Comparison of the product structures in algebraic and in topological K-theory

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Abstract

The compatibility up to sign of the product structures in algebraic K-theory and in topological K-theory of unital Banach algebras is established in total degree ≤ 2. This answers a question posed by Milnor.

1 Statement of the theorem and definition of the product structures in K-theories

As an application of the computations made in [7], we prove the following result.

1.1 Theorem. *Let A and B be two unital Banach algebras. Then the diagram*

$$\begin{tikzcd} K_p^{alg}(A) \otimes K_q^{alg}(B) \arrow[r, "\star"] \arrow[d, "\phi_p \otimes \phi_q"'] & K_{p+q}^{alg}(A \otimes_{\mathbb{Z}} B) \arrow[d, "(-1)^{pq}\hat{\phi}_{p+q}"] \\ K_p(A) \otimes K_q(B) \arrow[r, "\times"] & K_{p+q}(A \hat{\otimes} B) \end{tikzcd}$$

commutes for p, $q \geq 0$ satisfying $p+q \leq 2$. In other words, the external product structures in algebraic and in topological K-theory of unital Banach algebras are compatible in total degree ≤ 2, up to the sign $(-1)^{pq}$. In particular, for commutative unital Banach algebras, the internal product structures are also compatible in the same range and up to the same sign.

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Let us explain the notations. For a unital Banach algebras A (always over $\mathbb{C}$), we denote by $\mathrm{GL}(A)$ the infinite matrix group with the usual direct limit topology, by $E(A)$ the group of infinite elementary matrices, which coincides with the commutator subgroup $[\mathrm{GL}(A),\, \mathrm{GL}(A)]$ of $\mathrm{GL}(A)$, and by $\mathrm{St}(A)$ the infinite Steinberg group of A with standard generators $(x_{ij}(a))_{i\neq j,\, a\in A}$. The algebraic and topological K-theory groups are defined by:

- $K_0^{alg}(A) = K_0(A)$ is the Grothendieck group of the underlying ring A;
- $K_1^{alg}(A) := \mathrm{GL}(A)^{ab} = \mathrm{GL}(A)/E(A)$;
- $K_1(A) := \pi_0(\mathrm{GL}(A)) = \mathrm{GL}(A)/\,\mathrm{GL}(A)_0$, where $\mathrm{GL}(A)_0$ is the arc component of the identity in $\mathrm{GL}(A)$;
- $K_2^{alg}(A) := \mathrm{Ker}\left(\mathrm{St}(A)\overset{\varphi}{\twoheadrightarrow}E(A)\right)$, where the map $\mathrm{St}(A)\overset{\varphi}{\twoheadrightarrow}E(A)$ takes the standard generator $x_{ij}(a)$ of $\mathrm{St}(A)$ to the elementary matrix $e_{ij}(a)$;
- $K_2(A) := \pi_1(\mathrm{GL}(A))$.

By Bott periodicity, we have, for any Banach algebra A, $K_2(A) \cong K_0(A)$. We now depict the canonical and natural maps $\phi_i^A = \phi_i\colon K_i^{alg}(A) \longrightarrow K_i(A)$. For $i = 0$, ϕ_0^A is merely the identity of $K_0^{alg}(A)$, and the well-known inclusion $E(A) \subseteq \mathrm{GL}(A)_0$ allows to define the map ϕ_1^A taking, for $u \in \mathrm{GL}(A)$, the class $[u]$ in $K_1^{alg}(A)$ to the class $[u]$ in $K_1(A)$. Let us now describe ϕ_2^A. Let $\widetilde{\mathrm{GL}}(A)_0$ be the universal covering space of the topological group $\mathrm{GL}(A)_0$. As usual, we see the group $\widetilde{\mathrm{GL}}(A)_0$ as the set of homotopy classes (rel. to $\{0,\, 1\}$) of paths in $\mathrm{GL}(A)_0$ (parameterized by $t \in [0,\, 1]$) emanating from $\mathbb{1}$, with pointwise multiplication, and the projection $\widetilde{\mathrm{GL}}(A)_0 \twoheadrightarrow \mathrm{GL}(A)_0$ is given by evaluation at $t = 1$, and has its kernel equal to $\pi_1(\mathrm{GL}(A)_0) = \pi_1(\mathrm{GL}(A)) = K_2(A)$. Consider the map $\mathrm{St}(A) \longrightarrow \widetilde{\mathrm{GL}}(A)_0$ defined on the standard generators of $\mathrm{St}(A)$ by

$$\psi\colon x_{ij}(a) \longmapsto \Big[t \mapsto e_{ij}(t\cdot a)\Big]\,,$$

where $a \in A$, t ranges over $[0,\, 1]$, and the above brackets designate a homotopy class. One can easily check that the images of the $x_{ij}(a)$'s satisfy all the defining relations of $\mathrm{St}(A)$, consequently, the map ψ is a well-defined homomorphism. Now, the diagram

