t)^2\right), S^2\left((\Omega t)^2\right)\right)$.

Let now π be a representation of $\mathrm{SL}(2,\Omega[t]) \ltimes S^2\left((\Omega[t])^2\right)$ on H_π, $\xi \in H_\pi$ a unit vector which is (Q_t,δ)-invariant, $H_0 \subset H_\pi$ the space of $S^2\left((\Omega+\Omega t)^2\right)$-invariant vectors, and H_1 the orthogonal complement of H_0 in the space of $S^2(\Omega^2)$-invariant vectors, H_2 the orthogonal complement of H_0 in the space of $S^2\left((\Omega t)^2\right)$-invariant vectors, and H_3 the orthogonal complement of $H_0 \oplus H_1 \oplus H_2$ in H_π, then $H_0 \oplus H_1$ and $H_2 \oplus H_3$ are $\mathrm{SL}(2,\Omega) \ltimes S^2(\Omega^2)$-invariant and $H_0 \oplus H_2$ and $H_1 \oplus H_3$ are $\mathrm{SL}(2,\Omega) \ltimes S^2\left((\Omega t)^2\right)$-invariant. This yields the corresponding decomposition $\xi = \xi_0 + \xi_1 + \xi_2 + \xi_3$. Hence

$$\begin{aligned} \delta^2 &\geq \left\|\pi(g)\xi - \xi\right\|^2 \\ &= \left\|\pi(g)(\xi_0+\xi_1) - (\xi_0+\xi_1)\right\|^2 + \left\|\pi(g)(\xi_2+\xi_3) - (\xi_2+\xi_3)\right\|^2 \end{aligned}$$

for all $g \in Q$ and in a similar way

$$\delta^2 \geq \left\|\pi(g)(\xi_0+\xi_2) - (\xi_0+\xi_2)\right\|^2 + \left\|\pi(g)(\xi_1+\xi_3) - (\xi_1+\xi_3)\right\|^2$$

for all $g \in Q'$. As $H_2 \oplus H_3$ contains no $S^2(\Omega^2)$-invariant vector, there exists a $g_1 \in Q$ with

$$\varepsilon\left\|\xi_2+\xi_3\right\| \leq \left\|\pi(g_1)(\xi_2+\xi_3) - (\xi_2+\xi_3)\right\| \leq \delta.$$

So $\|\xi_2+\xi_3\| \leq \delta/\varepsilon$. Just as well $\|\xi_1+\xi_3\| \leq \delta/\varepsilon$ and hence

$$\|\xi_0\|^2 = \|\xi\|^2 - \|\xi_1+\xi_3\|^2 - \|\xi_2+\xi_3\|^2 + \|\xi_3\|^2 \geq 1 - 2(\delta/\varepsilon)^2,$$

i. e. $\|\xi_0\| \geq \sqrt{\left(1-\sqrt{2}\delta/\varepsilon\right)\left(1+\sqrt{2}\delta/\varepsilon\right)} > 1-\sqrt{2}\delta/\varepsilon$.

For all $g \in Q_t$,

$$\left\|\pi(g)\xi_0 - \xi_0\right\| \leq \left\|\pi(g)\xi - \xi\right\| + 2\left\|\xi_1+\xi_2+\xi_3\right\| \leq \delta + 2\sqrt{2}\delta/\varepsilon.$$

For $\eta = \xi_0/\|\xi_0\|$ this implies

$$\left\|\pi(g)\eta - \eta\right\| < \frac{1}{68}$$

for all $g \in Q_t$ by the choice of δ.

Let E be the spectral measure corresponding to the restriction of π to $S^2\left((\Omega[t])^2\right)$ on the dual group $S^2((\Omega[t])^2)$ and $\mu_\eta(B) = \langle E(B)\eta|\eta\rangle$ defined on $X^3 \cong \widehat{S^2((\Omega[t])^2)}$, then $(t^{-2}X)^3$ is the support of μ_η. If there were no $S^2\left((\Omega[t])^2\right)$-invariant vectors in H_π, it would hold that $\mu_\eta(\{0\}) = 0$.

