A ring homomorphism is enough to get nonstandard analysis

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Abstract

It is shown that assuming the existence of a suitable ring homomorphism is enough to get an algebraic presentation of nonstandard methods that is equivalent to the popular superstructure approach, including κ-saturation.

1 Introduction

Several approaches to nonstandard analysis have been presented in the literature which are aimed to avoid as much as possible a direct use of the formalism of mathematical logic. Two fundamental examples are H.J. Keisler's [10] and C.W. Henson's [6]. The goal of those approaches is to provide "elementary" presentations that make the nonstandard methods more easily accessible, especially to the mathematicians who do not have a background in logic.

Already since the early sixties, the existence was pointed out of a bijective correspondence between filters [ultrafilters] over a given set I, and ideals [maximal ideals, resp.] in the ring of functions $f : I \to D$, with D a given division ring (see [11] §8). Since ultrafilters, by means of the corresponding ultrapowers, are the basic ingredient in constructing models of nonstandard analysis, this fact strongly suggests that a purely algebraic presentation of the nonstandard methods should be found.

To the authors' knowledge, this idea was first explicitly pursued by W. Hatcher in [5], where the hyperreal numbers are introduced starting from a maximal ideal on the ring of real $\mathbb{N}$-sequences. In the same spirit, and independently, the first author

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[1] was able to get most of the elementary applications of real nonstandard analysis by assuming a homomorphism from the ring of real $\mathbb{N}$-sequences onto an ordered field. Unfortunately, although in both of the mentioned approaches an ultrafilter over $\mathbb{N}$ can be defined, the full strength of nonstandard methods was not obtained because neither the *Leibniz transfer principle*, nor the *κ-saturation property* were available.

The goal of this paper is to show that the algebraic approach actually has the full strength of the superstructure approach, and so it provides a sound and general foundational framework for the nonstandard methods. Starting from a suitable class of ring homomorphisms, we show that a nonstandard embedding between superstructures $* : V(\mathbb{R}) \to V(\mathbb{R}^*)$ can be naturally defined in such a way that both the Leibniz transfer principle and the κ-saturation property hold.

In the first section we review the algebraic approach, based on the so-called *hyperreal fields*, defined as homomorphic images of rings of functions. Starting from a given hyperreal field $\mathbb{F}$, in the second section we show that there is a natural way of defining an embedding $* : V(\mathbb{R}) \to V(\mathbb{F})$ that satisfies the Leibniz transfer principle. The possibility of obtaining nonstandard embeddings by the algebraic approach which are κ-saturated is proved the third section. Finally, the last section contains some remarks about the uniqueness problem for hyperreal fields.

We tried to make this paper self-contained, but some familiarity with the superstructure approach to nonstandard analysis is assumed (for a detailed presentation including a formal definition of the Leibniz principle see [2] §4.4). Some knowledge of ultrafilters and of the ultrapower construction can be useful, but is not necessary.

2 Hyper-homomorphisms

Let I be any set of indexes and let $\mathcal{F}(I,\mathbb{R}) = \{f \mid f : I \to \mathbb{R}\}$ denote the ring of I-sequences of real numbers where operations are defined pointwise.

Definition 2.1. A *hyper-homomorphism* is a ring homomorphism

$$\varphi : \mathcal{F}(I,\mathbb{R}) \twoheadrightarrow \mathbb{F}$$

onto a field $\mathbb{F}$. We say that $\mathbb{F}$ is the *hyperreal field* originating from φ. Sometimes we shall refer to $\mathbb{F}$ as a hyperreal field originating from the ring of real I-sequences.

It is a basic fact in algebra that the homomorphic image of a ring is a field if and only if the kernel is a maximal ideal. Thus there are plenty of hyper-homomorphisms, and $\mathbb{F}$ is a hyperreal field if and only if $\mathbb{F} \cong \mathcal{F}(I,\mathbb{R})/\mathcal{M}$ with $\mathcal{M}$ a maximal ideal.

Notice that if $f(i) \neq g(i)$ for all i, then $\varphi(f) \neq \varphi(g)$. In fact, let $h(i)$ be the inverse $(f(i) - g(i))^{-1}$. Then $h \cdot (f - g) = 1 \Rightarrow \varphi(h) \cdot (\varphi(f) - \varphi(g)) = 1$ and so $\varphi(f) - \varphi(g) \neq 0$. For every $r \in \mathbb{R}$, let c_r denote the constant sequence with value r. Clearly, $r \neq r' \Rightarrow \varphi(c_r) \neq \varphi(c_{r'})$. For simplicity, we shall identify each real number r with the corresponding $\varphi(c_r)$. In particular, we shall directly assume that $\mathbb{R} \subseteq \mathbb{F}$. To avoid the trivial case, we always assume that $\mathbb{R} \neq \mathbb{F}$.