$$\begin{tikzcd}
0 \arrow[r] & K_2^{alg}(A) \arrow[r] \arrow[d, dashed, "\phi_2^A"'] & \mathrm{St}(A) \arrow[r, "\varphi"] \arrow[d, "\psi"'] & E(A) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & K_2(A) \arrow[r] & \widetilde{\mathrm{GL}}(A)_0 \arrow[r] & \mathrm{GL}(A)_0 \arrow[r] & 0
\end{tikzcd}$$

commutes. Therefore, by restriction, ψ induces a homomorphism ϕ_2^A; explicitly,

$$\begin{aligned}
\phi_2^A\colon K_2^{alg}(A) &\longrightarrow K_2(A) = \pi_1(\mathrm{GL}(A)_0)\\
\textstyle\prod_s x_{i_sj_s}(a_s) &\longmapsto [e^{2\pi i t} \mapsto \textstyle\prod_s e_{i_sj_s}(t\cdot a_s)]\ .
\end{aligned}$$

1.2 Remark. *Algebraic and topological K-groups in higher degree ($p \geq 1$) can be defined by*

$$K_p^{alg}(A) := \pi_p(\mathrm{BGL}^\delta(A)^+) \text{ and } K_p(A) := \pi_{p-1}(\mathrm{GL}(A)) \cong \pi_p(\mathrm{BGL}(A)),$$

where $\mathrm{GL}^\delta(A)$ stands for $\mathrm{GL}(A)$ made discrete.

(The definition of K_p^{alg} makes sense for any unital ring). The map $\mathrm{B}(Id)\colon \mathrm{BGL}^\delta(A) \longrightarrow \mathrm{BGL}(A)$ induces at the level of fundamental groups a map taking $E(A) \subseteq \mathrm{GL}^\delta(A)$ to zero, since $\pi_1(\mathrm{BGL}(A)) = \pi_0(\mathrm{GL}(A)) = \mathrm{GL}(A)/\mathrm{GL}(A)_0$ and $E(R) \subseteq \mathrm{GL}(A)_0$. Consequently, $\mathrm{B}(Id)$ induces a map $\mathrm{B}(Id)^+\colon \mathrm{BGL}^\delta(A)^+ \longrightarrow \mathrm{BGL}(A)$. For any $p \geq 1$, this allows to define a canonical and natural map

$$\phi_p^A := \pi_p(\mathrm{B}(Id)^+)\colon K_p^{alg}(A) \longrightarrow K_p(A).$$

These definitions extend functorially to the non-unital situation. One can check that for $p = 1$ and 2, all these definitions coincide with the ones given above.

For two rings A and B (not necessarily unital), the external product in algebraic K-theory (see [6]) is denoted by

$$K_p^{alg}(A) \otimes K_q^{alg}(B) \xrightarrow{\star} K_{p+q}^{alg}(A \otimes_{\mathbb{Z}} B).$$

The internal product is defined for A commutative by composing the external product with the homomorphism $K_{p+q}^{alg}(A \otimes_{\mathbb{Z}} A) \longrightarrow K_{p+q}^{alg}(A)$, induced by the product map $\mu\colon A \otimes_{\mathbb{Z}} A \longrightarrow A$ (which is an ring homomorphism, precisely because A is commutative). It will be denoted by $\star_A$ or by $\star$. Note that this internal product is graded-commutative (see Theorem 2.1.12 in [6]).

As noticed by Loday in [6], the internal product he defines at the level of the plus construction (and of spectra) coincides, in total degree $p + q \leq 2$, with the product defined case by case by Milnor *only up to sign*. More precisely, both definitions coincide, except for $p = q = 1$, where Loday's product is minus Milnor's product (see Proposition 2.2.3 in [6]): for $x,\, y \in K_1^{alg}(A)$ with A commutative, the formula

$$x \star_A y = -\{x,\, y\} \in K_2^{alg}(A)$$

holds, where $\{x,\, y\}$ is the Steinberg symbol of x by y.

Let $A\hat{\otimes}B$ denote the completed projective tensor product (over $\mathbb{C}$) of two Banach algebras A and B. For a Banach algebra A and for $p \geq 1$, the p-fold suspension of A is defined by $S^pA := S(S^{p-1}A) \cong C_0(\mathbb{R}^p)\hat{\otimes}A$; note that it is not unital if so is A. The p-fold suspension isomorphism is a natural isomorphism

$$\sigma^p\colon K_p(A) \xrightarrow{\cong} K_0(S^pA).$$

(As a convenient notation, we also write $S^0A := A$ and $\sigma^0 := Id_{K_0(A)}$.) The equality of functors $K_0^{alg} = K_0$ and the suspension isomorphism uniquely define the external cross product

$$K_p(A) \otimes K_q(B) \xrightarrow{\times} K_{p+q}(A\hat{\otimes}B),$$

in topological K-theory, by requiring commutativity in the diagram

$$\begin{tikzcd}
K_p(A)\otimes K_q(B) \arrow[rr, dashed, "\times"] \arrow[d, "\cong"', "\sigma^p\otimes\sigma^q"] & & K_{p+q}(A\hat{\otimes}B) \arrow[d, "\sigma^{p+q}"', "\cong"] \\
K_0(S^pA)\otimes K_0(S^qB) \arrow[r, "\star"] & K_0(S^pA\otimes_{\mathbb{Z}}S^qB) \arrow[r, "\nu_*"] & K_0(S^{p+q}(A\hat{\otimes}B))
\end{tikzcd}$$