Like in the proof of Theorem 2.2 the $\frac{1}{68}$-invariance of η implies that for all measurable $B \subset X^3$ holds $|\mu_\eta(gB) - \mu_\eta(B)| < \frac{1}{34}$. As $gB \subset X^3$ for $B \subset (t^{-2}X)^3$ and $g \in \left\{u_{\pm 1}, u_{\pm 1}^T, u_{\pm t}, u_{\pm t}^T\right\}$ this yields a contradiction to Lemma 3.2. ∎

4 Invariant vectors

Theorem 4.1. *Let $m \geq 0$ and $\Omega_m = \mathbf{Z}[x_1, \ldots, x_m]$. Let Q consist of*

$$\alpha^{\pm} x_{j_1} \cdots x_{j_k}, \beta^{\pm} x_{j_1} \cdots x_{j_k}, \gamma^{\pm} x_{j_1} \cdots x_{j_k} \in S^2\left(\Omega_m^2\right),$$

$0 \leq k \leq m$, $1 \leq j_1 < \ldots < j_k \leq n$, and the $4(m+1)$ matrices $u_{\pm b}, u_{\pm b}^T$ in $\mathrm{SL}(2, \Omega_m)$ with $b = 1, x_1, \ldots, x_m$, then every unitary representation of $\mathrm{SL}(2, \Omega_m) \ltimes S^2(\Omega_m^2)$ with a $(Q, 6 \times 14^{-2m-2})$-invariant vector contains a vector distinct from 0 which is $S^2(\Omega_m^2)$-invariant.

Proof. The case $m = 0$ follows from Theorem 2.2. By induction and Theorem 3.3 this implies the theorem. Indeed,

$$\frac{\delta + 2\sqrt{2}\delta/\varepsilon}{1 - \sqrt{2}\delta/\varepsilon} = \frac{6 \times 14^{-2m-2} + 2\sqrt{2}}{14^2 - \sqrt{2}} < \frac{1}{68}$$

for $\varepsilon = 6 \times 14^{-2m-2}$ and $\delta = 6 \times 14^{-2m-4}$. ∎

The following yields a lower bound for the diameter of an orbit of the $S^2(\Omega^2)$-part.

Corollary 4.2. *Let Ω be a topological commutative ring with unit. Assume that there exist elements $\alpha_0 = 1, \alpha_1, \ldots, \alpha_m \in \Omega$ generating a dense subring $D \subset \Omega$. Let $Q \subset \mathrm{SL}(2, \Omega) \ltimes S^2(\Omega^2)$ be the subset in Theorem 4.1 where α_j replaces x_j. For a unitary representation π of $\mathrm{SL}(2, \Omega) \ltimes S^2(\Omega^2)$ on H_π and a unit vector $\xi \in H_\pi$ which is $(Q, 3 \times 14^{-2m-2}\varepsilon)$-invariant for an $\varepsilon > 0$ holds $\|\pi(g)\xi - \xi\| \leq \varepsilon$ for every $g \in S^2(\Omega^2)$.*

Proof. The mapping of $1, x_1, \ldots, x_m \in \Omega_m$ onto $1, \alpha_1, \ldots, \alpha_m \in D \subset \Omega$ can be canonically extended to a ring epimorphism inducing a group homomorphism

$$\Phi : \mathrm{SL}(2, \Omega_m) \ltimes S^2\left(\Omega_m^2\right) \to \mathrm{SL}(2, D) \ltimes S^2\left(D^2\right) \subset \mathrm{SL}(2, \Omega) \ltimes S^2\left(\Omega^2\right).$$