For every sequence f, denote by $Z(f) = \{i \mid f(i) = 0\}$ its *zero-set* and consider the following family of subsets of I:

$$\mathcal{U} = \{Z(f) \mid \varphi(f) = 0\}$$

Recall that a *filter* over a set I is a family of nonempty subsets of I that is closed under supersets and finite intersections. An *ultrafilter* U is a maximal filter, i.e. a filter U with the additional property that if a subset $a \notin U$ then its complement $I \setminus a \in U$. A *principal ultrafilter* is an ultrafilter of the form $U = \{a \subseteq I \mid i \in a\}$ for some fixed $i \in I$. Notice that an ultrafilter U is non-principal if and only if it contains no finite subset.

Proposition 2.2. *$\mathcal{U}$ is a non-principal ultrafilter over I.*

Proof. For each f, denote by f' the sequence such that $f'(i) = 0$ when $f(i) = 0$ and $f'(i) = 1$ otherwise. Notice that $\varphi(f) = 0 \Leftrightarrow \varphi(f') = 0$. In fact, on the one hand, $f = f \cdot f'$ and so if $\varphi(f') = 0$, then $\varphi(f) = \varphi(f) \cdot \varphi(f') = 0$. Vice versa, notice that $f' = f \cdot f''$ where $f''(i) = 0$ if $f(i) = 0$ and $f''(i) = 1/f(i)$ otherwise. Thus if $\varphi(f) = 0$, then also $\varphi(f') = \varphi(f) \cdot \varphi(f'') = 0$. Now let $a = Z(f)$ and $b = Z(g)$ where $\varphi(f) = \varphi(g) = 0$. Then $Z(f' + g') = Z(f') \cap Z(g') = Z(f) \cap Z(g) = a \cap b \in \mathcal{U}$ and $\mathcal{U}$ is closed under finite intersections. If $a \notin \mathcal{U}$, let f be the sequence with $f(i) = 1$ if $i \in a$ and $f(i) = 0$ otherwise. $f \cdot (1 - f) = 0 \Rightarrow \varphi(f) \cdot \varphi(1 - f) = 0$. Since $\varphi(f) \neq 0$ by hypothesis, we must have $\varphi(1 - f) = 0$, hence $Z(1 - f) = I \setminus Z(f) \in \mathcal{U}$. As a consequence, $\mathcal{U}$ is also closed under supersets. In fact, if we had $b \supseteq a \in \mathcal{U}$ for some $b \notin \mathcal{U}$, then $I \setminus b \in \mathcal{U}$ and $a \cap (I \setminus b) = \emptyset \in \mathcal{U}$. This is a contradiction because if $f(i) \neq 0$ for all i, then $\varphi(f) \neq 0$. We are left to show that $\mathcal{U}$ is non-principal. By contradiction, let us assume that there exists i^* such that $a \in \mathcal{U}$ if and only if $i^* \in a$. Then for every f, $\varphi(f) = f(i^*)$ (otherwise there is g with $[\varphi(f) - \varphi(c_{f(i^*)})] \cdot \varphi(g) = 1$, so $b = \{i \mid [f(i) - f(i^*)] \cdot g(i) = 1\} \in \mathcal{U}$, hence $i^* \in b$, which is impossible). Now, $\varphi(f) = f(i^*) \in \mathbb{R}$ for all f implies that $\operatorname{ran} \varphi = \mathbb{R}$, contradicting the assumption $\mathbb{R} \neq \mathbb{F}$. ∎

Corollary 2.3. $\varphi(f) = \varphi(g) \Leftrightarrow \{i \mid f(i) = g(i)\} \in \mathcal{U}$.

Proof. One direction is the very definition of $\mathcal{U}$. Vice versa, if $a = \{i \mid f(i) = g(i)\} \in \mathcal{U}$ but $\varphi(f) \neq \varphi(g)$, then take h with $h(i) = 1$ if $i \in a$ and $h(i) = 0$ otherwise. Then $\varphi(f - g) \cdot \varphi(h) = 0$ implies $\varphi(h) = 0$ and so $Z(h) = I \setminus a \in \mathcal{U}$, a contradiction. ∎

The reader who knows the ultrapower construction, can straightforwardly verify that the hyperreal field $\mathbb{F}$ is isomorphic to the ultrapower $\mathbb{R}^I/\mathcal{U}$. This fact was first pointed out in the classic work by L. Gillman and M. Jerison [4]. We do not prove it here because in this paper we are not assuming any knowledge of ultrapowers.