with $\nu\colon S^pA\otimes_{\mathbb{Z}}S^qB \longrightarrow S^pA\otimes_{\mathbb{C}}S^qB \hookrightarrow S^pA\hat{\otimes}S^qB \cong S^{p+q}(A\hat{\otimes}B)$ (compare with II.5.26 in [5]). As in the algebraic case, the internal product "$\cup$", called cup product, is defined for A commutative by composing with the homomorphism $K_{p+q}(A\hat{\otimes}A) \longrightarrow K_{p+q}(A)$, induced by the "completed product map" $\hat{\mu}\colon A\hat{\otimes}A \longrightarrow A$ (which is a Banach algebra morphism). Note that the cup product is graded-commutative (compare with Propositions II.4.10 and II.5.27 in [5]). Finally, for $p \geq 0$, $\hat{\phi}_p$ denotes the composition

$$K_p^{alg}(A\otimes_{\mathbb{Z}}B) \longrightarrow K_p^{alg}(A\otimes_{\mathbb{C}}B) \longrightarrow K_p^{alg}(A\hat{\otimes}B) \xrightarrow{\phi_p} K_p(A\hat{\otimes}B)\,.$$

(Notice that ν_* in the above diagram is just $\hat{\phi}_0$.) This makes all the notations used in Theorem 1.1 meaningful. Note that the statement amounts to the formula

$$\sigma^{p+q}\circ\hat{\phi}_{p+q}(x\star y) = (-1)^{pq}\Big(\sigma^p\circ\phi_p(x)\Big)\times\Big(\sigma^q\circ\phi_q(y)\Big)\in K_0(S^{p+q}(A\hat{\otimes}B))\,,$$

for all $x\in K_p^{alg}(A)$ and $y\in K_q^{alg}(B)$.

Before stating an important corollary of Theorem 1.1, for a compact Hausdorff space X, we let

$$\theta_*\colon K_*(C(X)) \xrightarrow{\cong} K^{-*}(X)$$

be the Swan-Serre isomorphism, where $C(X)$ is the commutative unital C^*-algebra of continuous complex valued functions on X, with the norm of uniform convergence.

1.3 Corollary. *For a compact Hausdorff space X, the diagram*

$$\begin{tikzcd}
K_p^{alg}(C(X))\otimes K_q^{alg}(C(X)) \arrow[r, "\star"] \arrow[d, "\phi_p\otimes\phi_q"'] & K_{p+q}^{alg}(C(X)) \arrow[d, "(-1)^{pq}\phi_{p+q}"] \\
K_p(C(X))\otimes K_q(C(X)) \arrow[r, "\cup"] \arrow[d, "\theta_p\otimes\theta_q"', "\cong"] & K_{p+q}(C(X)) \arrow[d, "\cong"', "\theta_{p+q}"] \\
K^{-p}(X)\otimes K^{-q}(X) \arrow[r, "\cup"] & K^{-(p+q)}(X)
\end{tikzcd}$$

commutes, for p, $q\geq 0$ satisfying $p+q\leq 2$, where the bottom horizontal map is the usual cup product in K-theory.

Proof. The product $\mu\colon C(X)\otimes_{\mathbb{Z}}C(X) \longrightarrow C(X)$ yields a commutative diagram

$$\begin{tikzcd}
K_{p+q}^{alg}(C(X)\otimes_{\mathbb{Z}}C(X)) \arrow[r, "K_{p+q}^{alg}(\mu)"] \arrow[d, "\hat{\phi}_{p+q}"'] & K_{p+q}^{alg}(C(X)) \arrow[d, "\phi_{p+q}"] \\
K_{p+q}(C(X)\hat{\otimes}C(X)) \arrow[r, "K_{p+q}(\hat{\mu})"] & K_{p+q}(C(X))
\end{tikzcd}$$

Consequently, commutativity of the upper square follows from Theorem 1.1. The bottom square commutes, since the Swan-Serre isomorphism is a ring map. ■

1.4 Remark. *i) Theorem 1.1 easily extends to the case of non-unital Banach algebras, and Corollary 1.3 to the more general situation of Hausdorff locally compact spaces, using the commutative C^*-algebra $C_0(X)$.*

ii) For the external cross product $K^{-p}(X) \otimes K^{-q}(Y) \xrightarrow{\times} K^{-(p+q)}(X \times Y)$, the result corresponding to Corollary 1.3 obviously holds (for Hausdorff locally compact spaces).

iii) Corollary 1.3 was an open question in Milnor's book [8] (see p. 67).

For the proof of Theorem 1.1, we can assume that $p \leq q$.