Let $H_0 \subset H_\pi$ be the subspace of $S^2(\Omega^2)$-invariant vectors and $H_1 \subset H_\pi$ its orthogonal complement, then for $\xi \in H_\pi$ there exist unique $\xi_0 \in H_0$ and $\xi_1 \in H_1$ with $\xi = \xi_0 + \xi_1$. As D is dense in Ω there are no nonzero $S^2(D^2)$-invariant vectors of the representation π on H_1 and hence also for the representation $\pi \circ \Phi$ of the group $\mathrm{SL}(2,\Omega_m) \ltimes S^2(\Omega_m^2)$. So Theorem 4.1 implies that there exists a $g_0 \in Q$ which fulfills $6 \times 14^{-2m-2}\|\xi_1\| \leq \|\pi(g_0)\xi_1 - \xi_1\|$. On the other hand by assumption

$$\begin{aligned}\|\pi(g_0)\xi_0 - \xi_0\|^2 + \|\pi(g_0)\xi_1 - \xi_1\|^2 &= \|\pi(g_0)\xi - \xi\|^2 \\ &\leq \left(3 \times 14^{-2m-2}\varepsilon\right)^2\end{aligned}$$

and hence $6 \times 14^{-2m-2}\|\xi_1\| \leq 3 \times 14^{-2m-2}\varepsilon$, i. e. $\|\xi_1\| \leq \frac{\varepsilon}{2}$. Finally the $S^2(\Omega^2)$-invariance of ξ_0 implies $\|\pi(g)\xi - \xi\| = \|\pi(g)\xi_1 - \xi_1\| \leq 2\|\xi_1\| \leq \varepsilon$ for all $g \in S^2(\Omega^2)$. ■

5 Proof of Theorem 1.1

To prove Theorem 1.1 two more results are needed.

Let $\rho_{j,k} : \mathrm{SL}(2,\Omega) \to \mathrm{Sp}(n,\Omega)$ be the homomorphisms

$$\rho_{j,k}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} I_n + (a-1)(E_{j,j} + E_{k,k}) & b(E_{j,k} + E_{k,j}) \\ c(E_{j,k} + E_{k,j}) & I_n + (d-1)(E_{j,j} + E_{k,k}) \end{pmatrix}$$

and $\tilde{\rho}_{j,k} : \mathrm{SL}(2,\Omega) \to \mathrm{SL}(n,\Omega)$ the homomorphisms

$$\tilde{\rho}_{j,k}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = I_n + (a-1)E_{j,j} + bE_{j,k} + cE_{k,j} + (d-1)E_{k,k}$$

for $j,k = 1,\ldots,n$. Let

$$\begin{aligned} G_{j,k} &= \rho_{j,k}(\mathrm{SL}(2,\Omega)), \\ \tilde{G}_{j,k} &= \left\{\begin{pmatrix} \tilde{\rho}_{j,k}(g) & 0 \\ 0 & \left(\tilde{\rho}_{j,k}(g)^T\right)^{-1} \end{pmatrix} : g \in \mathrm{SL}(2,\Omega)\right\} \end{aligned}$$

for $j,k = 1,\ldots,n$ be the corresponding subgroups.

Lemma 5.1. *Let $n \geq 2$, then for every elementary symplectic matrix in* $\mathrm{Sp}(n,\Omega)$ *there exists a copy of a subgroup isomorphic to $S^2(\Omega^2)$ canonically contained in a semi-direct product isomorphic to* $\mathrm{SL}(2,\Omega) \ltimes S^2(\Omega^2)$. *There the* $\mathrm{SL}(2,\Omega)$*-part can be chosen to be a $\tilde{G}_{k,k+1}$ or a $G_{k,k+1}$ with $1 \leq k \leq n-1$.*

Proof. For $n = 2$ there are four natural embeddings of $\mathrm{SL}(2,\Omega) \ltimes S^2(\Omega^2)$ in $\mathrm{Sp}(2,\Omega)$ where the union of the $S^2(\Omega^2)$-parts contains every elementary symplectic matrix. The subgroups contain the following matrices with $x,y,z \in \Omega$

$$\begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & y & z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ x & y & 1 & 0 \\ y & z & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & y & x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & z & -y & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ y & 1 & 0 & z \\ x & 0 & 1 & -y \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For example the first of these are contained in the $S^2(\Omega^2)$-part of the embedding mapping (a,b) to $\begin{pmatrix} a & ab \\ 0 & \left(a^T\right)^{-1} \end{pmatrix} \in \mathrm{Sp}(2,\Omega)$ for $a \in \mathrm{SL}(2,\Omega)$ and $b \in S^2(\Omega^2)$.