3 Getting a nonstandard embedding

Starting from a hyper-homomorphism φ, we now want to define a nonstandard embedding

$$* : V(\mathbb{R}) \longrightarrow V(\mathbb{F})$$

where $V(\mathbb{R})$ and $V(\mathbb{F})$ are the superstructures over $\mathbb{R}$ and $\mathbb{F}$, respectively. Recall that the *superstructure* $V(X)$ over a set X is the union $\bigcup_n V_n(X)$ where $V_0(X) = X$ and $V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X))$ is the union of $V_n(X)$ and all of its subsets. As customary in the superstructure approach, we assume that $\mathbb{F} \supset \mathbb{R}$ is a set of atoms. Now consider the following set of functions

$$\mathcal{F} = \{f \mid f : I \to A \text{ for some } A \in V(\mathbb{R})\}$$

Definition 3.1. Let $f, g \in \mathcal{F}$. We say that f and g are equal *almost everywhere (a.e.)*, and write $f \sim g$, if $\{i \mid f(i) = g(i)\} \in \mathcal{U}$. Similarly, we say that g belongs to f a.e., and write $g \lhd f$, when the set $\{i \mid f(i) \in g(i)\} \in \mathcal{U}$.

By the properties of a filter, $\sim$ is an equivalence relation. Denote by $[f] = \{g \in \mathcal{F} \mid f \sim g\}$ the $\sim$-equivalence class of f. We now extend the hyper-homomorphism φ to a mapping Ψ defined on the quotient $\mathcal{F}/\!\sim\; = \{[f] \mid f \in \mathcal{F}\}$ and taking values in $V(\mathbb{F})$. More precisely:

Proposition 3.2. *There exists a unique 1-1 mapping $\Psi : \mathcal{F}/\!\sim\; \to V(\mathbb{F})$ such that, for every $f \in \mathcal{F}$,*

$$\Psi([f]) = \varphi(g) \text{ if } f \sim g \in \mathcal{F}(I, \mathbb{R}); \quad \Psi([f]) = \{\Psi([g]) \mid g \lhd f\} \text{ otherwise.}$$

Proof. For each n, let $\mathcal{F}_n = \{[f] \in \mathcal{F}/\!\sim\; \mid \{i \mid f(i) \in V_n(\mathbb{R})\} \in \mathcal{U}\}$. Proceeding by induction on n, we shall show that there exists a unique 1-1 mapping $\Psi_n : \mathcal{F}_n \to V_n(\mathbb{F})$ that satisfies the above properties. Then the union $\Psi = \bigcup_n \Psi_n$ will satisfy the requirements (indeed $\mathcal{F}/\!\sim\; = \bigcup_n \mathcal{F}_n$). For $f \in \mathcal{F}_0$, pick $g \in \mathcal{F}(I, \mathbb{R})$ with $f \sim g$ and let $\Psi_0([f]) = \varphi(g)$. By Corollary 2.3, the definition of $\Psi_0 : \mathcal{F}_0 \to \mathbb{F} = V_0(\mathbb{F})$ is well-posed and Ψ_0 is 1-1. Now let $[f] \in \mathcal{F}_{n+1}$. Notice that $g \lhd f \Rightarrow g \in \mathcal{F}_n$, so we can apply the induction hypothesis and put $\Psi_{n+1}([f]) = \{\Psi_n([g]) \mid g \lhd f\}$. It is easily verified that this definition is well-posed. To show that Ψ_{n+1} is 1-1, assume that $[f] \neq [f']$ are different elements of $\mathcal{F}_{n+1}$. Without loss of generality we can assume that for all i, both $f(i)$ and $f'(i)$ are sets (i.e. $f(i), f'(i) \notin \mathbb{R}$). For each $i \in a = \{j \mid f(j) \neq g(j)\}$ let x_i be an element that witnesses that $f(i)$ and $g(i)$ are different, i.e. either $x_i \in f(i)$ and $x_i \notin f'(i)$, or $x_i \in f'(i)$ and $x_i \notin f(i)$. Thus $b = \{i \in a \mid x_i \in f(i) \setminus f'(i)\}$ and $b' = \{i \in a \mid x_i \in f'(i) \setminus f(i)\}$ are disjoint sets with $b \cup b' = a \in \mathcal{U}$. By the properties of a ultrafilter, either $b \in \mathcal{U}$ or $b' \in \mathcal{U}$. In both cases, if ξ is any function with $\xi(i) = x_i$ for $i \in a$, then $[\xi] \in \mathcal{F}_n$ witnesses that $\Psi_{n+1}([f]) \neq \Psi_{n+1}([f'])$. The uniqueness of Ψ_{n+1} trivially follows from the definition. ∎

Definition 3.3. The *nonstandard embedding* $* : V(\mathbb{R}) \to V(\mathbb{F})$ (determined by the hyper-homomorphism φ) is the mapping where $r^* = r$ if $r \in \mathbb{R}$ is an atom, and $A^* = \{\Psi([f]) \mid f : I \to A\}$ otherwise. The set A^* is called the *nonstandard extension* of A.

Notice that $\mathbb{R}^* = \{\Psi([f]) \mid f : I \to \mathbb{R}\} = \{\varphi(f) \mid f : I \to \mathbb{R}\} = \mathbb{F}$. The nonstandard extension $\mathbb{N}^*$ is called the set of *hypernaturals*. In the next Proposition we prove that our overall assumption $\mathbb{R} \neq \mathbb{R}^*$ is enough to bring about the usual non-triviality property for nonstandard embeddings.