This paper is organized as follows. In Section 2, we prove Theorem 1.1 for $p = 0$. The most difficult case, namely $p = q = 1$, is dealt with in Section 3, applying results of [7] (coping with the C^*-algebra $C^*\mathbb{Z}^2 \cong C(\mathbb{T}^2)$).

2 The cases $p = 0$

By direct computation, we prove Theorem 1.1 for $p = 0$.

Recall that the algebraic and the topological K-theory groups are Morita invariant: for $i \geq 0$ and $n \geq 1$, there are isomorphisms

$$K_i^{alg}(A) \cong K_i^{alg}(M_n(A)) \text{ and } K_i(A) \cong K_i(M_n(A)),$$

induced by the (non-unital) inclusion $A \hookrightarrow M_n(A)$, $a \longmapsto \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$. In particular, the products being natural, they are compatible with Morita equivalence. We can therefore reduce to the case of idempotent (1×1)-matrices and invertible (1×1)-matrices. Let $x \in K_0^{alg}(A)$ and $y \in K_q^{alg}(B)$. We have to show that

$$\sigma^q \circ \hat{\phi}_q(x \star y) = x \times (\sigma^q \circ \phi_q(y)) \in K_0(S^q(A \hat{\otimes} B)).$$

Let x be the class of an idempotent $\varepsilon \in A$. For $q = 0$, there is nothing to prove. For $q = 1$, suppose that y is the class of an invertible element $u \in B$. By definition of the $\star$-product (see [8]), one has

$$x \star y = \Big[\varepsilon \otimes u + (1 - \varepsilon) \otimes 1\Big] \in K_1^{alg}(A \otimes_{\mathbb{Z}} B).$$

(The inverse of this matrix is $\varepsilon \otimes u^{-1} + (1 - \varepsilon) \otimes 1$.) The suspension isomorphism is given by

$$\sigma = \sigma^1 \colon K_1(A) \xrightarrow{\cong} K_0(SA), \ [v] \longmapsto \Big[t \mapsto R_t \cdot P \cdot R_t^{-1}\Big] - \Big[P\Big],$$

where $v \in \mathrm{GL}_n(A)$, $P := \mathrm{Diag}(\mathbb{1}_n, \mathbb{0}_n)$, and $R_t = R_t(v)$ is a homotopy (i.e. a path) in $\mathrm{GL}_{2n}(A)$ from $\mathbb{1}_{2n}$ to the matrix $\mathrm{Diag}(v, v^{-1})$ which, by the Whitehead Lemma, belongs to the arc component of $\mathbb{1}_{2n}$ in $\mathrm{GL}_{2n}(A)$. The suspension isomorphism is independent of the chosen homotopy. If R_t is a path from $\mathbb{1}_2$ to $\mathrm{Diag}(u, u^{-1})$, then

$S_t := \varepsilon\hat{\otimes}R_t(u) + (1-\varepsilon)\hat{\otimes}\mathbb{1}_2$ (tensor product of matrices) is a path from $1\hat{\otimes}\mathbb{1}_2 = \mathbb{1}_2$ to $\mathrm{Diag}\Big(\varepsilon\hat{\otimes}u + (1-\varepsilon)\hat{\otimes}1,\ \varepsilon\hat{\otimes}u^{-1} + (1-\varepsilon)\hat{\otimes}1\Big)$, so that

$$\sigma \circ \hat{\phi}_1(x \star y) = \Big[t \mapsto S_t \cdot Q \cdot S_t^{-1}\Big] - \Big[Q\Big]\,,$$

with $Q := \mathrm{Diag}(1\hat{\otimes}1,\, 0\hat{\otimes}0)$. On the other hand, letting $P := \mathrm{Diag}(1,\, 0)$,

$$\begin{aligned} x \times (\sigma \circ \phi_1(y)) &= \Big[t \mapsto \varepsilon\hat{\otimes}(R_t \cdot P \cdot R_t^{-1})\Big] - \Big[\varepsilon\hat{\otimes}P\Big] \\ &= \Big[t \mapsto \varepsilon\hat{\otimes}(R_t \cdot P \cdot R_t^{-1}) + (1-\varepsilon)\hat{\otimes}P\Big] - \Big[Q\Big] \end{aligned}$$

holds. Now, observe that the matrices $S_t \cdot Q \cdot S_t^{-1}$ and $\varepsilon\hat{\otimes}(R_t \cdot P \cdot R_t^{-1}) + (1-\varepsilon)\hat{\otimes}P$ are equal (and not just equivalent). This proves Theorem 1.1 for $p = 0$ and $q = 1$.

2.1 Remark. *We deduce from this computation that*

$$\times \colon K_0(A) \otimes K_1(B) \longrightarrow K_1(A\hat{\otimes}B)\,,\ [\varepsilon] \otimes [u] \longmapsto \Big[\varepsilon\hat{\otimes}u + (\mathbb{1}_m - \varepsilon)\hat{\otimes}\mathbb{1}_n\Big]\,,$$

provided that $\varepsilon = \varepsilon^2 \in M_m(A)$ *and* $u \in \mathrm{GL}_n(B)$.