Embedding $\mathrm{Sp}(n-1,\Omega)$ into $\mathrm{Sp}(n,\Omega)$ and induction finishes the proof. ■

Lemma 5.2. *Let π be a unitary representation of G on H_π and for a unit vector $\xi \in H_\pi$, let $\|\pi(g)\xi - \xi\| \leq 1$ for all $g \in G$, then there exists a G-invariant unit vector in H_π.*

Proof. The assumption is equivalent to $\operatorname{Re}\langle\pi(g)\xi|\xi\rangle \geq 1/2$ for all $g \in G$. Let C be the closed convex hull of $\pi(G)\xi$, and let η be the point of minimal norm in C. Then η is G-invariant and $\eta \neq 0$ since $\operatorname{Re}\langle\eta|\xi\rangle \geq 1/2$. ■

Now Theorem 1.1 can be proved.

Proof. As $\|\pi(h)\xi - \xi\| = \|\pi(h^{-1})\xi - \xi\|$ it can be supposed that Q is symmetric. So replace Q by $Q \cup Q^{-1}$.

Let now $\xi \in H_\pi$ be a $\left(Q, 3 \times 14^{-2m-2}(\nu_n(\Omega))^{-1}\right)$-invariant vector and $h \in \operatorname{Sp}(n, \Omega)$ an elementary symplectic matrix, then by Lemma 5.1 there exists a subgroup isomorphic to $\mathrm{SL}(2,\Omega) \ltimes S^2(\Omega^2)$ such that $h \in S^2(\Omega^2)$. Hence $\|\pi(h)\xi - \xi\| \leq (\nu_n(\Omega))^{-1}$ by Corollary 4.2.

Let $\nu = \nu_n(\Omega)$, $g \in \operatorname{Sp}(n,\Omega)$, and $g = g_0 g_1 \cdots g_\nu$ where $g_0 = I_{2n}$ and $g_1, \ldots, g_\nu$ are elementary symplectic matrices, then

$$\pi(g)\xi - \xi = \sum_{j=0}^{\nu-1} \pi(g_0 g_1 \cdots g_{\nu-j})\xi - \pi(g_0 g_1 \cdots g_{\nu-j-1})\xi$$

and

$$\begin{aligned} \|\pi(g)\xi - \xi\| &\leq \sum_{j=0}^{\nu-1} \|\pi(g_0 g_1 \cdots g_{\nu-j})\xi - \pi(g_0 g_1 \cdots g_{\nu-j-1})\xi\| \\ &= \sum_{j=0}^{\nu-1} \|\pi(g_{\nu-j})\xi - \xi\| \leq \nu\nu^{-1} = 1. \end{aligned}$$

By Lemma 5.2 there exists a nonzero G-invariant vector.

Let π be a representation of $\operatorname{Sp}(n,\Omega)$ on H_π without nonzero invariant vectors, $\xi \in H_\pi$ a unit vector, and U a neighborhood of $0 \in \Omega$ such that $\|\pi(g(t))\xi - \xi\| < \varepsilon = 3 \times 14^{-2m-2}(\nu_n(\Omega))^{-1}$ for every elementary symplectic matrix $g(t)$ with $t \in U$, then there is a $g \in Q_1$ with $\|\pi(g)\xi - \xi\| \geq \varepsilon$. ■

6 The loop group

This section contains the proof of Theorem 1.2. Let in the following Ω be the ring of continuous functions $f : S^1 \to \mathbf{C}$. For preparation two results are needed.