Proposition 3.4. *There are "nonstandard" hypernaturals, i.e. elements* $\xi \in \mathbb{N}^* \setminus \mathbb{N}$.

Proof. Assume by contradiction that $\mathbb{N}^* = \mathbb{N}$. Then also $\mathcal{P}(\mathbb{N})^* = \mathcal{P}(\mathbb{N})$. In fact, on the one hand, $A = A^* \in \mathcal{P}(\mathbb{N})^*$ for all $A \subseteq \mathbb{N}$. Vice versa, if $x \in y \in \mathcal{P}(\mathbb{N})^*$ then it is easily checked from the definitions that $x \in \mathbb{N}^* = \mathbb{N}$, and so $y \in \mathcal{P}(\mathbb{N})$. Now fix a bijection $\chi : \mathbb{R} \to \mathcal{P}(\mathbb{N})$. For every $f : I \to \mathbb{R}$, consider the composition $\chi \circ f : I \to \mathcal{P}(\mathbb{N})$. Then $\Psi(\chi \circ f) \in \mathcal{P}(\mathbb{N})^* = \mathcal{P}(\mathbb{N})$ and so there is $A \subseteq \mathbb{N}$ with $\Psi(\chi \circ f) = A = A^*$, i.e. $\{i \mid \chi(f(i)) = A\} = \{i \mid f(i) = \chi^{-1}(A)\} \in \mathcal{U}$. We conclude that $\Psi([f]) = \chi^{-1}(A) \in \mathbb{R}$. Since this is true for all $f : I \to \mathbb{R}$, we have proved that $\mathbb{R}^* \subseteq \mathbb{R}$, a contradiction. ∎

We now show that $*$ satisfies the *Leibniz transfer principle.* This principle states that every "elementary" property holds for elements $a_1, \ldots, a_n$ if and only if the same property holds for the corresponding nonstandard extensions $a_1^*, \ldots, a_n^*$. The notion of "elementary" property is formalized as a *bounded quantifier formula* of first-order logic. Loosely speaking, a bounded quantifier formula is a formula where all quantifiers occur in the *bounded* forms $\forall x\,(x \in y \to \ldots)$ or $\exists x, (x \in y \wedge \ldots)$. See [2] §4.4 for a precise definition.

Theorem 3.5.
For every bounded quantifier formula $\sigma(x_1, \ldots, x_n)$ *in the language of set theory, and for every* $a_1, \ldots, a_n \in V(\mathbb{R})$,

$$\sigma(a_1, \ldots, a_n) \Longleftrightarrow \sigma(a_1^*, \ldots, a_n^*)$$

Proof. First we prove a more general fact, which is a version of the fundamental *Los theorem* of ultrapowers (see [2] §4.1). More precisely, if $[f_1], \ldots, [f_n] \in \mathcal{F}/\!\sim$, and $\sigma(x_1, \ldots, x_n)$ is a bounded quantifier formula, then:

$$\sigma\left(\Psi([f_1]), \ldots, \Psi([f_n])\right) \Longleftrightarrow \{i \mid \sigma\left(f_1(i), \ldots, f_n(i)\right)\} \in \mathcal{U}$$

This implies the theorem because by definition every $a_j^* = \Psi([c_{a_j}])$ where c_{a_j} is the constant I-sequence with value a_j, and trivially

$$\{i \mid \sigma(c_{a_1}(i), \ldots, c_{a_n}(i))\} \in \mathcal{U} \Longleftrightarrow \sigma(a_1, \ldots, a_n)$$

We proceed by induction on the complexity of formulas. Since Ψ is 1-1, $\Psi([f_1]) = \Psi([f_2]) \Leftrightarrow [f_1] = [f_2] \Leftrightarrow \{i \mid f_1(i) = f_2(i)\} \in \mathcal{U}$. Besides, by definition of Ψ,

$$\Psi([f_1]) \in \Psi([f_2]) \Leftrightarrow f_1 \lhd f_2 \Leftrightarrow \{i \mid f_1(i) \in f_2(i)\} \in \mathcal{U}.$$

Now notice that σ and σ' satisfy the assertion in the theorem if and only if the disjunction $\sigma \wedge \sigma'$ does. This is a straightforward consequence of the following property of ultrafilters. $a \cap b \in \mathcal{U}$ if and only if both $a \in \mathcal{U}$ and $b \in \mathcal{U}$. Similarly, since $a \in \mathcal{U}$ if and only if its complement $I \setminus a \notin \mathcal{U}$, it is easily seen that σ satisfies the assertion in the theorem if and only if its negation $\neg\sigma$ does. Now let us concentrate on the existential quantifier. Assume first that $\exists x \in \Psi([g])\, \sigma(x, \Psi([f_1]), \ldots, \Psi([f_n]))$. Then there is $h \lhd g$ with $\sigma(\Psi([h]), \Psi([f_1]), \ldots, \Psi([f_n]))$. We conclude that the set

$$\{i \mid \exists x \in g(i)\ \sigma\left(x, f_1(i), \ldots, f_n(i)\right)\} \in \mathcal{U}$$

because it includes $\{i \mid h(i) \in g(i)\} \cap \{i \mid \sigma(h(i), f_1(i), \ldots, f_n(i))\}$, the intersection of two sets in $\mathcal{U}$. Vice versa, assume

$$a = \{i \mid \exists x \in g(i)\ \sigma(x, f_1(i), \ldots, f_n(i))\} \in \mathcal{U}.$$