Now, let us prove Theorem 1.1 for $p = 0$ and $q = 2$. Let $x \in K_0^{alg}(A)$; using Morita invariance, we can assume that x is represented by an idempotent $\varepsilon \in A$. First, we give explicit formulas for the corresponding products by x in algebraic and in topological K_2-theory. If A is commutative, following the definition given by Milnor (see [8], p. 67), one easily checks that the product

$$x\star \colon K_2^{alg}(A) \longrightarrow K_2^{alg}(A)\,,\ y \longmapsto x \star y$$

is given by the endomorphism $(\gamma_x)_*$ of $H_2(E(R);\, \mathbb{Z}) \cong K_2^{alg}(A)$ induced by

$$\gamma_x \colon E(A) \longrightarrow E(A)\,,\ E_n(A) \ni X \longmapsto \varepsilon \cdot X + (1-\varepsilon) \cdot \mathbb{1}_n\,.$$

We need to express the map $(\gamma_x)_*$ explicitly on $K_2^{alg}(A)$ considered as the kernel in the universal central extension $0 \longrightarrow K_2^{alg}(A) \longrightarrow \mathrm{St}(A) \xrightarrow{\varphi} E(A) \longrightarrow 0$. Let $X = \prod_s e_{i_s j_s}(a_s) \in E_n(A)$ (a finite product of elementary matrices). Since $\varepsilon = \varepsilon^2$, one has clearly

$$\varepsilon \cdot X + (1-\varepsilon) \cdot \mathbb{1}_n = \prod_s \Big(\varepsilon \cdot e_{i_s j_s}(a_s) + (1-\varepsilon) \cdot \mathbb{1}_n\Big) = \prod_s e_{i_s j_s}(\varepsilon a_s)\,.$$

This means that the map γ_x is simply given by $e_{ij}(a) \longmapsto e_{ij}(\varepsilon a)$. We can therefore lift this map to $\mathrm{St}(A)$ by defining

$$\bar{\gamma}_x \colon\ \mathrm{St}(A) \longrightarrow \mathrm{St}(A)\,,\ x_{ij}(a) \longmapsto x_{ij}(\varepsilon a)\,.$$

We obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K_2^{alg}(A) \arrow[r] \arrow[d, dashed, "(\gamma_x)_*"'] & \mathrm{St}(A) \arrow[r, "\varphi"] \arrow[d, "\bar{\gamma}_x"'] & E(A) \arrow[r] \arrow[d, "\gamma_x"] & 0 \\ 0 \arrow[r] & K_2^{alg}(A) \arrow[r] & \mathrm{St}(A) \arrow[r, "\varphi"] & E(A) \arrow[r] & 0 \end{tikzcd}$$

This shows that $(\gamma_x)_* = \bar{\gamma}_{x|K_2^{alg}(A)}$, and gives a satisfactory description of the product in question, namely

$$x\star\colon K_2^{alg}(A) \longrightarrow K_2^{alg}(A)\,,\ \prod_s x_{i_sj_s}(a_s) \longmapsto \prod_s x_{i_sj_s}(\varepsilon a_s)\,.$$

For A and B two unital rings, this generalizes to give

$$x\star\colon K_2^{alg}(B) \longrightarrow K_2^{alg}(A\otimes_{\mathbb{Z}} B)\,,\ \prod_s x_{i_sj_s}(b_s) \longmapsto \prod_s x_{i_sj_s}(\varepsilon\otimes b_s)\,.$$

Now, for a unital commutative Banach algebra A, we would like to describe the product $x\cup\colon K_2(A) \longrightarrow K_2(A)$. First, observe that by definition of the cup product and naturality of the suspension isomorphism, the diagram

$$\begin{tikzcd}
K_0(A)\times K_2(A) \arrow[r, "\cup"] \arrow[d, "\cong"'] & K_2(A\hat{\otimes}A) \arrow[r, "K_2(\hat{\mu})"] \arrow[d, "\cong"'] & K_2(A) \arrow[d, "\cong"] \\
K_0(A)\times K_1(SA) \arrow[r, "\cup"] \arrow[d, "\cong"'] & K_1(S(A\hat{\otimes}A)) \arrow[r, "K_1(S\hat{\mu})"] \arrow[d, "\cong"'] & K_1(SA) \\
K_0(A)\times K_0(S^2A) \arrow[r, "\cup"] & K_0(S^2(A\hat{\otimes}A)) &
\end{tikzcd}$$

commutes, where $S\hat{\mu}$ is induced by $\hat{\mu}\colon A\hat{\otimes}A \longrightarrow A$ and is explicitly given by

$$S\hat{\mu}\colon S(A\hat{\otimes}A) \longrightarrow SA\,,\ \Big(t\mapsto a(t)\hat{\otimes}b(t)\Big) \longmapsto \Big(t\mapsto a(t)\cdot b(t)\Big)\,.$$