The following lemma is also proven in [4]. But the proof here is different and the statement a little bit stronger.

Lemma 6.1. *Let $f, \tilde{f} : S^1 \to \mathbf{C}$ be two continuous functions with no common zero. Then there exists a continuous function $\Phi : S^1 \to S^1 \subset \mathbf{C}$ such that $\tilde{f} + \Phi f$ has no zero.*

Proof. Let $A = \left\{x \in S^1 : \left|\tilde{f}(x)\right| = |f(x)|\right\}$, then A is closed. For $x \in A$ let $\Phi(x) = \frac{\tilde{f}(x)}{f(x)} \in S^1$. If $A \neq S^1$ there is a continuous function $\tilde{\Phi} : A \to \mathbf{R}$ such that $\Phi(x) = \exp\left(i\tilde{\Phi}(x)\right)$ for $x \in A$. As A is compact and $\tilde{\Phi}$ is continuous $\tilde{\Phi}(A)$ is compact and in particular contained in a bounded interval $[a, b]$. A theorem of Tietze and Urysohn, see for example [3, page 83], assures that $\tilde{\Phi}$ can be extended to a continuous function on S^1.

This Φ proves the lemma. Indeed, $|f(x)| \neq \left|\tilde{f}(x)\right|$ for $x \notin A$ and hence $\left|\tilde{f}(x) + \Phi(x) f(x)\right| \geq \left||f(x)| - \left|\tilde{f}(x)\right|\right| > 0$. In the other case $x \in A$ holds that $\tilde{f}(x) + \Phi(x) f(x) = 2\tilde{f}(x) \neq 0$. ∎

Theorem 6.2. *For $n \geq 2$ and the ring Ω the group G is boundedly elementary generated and $\nu_n(\Omega) \leq 3n^2 + 4n$.*

Proof. Let $g \in \mathrm{Sp}(n, \Omega)$ be arbitrary and $(f_1, \ldots, f_{2n})$ the first column of the matrix g, then the functions $f_1, \ldots, f_{2n}$ have no common zero as g is invertible. In particular the functions $\tilde{f} = f_1$ and $f = |f_2|^2 + \cdots + |f_{2n}|^2$ have no common zero. Let $\Phi : S^1 \to \mathbf{C}$ be as in Lemma 6.1, then add to the first row the second multiplied by $\Phi\overline{f_2}$ and so on to the last row. After these $2n - 1$ steps the entry in the first row and first column is a function without zero. Four further steps yield the function constant 1 in this entry. Then in $2n-1$ steps the second to the $2n$-th entry becomes the zero function. For the first row this employs a total of $2(2n-1) + 4 = 4n + 2$ steps.

Similarly the next $n-1$ columns yield $4(n+1-k)+2$ steps in the k-th column. In total $\sum_{k=1}^{n}(4(n+1-k)+2) = 2n^2 + 4n$. The remaining entries can be made to zero in $\sum_{k=1}^{n}(k-1) = (n^2 - n)/2$ and $\sum_{k=1}^{n} k = (n^2 + n)/2$ steps. This yields a total of $3n^2 + 4n = 2n^2 + 4n + (n^2 - n)/2 + (n^2 + n)/2$ steps. ∎

The ring Ω has a dense subring generated by the functions $x \mapsto x$, $x \mapsto \overline{x}$, and $x \mapsto \sqrt{2} + i$. This can be deduced using the Stone–Weierstrass theorem and the fact that $\sqrt{2} + i$ generates a dense subring of $\mathbf{C}$. By choosing the functions $x \mapsto n^{-1}x$, $x \mapsto n^{-1}\overline{x}$ and $x \mapsto n^{-1}\left(\sqrt{2} + i\right)$ for any positive integer n also the last statement of Theorem 1.1 holds. This proves Theorem 1.2.

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