For each $i \in a$, pick $x_i \in g(i)$ such that $\sigma(x_i, f_1(i), \ldots, f_n(i))$ and consider any I-sequence ξ such that $\xi(i) = x_i$ for $i \in a$. Then, by the induction hypothesis,

$$\Psi([\xi]) \in \Psi([g]) \wedge \sigma(\Psi([\xi]), \Psi([f_1]), \ldots, \Psi([f_n]))$$

and so, in particular, $\exists x \in \Psi([g])\ \sigma(x, \Psi([f_1]), \ldots, \Psi([f_n]))$. ■

4 The saturation property

In the usual language of nonstandard analysis, an object $x \in V(\mathbb{R}^*)$ is called *internal* if it belongs to some nonstandard extension, i.e. if $x \in A^*$ for some A. Thus, in our context, x is internal if and only if $x = \Psi([f])$ for some $f \in \mathcal{F}$. Any nonstandard extension A^* is itself an internal set.

We say that a family $\mathcal{B}$ of sets has the *finite intersection property (f.i.p.)* if $B_1 \cap \ldots \cap B_n \neq \emptyset$ for any choice of finitely many $B_1, \ldots, B_n \in \mathcal{B}$. Now let an infinite cardinal κ be given. A fundamental notion in nonstandard analysis is the following.

Definition 4.1. A nonstandard embedding $*$ is *κ-saturated* (satisfies the *κ-saturation property*) if every bounded family $\mathcal{B}$ of internal sets of cardinality $|\mathcal{B}| < \kappa$ that has the f.i.p., has nonempty intersection $\bigcap \mathcal{B} \neq \emptyset$.

A family $\mathcal{B}$ is *bounded* if $\mathcal{B} \subseteq A^*$ for some A.

To get the saturation property we need to consider hyper-homomorphisms with additional properties. Denote by $\mathrm{Fin}(I)$ the collection of all finite subsets of I.

Definition 4.2. A hyper-homomorphism $\varphi : \mathcal{F}(I, \mathbb{R}) \twoheadrightarrow \mathbb{F}$ is *good* if for every family $\{f_a \mid a \in \mathrm{Fin}(I)\} \subseteq \ker \varphi$ such that $Z(f_b) \subseteq Z(f_a)$ whenever $b \supseteq a$, there exists a family $\{g_a : a \in \mathrm{Fin}(I)\} \subseteq \ker \varphi$ with $Z(g_a) \subseteq Z(f_a)$ and $Z(g_{a \cup b}) = Z(g_a) \cap Z(g_b)$ for all a, b.

The above definition is the counter-part in our context of the property of *goodness* for ultrafilters, which was introduced by H.J. Keisler [8], [9] in the sixties in order to get saturated ultraproducts. The motivation for considering good hyper-homomorphisms is given by the following result.

Theorem 4.3.
Let $\varphi : \mathcal{F}(I, \mathbb{R}) \twoheadrightarrow \mathbb{F}$ be a hyper-homomorphism, and let $\kappa = |I|$ be the cardinality of the set of indexes. If φ is good then the corresponding nonstandard embedding is κ^+-saturated.[1]

[1]By κ^+ we denote the successor cardinal of κ.

Proof. Let a bounded family $\mathcal{B}$ of internal sets with the f.i.p. be given, and assume $|\mathcal{B}| \leq \kappa$. Pick $f : I \to \mathbb{N}$ with $\Psi([f]) \in \mathbb{N}^* \setminus \mathbb{N}$. For every n, the set $u_n = \{i \mid f(i) \neq n\} \in \mathcal{U}$ but clearly $\bigcap_n u_n = \emptyset$. Since $|\mathcal{B}| \leq |I|$, we can enumerate $\mathcal{B} = \{\Psi([f_j]) \mid j \in I\}$ (possibly with repetitions). For each $a \in \mathrm{Fin}(I)$, the set

$$u_a = \{i \mid \bigcap_{j\in a} f_j(i) \neq \emptyset\} \in \mathcal{U}$$

because $\bigcap_{j\in a} \Psi([f_j]) \neq \emptyset$ by the f.i.p. Now take f_a with $f_a(i) = 0$ if and only if $i \in \bigcap_{n\leq m} u_n \cap u_a$, where m is the (finite) cardinality of a. Clearly $Z(f_a) \in \mathcal{U}$ because it is a finite intersection of elements in $\mathcal{U}$. Besides, trivially $Z(f_b) \subseteq Z(f_a)$ whenever $b \supseteq a$. By hypothesis, there is a family $\{g_a \mid a \in \mathrm{Fin}(I)\} \subseteq \ker\varphi$ such that $Z(g_a) \subseteq Z(f_a)$ and $Z(g_{a\cup b}) = Z(g_a) \cap Z(g_b)$ for all a, b. For each i, consider the set