The map $K_2(A) = \pi_1(\mathrm{GL}(A)) \xrightarrow{\cong} K_1(SA)$, $\Big[e^{2\pi it}\mapsto v(t)\Big] \longmapsto \Big[t\mapsto v(t)\Big]$ is the isomorphism indicated on the right above. This explicit description and the one of the product $K_0\times K_1 \longrightarrow K_1$ given in Remark 2.1, allows to compute

$$\begin{gathered} x\cup\colon K_2(A) \longrightarrow K_2(A) \\ \Big[e^{2\pi it}\mapsto \textstyle\prod_s e_{i_sj_s}(t\cdot a_s)\Big] \longmapsto \Big[e^{2\pi it}\mapsto \textstyle\prod_s e_{i_sj_s}(t\cdot\varepsilon a_s)\Big]\,. \end{gathered}$$

For two unital Banach algebras A and B, this generalizes to yield

$$\begin{gathered} x\times\colon K_2(B) \longrightarrow K_2(A\hat{\otimes}B) \\ \Big[e^{2\pi it}\mapsto \textstyle\prod_s e_{i_sj_s}(t\cdot b_s)\Big] \longmapsto \Big[e^{2\pi it}\mapsto \textstyle\prod_s e_{i_sj_s}(t\cdot \varepsilon\hat{\otimes}b_s)\Big]\,. \end{gathered}$$

We are now in position to prove Theorem 1.1 for $p=0$ and $q=2$. For an element $y=\prod_s x_{i_sj_s}(b_s)\in K_2^{alg}(B)$, one has $\phi_2(y) = \Big[e^{2\pi it}\mapsto \prod_s e_{i_sj_s}(t\cdot b_s)\Big]$ (see Section 1 for the explicit description of ϕ_2). For $x=[\varepsilon]\in K_0^{alg}(A)$, with $\varepsilon=\varepsilon^2\in A$, we deduce from the above considerations that

$$\begin{gathered} \hat{\phi}_2\colon K_2^{alg}(A\otimes_{\mathbb{Z}}B) \longrightarrow K_2^{alg}(A\hat{\otimes}B) \xrightarrow{\phi_2} K_2(A\hat{\otimes}B) \\ \underbrace{\textstyle\prod_s x_{i_sj_s}(\varepsilon\otimes b_s)}_{=x\star y} \longmapsto \textstyle\prod_s x_{i_sj_s}(\varepsilon\hat{\otimes}b_s) \longmapsto \underbrace{\Big[e^{2\pi it}\mapsto \textstyle\prod_s e_{i_sj_s}(t\cdot\varepsilon\hat{\otimes}b_s)\Big]}_{=x\times\phi_2(y)}\,, \end{gathered}$$

i.e. $\hat{\phi}_2(x\star y) = x\times\phi_2(y)$, as was to be shown.

3 The case $p = q = 1$

In this section, we prove Theorem 1.1 for $p = q = 1$. It is the most difficult case, although the difficulty is not conspicuous here, since it is almost completely contained in the lengthy computations of [7].

Here, we use the same notation for an invertible matrix and for its K_1^{alg}-theory class. Roughly speaking, the following lemma tells us that we can restrict to the commutative case and the internal products $\star_A$ and $\cup$.

3.1 Lemma. *Let A and B be two unital Banach algebras, and let $x \in \mathrm{GL}_1(A)$ and $y \in \mathrm{GL}_1(B)$ be two invertibles. Consider $C := \overline{\langle 1, \hat{x}, \hat{y} \rangle}$ the unital Banach subalgebra of $A \hat{\otimes} B$ generated by $\hat{x} := x \hat{\otimes} 1$ and $\hat{y} := 1 \hat{\otimes} y$. Denote by i the inclusion of C in $A \hat{\otimes} B$, and by $j\colon A \otimes_{\mathbb{Z}} B \longrightarrow A \hat{\otimes} B$ the canonical map. Then, C is a commutative unital Banach algebra and the following formulas hold:*

$$\begin{aligned} &i)\ \ j_*(x \star y) = i_*(\hat{x} \star_C \hat{y}) \in K_2^{alg}(A \hat{\otimes} B)\,; \\ &ii)\ \ \phi_1(x) \times \phi_1(y) = i_*\Big(\phi_1(\hat{x}) \cup \phi_1(\hat{y})\Big) \in K_2(A \hat{\otimes} B)\,. \end{aligned}$$

Proof. Recall that the products for algebraic K_1-theory are given by

$$x \star y = -\{x \otimes 1, 1 \otimes y\} \ \text{ and } \ \hat{x} \star_C \hat{y} = -\{\hat{x}, \hat{y}\}\,.$$