$$\chi(i) = \left\{j \mid i \in Z(g_{\{j\}})\right\}.$$

Notice that if $\chi(i)$ contains m elements, say $j_1, \ldots, j_m \in \chi(i)$, then

$$i \in Z(g_{\{j_1\}}) \cap \ldots \cap Z(g_{\{j_m\}}) = Z(g_{\{j_1,\ldots,j_m\}}) \subseteq Z(f_{\{j_1,\ldots,j_m\}}) \subseteq \bigcap_{n\leq m} u_n.$$

Since $\bigcap_n u_n = \emptyset$, it follows that $\chi(i)$ must be finite. Now pick an element $h(i) \in \bigcap\{f_j(i) \mid j \in \chi(i)\}$. This is possible because, by definition of $\chi(i)$,

$$i \in \bigcap\left\{Z(g_{\{j\}}) \mid j \in \chi(i)\right\} = Z(g_{\chi(i)}) \subseteq Z(f_{\chi(i)}) \subseteq u_{\chi(i)},$$

and so $\bigcap\{f_j(i) \mid j \in \chi(i)\} \neq \emptyset$. As the family $\mathcal{B}$ is bounded, the I-sequence h belongs to $\mathcal{F}$. We get the assertion by showing that for every j, $\Psi([h]) \in \Psi([f_j])$. This is true because $i \in Z(g_{\{j\}}) \Leftrightarrow j \in \chi(i) \Rightarrow h(i) \in f_j(i)$ and so $Z(g_{\{j\}}) \subseteq \{i \mid h(i) \in f_j(i)\} \in \mathcal{U}$. ∎

If κ is not too large, then the converse holds. Recall that the cardinal $\beth_\omega$ (*beth-omega*) is defined as $\sup\{\beth_n \mid n \in \mathbb{N}\}$ where $\beth_0 = \aleph_0$ and $\beth_{n+1} = 2^{\beth_n}$.

Theorem 4.4.
Let $\varphi : \mathcal{F}(I, \mathbb{R}) \twoheadrightarrow \mathbb{F}$ be a hyper-homomorphism which originates a κ^+-saturated nonstandard embedding, where $\kappa = |I|$. If $\kappa < \beth_\omega$ (or, equivalently, if $\kappa = |A|$ for some $A \in V(\mathbb{R})$) then φ is good.

Proof. Let $\{f_a \mid a \in \mathrm{Fin}(I)\} \subseteq \ker\varphi$ where $Z(f_b) \subseteq Z(f_a)$ for $b \supseteq a$. We have to show that there exists a family $\{g_a \mid a \in \mathrm{Fin}(I)\} \subseteq \ker\varphi$ with $Z(g_a) \subseteq Z(f_a)$ and $Z(g_{a\cup b}) = Z(g_a) \cap Z(g_b)$ for all a, b. By the hypothesis on κ, there is a bijection $\chi : \mathrm{Fin}(\kappa) \to A$ for some $A \in V(\mathbb{R})$. For every $i \in I$ and $a \in \mathrm{Fin}(\kappa)$, let $G_a(i) = \{\chi(a') \mid a \subseteq a' \text{ and } f_{a'}(i) = 0\}$. We shall need the following facts, that can be proved in a straightforward manner.

1. $G_a(i) \neq \emptyset \Leftrightarrow f_a(i) = 0$;
2. $G_{a\cup b}(i) = G_a(i) \cap G_b(i)$;
3. $\{i \mid \chi(a) \in G_a(i)\} = Z(f_a)$.

Consider the family of internal sets $\mathcal{B} = \left\{\Psi([G_{\{\xi\}}]) \mid \xi \in \kappa\right\}$. Notice that $\mathcal{B}$ is bounded since $\mathcal{B} \subseteq \mathcal{P}(A)^*$. For every $\{\xi_1, \ldots, \xi_n\} = a \in \mathrm{Fin}(\kappa)$, $\chi(a)^* \in \Psi([G_{\{\xi_1\}}]) \cap \ldots \cap \Psi([G_{\{\xi_n\}}])$ because

$$\left\{i \mid \chi(a) \in G_{\{\xi_1\}}(i) \cap \ldots \cap G_{\{\xi_n\}}(i) = G_a(i)\right\} = Z(f_a) \in \mathcal{U}.$$

Thus $\mathcal{B}$ has the f.i.p. By κ^+-saturation, there is an element $\Psi([h]) \in \bigcap \mathcal{B}$. For each $a \in \mathrm{Fin}(\kappa)$, take g_a such that $g_a(i) = 0$ if and only if $h(i) \in G_a(i)$. Notice that for every $\xi \in a$, $\Psi([h]) \in \Psi([G_{\{\xi\}}])$, hence $u_\xi = \left\{i \mid h(i) \in G_{\{\xi\}}(i)\right\} \in \mathcal{U}$. But then $Z(g_a) = \{i \mid h(i) \in G_a(i)\} = \bigcap_{\xi \in a} u_\xi \in \mathcal{U}$ and so $\varphi(g_a) = 0$. This proves that the family $\{g_a \mid a \in \mathrm{Fin}(\kappa)\} \subseteq \ker\varphi$. Notice that $Z(g_a) \subseteq Z(f_a)$ because if $f_a(i) \neq 0$ then $G_a(i) = \emptyset$ and so $g_a(i) \neq 0$. Finally, $g_{a\cup b}(i) = 0 \Leftrightarrow h(i) \in G_{a\cup b}(i) = G_a(i) \cap G_b(i) \Leftrightarrow g_a(i) = g_b(i) = 0$, i.e. $Z(g_{a\cup b}) = Z(g_a) \cap Z(g_b)$. This completes the proof. ∎

If the index set I is countable, then the goodness property comes for free.

Proposition 4.5. *If I is countable, all hyper-homomorphisms $\varphi : \mathcal{F}(I, \mathbb{R}) \twoheadrightarrow \mathbb{F}$ are good.*

Proof. Without loss of generality, we directly assume $I = \mathbb{N}$. For each finite $a \subset \mathbb{N}$, let $g_a = f_{a'}$ where $a' = \{0, 1, \ldots, \max a\}$. Clearly, $Z(g_a) \subseteq Z(f_a)$ because $a \subseteq a'$. Now let $x = \max(a \cup b)$ and assume $x \in a$ (if $x \in b$ the proof is similar). Notice that $b' \subseteq (a \cup b)' = a'$, hence $Z(g_{a\cup b}) = Z(f_{(a\cup b)'}) = Z(f_{a'}) = Z(f_{a'}) \cap Z(f_{b'}) = Z(g_a) \cap Z(g_b)$. ∎

Thus every nonstandard embedding originating from a hyper-homomorphism on the ring of real $\mathbb{N}$-sequences satisfies the saturation property for countable families (i.e. it is $\aleph_1$-saturated). The general (uncountable) case is much harder and is implied by the following existence result about good ultrafilters.

Proposition 4.6. *For every infinite cardinal κ there exists a good hyper-homomorphism $\varphi : \mathcal{F}(\kappa, \mathbb{R}) \twoheadrightarrow \mathbb{F}$.*

Proof. Recall the following definitions (see for instance [2] §4.3 and §6.1). An ultrafilter $\mathcal{U}$ is *countably incomplete* if there is a countable family $\{u_n \mid n \in \mathbb{N}\} \subseteq \mathcal{U}$ with empty intersection $\bigcap_n u_n = \emptyset$. A filter $\mathcal{U}$ over κ is *good* if for every anti-monotonic function $\eta : \mathrm{Fin}(\kappa) \to \mathcal{U}$ there exists an anti-additive function $\theta : \mathrm{Fin}(\kappa) \to \mathcal{U}$ such that $\theta(a) \subseteq \eta(a)$ for all a. By definition, η is *anti-monotonic* if $a \subseteq b \Rightarrow \eta(a) \supseteq \eta(b)$; and θ is *anti-additive* if $\theta(a \cup b) = \theta(a) \cap \theta(b)$. Under the generalized continuum hypothesis, H.J. Keisler [8] proved that there exist countably incomplete good ultrafilters over any given cardinal κ. Subsequently, K. Kunen [12] showed that the generalized continuum hypothesis is not needed. We shall get the assertion by proving the following fact. If $\mathcal{U}$ is a countably incomplete good ultrafilter over κ, then the canonical projection

$$\varphi : \mathcal{F}(\kappa, \mathbb{R}) \twoheadrightarrow \mathcal{F}(\kappa, \mathbb{R})/\mathcal{M}$$

modulo the ideal $\mathcal{M} = \{f \mid Z(f) \in \mathcal{U}\}$ is a good hyper-homomorphism. First, it is easily seen that $\mathcal{M}$ is maximal because $\mathcal{U}$ is an ultrafilter. Hence the quotient