Naturality of the Steinberg symbol yields

$$j_*\Big(\{x \otimes 1,\, 1 \otimes y\}\Big) = \{j_*(x \otimes 1),\, j_*(1 \otimes y)\} = \{i_*(\hat{x}),\, i_*(\hat{y})\} = i_*(\{\hat{x},\, \hat{y}\})\,,$$

establishing i). Using the suspension isomorphism (for x) and Remark 2.1, the product $\phi_1(x) \times \phi_1(y)$ equals the homotopy class of the map taking $e^{2\pi i t}$ to

$$X_t := \Big((R_t \cdot P \cdot R_t^{-1}) \hat{\otimes} y + (1\!\!1_2 - R_t \cdot P \cdot R_t^{-1}) \hat{\otimes} 1\Big) \cdot \Big(P \hat{\otimes} y + (1\!\!1_2 - P) \hat{\otimes} 1\Big)^{-1}\,,$$

where $P := \mathrm{Diag}(1, 0)$, and $R_t = R_t(x)$ is a homotopy in $\mathrm{GL}_2(A)$ from $1\!\!1_2$ to $\mathrm{Diag}(x, x^{-1})$. Similarly, $\phi_1(\hat{x}) \cup \phi_1(\hat{y})$ is determined by

$$\Big(R_t(\hat{x}) \cdot Q \cdot R_t(\hat{x})^{-1} \cdot \hat{y} + (1\!\!1_2 - R_t(\hat{x}) \cdot Q \cdot R_t(\hat{x}))^{-1}\Big) \cdot \Big(Q \cdot \hat{y} + (1\!\!1_2 - Q)\Big)^{-1}\,,$$

where $Q := \mathrm{Diag}(1 \hat{\otimes} 1,\, 0 \hat{\otimes} 0)$. Since i_* takes this element to X_t, ii) follows. ∎

The final lemma deals with the case of internal products.

3.2 Lemma. *Let A be a commutative unital Banach algebra. Then, for two invertibles $x, y \in \mathrm{GL}_1(A)$, one has*

$$\phi_2(x \star_A y) = -\phi_2(\{x,\, y\}) = -\phi_1(x) \cup \phi_1(y) \in K_2(A)\,.$$

Proof. The lemma is a consequence of the computations we made to prove the main result in [7]. In fact, Proposition 6.1 in *loc. cit.* is precisely the content of Lemma 3.2 for the particular Banach algebra $C^*\mathbb{Z}^2 \cong C(\mathbb{T}^2)$ and for the product $a \star_{C^*\mathbb{Z}^2} b$, where a and b are prescribed generators of $\mathbb{Z}^2$, viewed as unitaries in

$C^*\mathbb{Z}^2$. (Indeed, $\phi_1(a) \cup \phi_1(b)$ is well-known to be the Bott element $\hat{\delta}$ of $K_2(C^*\mathbb{Z}^2) \cong K^0(\mathbb{T}^2)$.) Now, we claim that by naturality and by classical results on the K-theory of commutative Banach algebras, the general case follows. To prove this, we first consider the sub-algebra

$$\mathcal{A}_\rho := \left\{ (\lambda_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} \,\middle|\, \sum_{n\in\mathbb{Z}} \rho^{|n|} \cdot |\lambda_n| < \infty \right\}$$

of $\ell^1\mathbb{Z}$, where $\rho \geq 1$ is a real number. In other words, $\mathcal{A}_\rho$ is the completion of the algebra $\mathbb{C}[\mathbb{Z}]$ for the norm

$$\left|\left| \sum_{n\in\mathbb{Z}} \lambda_n \cdot a^n \right|\right|_\rho := \sum_{n\in\mathbb{Z}} \rho^{|n|} \cdot |\lambda_n| \,,$$

where a is a prescribed generator of the group $\mathbb{Z}$. So, $\mathcal{A}_\rho$ is a unital Banach algebra for this norm, with the following "universal property" : given $u \in \mathrm{GL}_1(A)$, where A is any unital Banach algebra, one has $1 = ||1||_A \leq ||u||_A \cdot ||u^{-1}||_A$, therefore $\rho_u := \max\{||u^{-1}||_A,\, ||u||_A\}$ is ≥ 1, and the inequalities

$$\left|\left| \sum_{n\in\mathbb{Z}} \lambda_n \cdot u^n \right|\right|_A \leq \sum_{n<0} |\lambda_n| \cdot ||u^{-1}||_A^{|n|} + \sum_{n\geq 0} |\lambda_n| \cdot ||u||_A^n \leq \left|\left| \sum_{n\in\mathbb{Z}} \lambda_n \cdot a^n \right|\right|_{\rho_u}$$

imply that the algebra map $\nu_u \colon \mathbb{C}[\mathbb{Z}] \longrightarrow A$, $a \longmapsto u$ extends uniquely to a unital Banach algebra morphism $\bar{\nu}_u \colon \mathcal{A}_{\rho_u} \longrightarrow A$. Applying this result twice, by the universal property of the projective tensor product of Banach algebras, we obtain a unital Banach algebra morphism

$$\bar{\nu}_{x,\,y} \colon \mathcal{A}_{\rho_x} \hat{\otimes} \mathcal{A}_{\rho_y} \longrightarrow A \,, \xi \otimes \eta \longmapsto \bar{\nu}_x(\xi) \cdot \bar{\nu}_y(\eta) \,.$$

It is clear that $\bar{\nu}_{x,\,y}(a) = x$ and $\bar{\nu}_{x,\,y}(b) = y$, where a and b designate the prescribed generators of $\mathbb{Z}^2$, considered as elements of $\mathrm{GL}_1(\mathcal{A}_{\rho_x} \hat{\otimes} \mathcal{A}_{\rho_y})$ via the map $\mathbb{Z}[\mathbb{Z}^2] \cong \mathbb{Z}[\mathbb{Z}] \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}] \hookrightarrow \mathcal{A}_{\rho_x} \hat{\otimes} \mathcal{A}_{\rho_y}$.