$\mathcal{F}(\kappa, \mathbb{R})/\mathcal{M} = \mathbb{F}$ is a field. Let us turn to the non-triviality condition $\mathbb{R} \neq F$. Take a countable family $\{u_n \mid n \in \mathbb{N}\}$ as given by the property of countable incompleteness. Without loss of generality we assume that $u_n \supseteq u_{n+1}$ for all n (otherwise take $u'_n = \bigcap_{k \leq n} u_n$). Let $a_0 = I \setminus u_0$; $a_{n+1} = u_n \setminus u_{n+1}$ and let f be the function such that $f(i) = n$ if and only if $i \in a_n$. For every n, $\varphi(f) \neq n$ because $a_n \notin \mathcal{U}$, and so $\varphi(f)$ is different from the image of any constant function. Now let a family $\{f_a \mid a \in \mathrm{Fin}(\kappa)\} \subseteq \ker \varphi = \mathcal{M}$ be given where $Z(f_b) \subseteq Z(f_a)$ for $b \supseteq a$. The function $\eta : \mathrm{Fin}(\kappa) \to \mathcal{U}$ where $\eta(a) = Z(f_a)$ is anti-monotonic, and so by hypothesis there is an anti-additive $\theta : \mathrm{Fin}(\kappa) \to \mathcal{U}$ with $\theta(a) \subseteq \eta(a)$. For each $a \in \mathrm{Fin}(\kappa)$, let g_a be a function with $g_a(i) = 0$ if and only if $i \in \theta(a)$. Then $\{g_a \mid a \in \mathrm{Fin}(\kappa)\} \subseteq \ker \varphi$ is the family we were looking for. ■

5 On the uniqueness of the hyperreals

We conclude this paper with some remarks about the uniqueness problem of the hyperreals. Since a homomorphic image of a ring is a field if and only if the kernel is a maximal ideal, in principle there are as many hyperreal fields $\mathbb{R}^* \cong \mathcal{F}(I, \mathbb{R})/\mathcal{M}$ as there are maximal ideals $\mathcal{M}$ in $\mathcal{F}(I, \mathbb{R})$. J. Roitman [13] proved that the following is consistent with ZFC: "there are $2^{\aleph_0}$ non-isomorphic hyperreal fields originating from the ring $\mathcal{F}(\mathbb{N}, \mathbb{R})$ of real $\mathbb{N}$-sequences". On the other hand, the goodness property yields the following.

Theorem 5.1.
Let κ be a cardinal such that $2^\kappa = \kappa^+$. Then all hyperreal fields originating from good hyper-homomorphisms on rings $\mathcal{F}(I, \mathbb{R})$ where $|I| = \kappa$ are isomorphic (as ordered fields).

Proof. Recall that a linearly ordered set $\langle L, \leq \rangle$ is an *η_α-set* if: (i) no subset of cardinality $< \aleph_\alpha$ is unbounded (above or below) in L; (ii) for every $A, B \subset L$ of cardinality $< \aleph_\alpha$ with $A < B$ (i.e. $a < b$ for all $a \in A$ and $b \in B$) there exists x with $A < \{x\} < B$. A classic result by E. Erdös, L. Gillman and M. Henriksen [3] states that any two real-closed fields that are η_α-sets of cardinality $\aleph_\alpha$ are isomorphic. We shall get the assertion by proving the following. If the hyper-homomorphism $\varphi : \mathcal{F}(\kappa, \mathbb{R}) \twoheadrightarrow \mathbb{R}^*$ is good, then the real-closed field $\mathbb{R}^*$ is an $\eta_{\alpha+1}$-set of cardinality $\aleph_{\alpha+1} = \kappa^+$. Clearly $\mathbb{R}^*$ is real-closed because $\mathbb{R}$ is (use Leibniz principle). If $A \subset \mathbb{R}^*$ has cardinality $< \kappa^+$, then $\mathcal{B} = \{[a, +\infty) \mid x \in A\}$ is a bounded family of internal sets with the f.i.p. and $|\mathcal{B}| = |A| < \kappa^+$. If, by contradiction, A is unbounded above, then $\bigcap \mathcal{B} = \emptyset$, contradicting the saturation property. Similarly, A is proved to be also bounded below. As for (ii), let us consider the family $\mathcal{B} = \{[a, b] \mid a \in A \text{ and } b \in B\}$. Clearly $\mathcal{B}$ has the f.i.p. and its cardinality $|\mathcal{B}| = |A \times B| = \max\{|A|, |B|\} < \kappa^+$. Then by saturation there is $x \in \bigcap \mathcal{B}$ and so $A < \{x\} < B$. We are left to show that $|\mathbb{R}^*| = \kappa^+$. Notice that $\mathcal{B} = \{\mathbb{R}^* \setminus \{x\} \mid x \in \mathbb{R}^*\}$ is a bounded family of internal sets with the f.i.p. and with empty intersection. Hence its cardinality must be $|\mathcal{B}| = |\mathbb{R}^*| \geq \kappa^+$. On the other hand, the map φ is onto, so $|\mathbb{R}^*| \leq |\mathcal{F}(\kappa, \mathbb{R})| = \left(2^{\aleph_0}\right)^\kappa = 2^\kappa = \kappa^+$. ■

Putting together Proposition 4.5 and the previous Theorem, we get the following fact, which was first pointed out by W.A.J. Luxemburg in his lecture notes [7].

Corollary 5.2. *Under the continuum hypothesis, all hyperreal fields originating from the ring of* $\mathbb{N}$*-sequences are isomorphic (as ordered fields).*

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