In our context, the second important feature of the algebra $\mathcal{A}_\rho$ is that it is dense in $\ell^1\mathbb{Z}$ and that the inclusions

$$\mathcal{A}_\rho \overset{\text{incl}}{\hookrightarrow} \ell^1\mathbb{Z} \hookrightarrow C^*\mathbb{Z}$$

induce isomorphisms in topological K-theory, for any $\rho \geq 1$. For the second inclusion, this follows from the Wiener Lemma (see [9], 11.6) and the Density Theorem (see [3], Proposition 3, pp. 285–286), and the first is a consequence of the Oka Principle in K-theory established by Bost in [2] (see Theorem 1.1.1 and Example 1.1.3 therein). This also follows from a theorem of Arens, Eidlin and Novodvorskii : let B be a *commutative* unital Banach algebra, and let $\mathrm{Spec}(B)$ be its spectrum (it is a compact Hausdorff space); then, the Gelfand transform

$$\mathscr{G}^B \colon B \longrightarrow C(\mathrm{Spec}(B))$$

is a natural morphism and induces an isomorphism in topological K-theory (see [2], Theorem 1.3.2). It is clear that $\mathrm{Spec}(\ell^1\mathbb{Z})$ identifies with the unit circle S^1 and is included in $\operatorname{Spec}(\mathcal{A}_\rho)$, that correspondingly identifies with the closed annulus with radii ρ^{-1} and ρ. This inclusion is a homotopy equivalence, hence the isomorphism $\mathrm{incl}_* = (\mathscr{G}_*^{\ell^1\mathbb{Z}})^{-1} \circ \mathscr{G}_*^{\mathcal{A}_\rho} \colon K_*(\mathcal{A}_\rho) \xrightarrow{\cong} K_*(\ell^1\mathbb{Z})$. Similarly, the inclusions $\mathcal{A}_{\rho_x} \hat{\otimes} \mathcal{A}_{\rho_y} \hookrightarrow \ell^1\mathbb{Z}\hat{\otimes}\ell^1\mathbb{Z} \cong \ell^1\mathbb{Z}^2 \hookrightarrow C^*\mathbb{Z}^2$ induce isomorphisms

$$K_*(\mathcal{A}_{\rho_x} \hat{\otimes} \mathcal{A}_{\rho_y}) \xrightarrow{\cong} K_*(\ell^1\mathbb{Z}^2) \xrightarrow{\cong} K_*(C^*\mathbb{Z}^2)\,,$$

since for two commutative unital Banach algebras B_1 and B_2, there is a canonical homeomorphism ([4], Proposition IV.1.20)

$$\operatorname{Spec}(B_1 \hat{\otimes} B_2) \cong \operatorname{Spec}(B_1) \times \operatorname{Spec}(B_2)\,.$$

We denote $\mathcal{A}_{\rho_x} \hat{\otimes} \mathcal{A}_{\rho_y}$ simply by $\mathcal{A}$. By naturality of the internal $\star$-product, of the cup product and of the maps ϕ_1 and ϕ_2, we deduce from this argument that

$$\phi_2^{\mathcal{A}}(a \star_{\mathcal{A}} b) = -\phi_1^{\mathcal{A}}(a) \cup \phi_1^{\mathcal{A}}(b)\,.$$

By naturality, $\phi_2^A(x \star_A y) = -\phi_1^A(x) \cup \phi_1^A(y)$ holds, as was to be shown. ∎

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We now prove Theorem 1.1 for $p = q = 1$. Let $x \in K_1^{alg}(A)$ and $y \in K_1^{alg}(B)$. We have to establish that $\hat{\phi}_2(x \star y) = -\phi_1(x) \times \phi_1(y)$. By Morita invariance of the products, we can assume that $x \in \mathrm{GL}_1(A)$ and $y \in \mathrm{GL}_1(B)$. We have, with the notations of Lemma 3.1,

$$\begin{aligned}\hat{\phi}_2(x \star y) &= \phi_2^{A\hat{\otimes}B} \circ j_*(x \star y) = \phi_2^{A\hat{\otimes}B} \circ i_*(\hat{x} \star_C \hat{y}) = i_* \circ \phi_2^C(\hat{x} \star_C \hat{y}) = \\ &= -i_*\Big(\phi_1(\hat{x}) \cup \phi_1(\hat{y})\Big) = -\phi_1(x) \times \phi_1(y)\,,\end{aligned}$$

where the first equality follows from the definition of $\hat{\phi}_2$, the second from Lemma 3.1, the third from naturality of ϕ_2, the fourth from Lemma 3.2 for C, and the last one from Lemma 3.1 again.

Now, the proof of Theorem 1.1 is complete. ∎

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