On the flatness of a class of metric f–manifolds

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Abstract

We consider a metric f–structure on a manifold M of dimension $2n+s$. We suppose that its kernel is paralellizable by global orthonormal vector fields $\xi_1, \dots, \xi_s$ and that the dual 1–forms satisfy $d\eta^k = F$ where F is the associated Sasaki 2–form and $k = 1, \dots, s$. We prove that if n is greater than one then M cannot be flat. This is a generalization of a result by D.E.Blair proved for contact metric manifolds. We also give a counterexample in the case $n = 1$.

1 Introduction

In recent years we have observed a rapid development of symplectic geometry and then also of contact geometry. We are interested in the Riemannian aspect of contact manifolds and their generalizations. Many results are due to the Japanese school. As a main reference for contact Riemannian manifolds we refer to the books by D.E. Blair [3, 4] and to the vast bibliography therein.

D.E. Blair proved that a contact metric manifold cannot be flat if its dimension is $2n+1$ with n greater than one, cf. [2]. He also constructed an example of a 3–dimensional contact metric manifold with vanishing curvature tensor.

From the same point of view we study a certain generalization of a contact metric structure, cf. [1, 9]. We consider a manifold M of dimension $2n+s$ with $n > 0$ and $s > 0$, equipped with an f–structure as introduced in [10], i.e. a tensor field φ of type (1,1) such that $\varphi^3 + \varphi = 0$. We suppose that the kernel of φ is a parallelizable subbundle of TM. Hence there exist global vector fields $\xi_1, \dots, \xi_s$ which span the kernel of φ. Let $\eta^1, \dots, \eta^s$ be their dual 1–forms. According to the definitions of [8], the set consisting of M with the geometric structures $(\varphi, \xi_1, \dots, \xi_s, \eta^1, \dots, \eta^s, g)$, g

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a compatible metric, is called an *almost $\mathcal{S}$–manifold* if $d\eta^k = F$ for all $k = 1, \dots, s$ where F is the Sasaki 2–form defined by g and φ. Examples of such manifolds may be constructed using the suspension method [6] or using the pull-back of toroidal bunldles, cf. [5, 8]. The almost $\mathcal{S}$–structure may be also viewed as a Riemannian almost CR–manifold of codimension s such that the orthogonal bundle is parallelizable; however we shall not use this approach in the current paper.

In the present paper we extend the result proved in [2] to almost $\mathcal{S}$–manifolds. In Section 3 we obtain preparatory identities of the curvature tensor of an almost $\mathcal{S}$–manifold. Then in Section 4 we apply these identities to prove Theorem 4.1 which says that an almost $\mathcal{S}$–manifold of dimension $2n + s$ cannot be flat if $n > 1$.

In Section 5 we consider the geometry of the pull-back of a fibration. Under certain conditions on the fibration we get a natural structure of a Riemannian fibration on the pull-back bundle. Moreover we give conditions for the pull–back bundle to be totally geodesically immersed in the original fibration. In Section 6 we use the general construction on the pull-back bundle to obtain examples of flat almost $\mathcal{S}$–manifolds of dimension $2 + s$, for all $s > 0$.

2 Preliminaries

Let M be a $(2n + s)$–dimensional manifold equipped with an *f.pk–structure*, that is an f–structure φ with a parallelizable kernel. This means that there are s global vector fields $\xi_1, \dots, \xi_s$ and 1–forms $\eta^1, \dots, \eta^s$ on M satisfying the following conditions:

$$\varphi(\xi_i) = 0\ ,\quad \eta^i \circ \varphi = 0\ ,\quad \varphi^2 = -I + \sum_{j=1}^{s} \eta^j \otimes \xi_j,\quad \eta^i(\xi_j) = \delta^i_j$$

for all $i, j = 1, \dots, s$. We denote by $\mathcal{X}(M)$ the module of differentiable vector fields on M. On such a manifold there always exists a *compatible* Riemannian metric g, in the sense that for each $X, Y \in \mathcal{X}(M)$

$$g(X, Y) = g(\varphi X, \varphi Y) + \sum_{j=1}^{s} \eta^j(X)\eta^j(Y).$$

Fixed such a metric on M, let F be the Sasaki form of φ defined by $F(X, Y) = g(X, \varphi Y)$ for $X, Y \in \mathcal{X}(M)$. We denote by $\mathcal{D}$ the bundle $\mathrm{Im}\varphi$ which is the orthogonal complement of the bundle $\ker\varphi =< \xi_1, \dots, \xi_s >$.

We assume also the following condition

$$F = d\eta^1 = \cdots = d\eta^s\ . \tag{2.1}$$

Then the metric *f.pk*–structure $(\varphi, \xi_i, \eta^i, g)$ is called an *almost $\mathcal{S}$–structure* and M an *almost $\mathcal{S}$–manifold*, cf. [8]. A metric *f.pk*–structure is said to be a *$\mathcal{K}$–structure* if F is closed and the structure is normal, i.e. $N = [\varphi, \varphi] + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i = 0$, where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. A normal almost $\mathcal{S}$–manifold is called an *$\mathcal{S}$–manifold*. Let ∇ be the Levi-Civita connection of g. We recall some formulas which will be used in the present paper, cf. [8],

$$\nabla_{\xi_i}\xi_j = 0 \tag{2.2}$$

$$\nabla_{\xi_i}\varphi = 0 \tag{2.3}$$

for $i, j \in \{1, \dots, s\}$ and

$$2g((\nabla_X \varphi)Y, Z) = g(N(Y, Z), \varphi X) + 2g(\varphi Y, \varphi X)\overline{\eta}(Z) - 2g(\varphi Z, \varphi X)\overline{\eta}(Y). \tag{2.4}$$

where $X, Y, Z \in \mathcal{X}(M)$ and $\overline{\eta} = \sum_{j=1}^{s} \eta^j$. We also consider the self-adjoint operators

$$h_i := \frac{1}{2}\mathcal{L}_{\xi_i}\varphi \tag{2.5}$$

where $i = 1, \dots, s$. In [8] many properties of these operators are proved. We list below those which will be used in the paper.

For each $i, j = 1, \dots, s$

$$h_i\xi_j = 0; \ \eta^j \circ h_i = 0. \tag{2.6}$$

Furthermore for each $X \in \mathcal{X}(M)$ and each $i = 1, \dots, s$

$$h_i \circ \varphi = -\varphi \circ h_i \tag{2.7}$$

$$\nabla_X \xi_i = -\varphi X - \varphi h_i X \tag{2.8}$$

so that

$$\nabla_X \xi_i \in \mathcal{D}. \tag{2.9}$$

Moreover, for each $X \in \mathcal{D}$ and for each $i, j = 1, \dots, s$ we have

$$\eta^j(\nabla_{\xi_i} X) = \eta^j([\xi_i, X]) = -2d\eta^j(\xi_i, X) = -2F(\xi_i, X) = 0. \tag{2.10}$$

Finally, for each $X, Y \in \mathcal{D}$ we have

$$(\nabla_X \varphi)Y + (\nabla_{\varphi X}\varphi)\varphi Y = 2g(X, Y)\overline{\xi} \tag{2.11}$$

where $\overline{\xi} = \sum_{j=1}^{s} \xi_j$.

3 Curvature identities

Let $(M, \varphi, \xi_1, \dots, \xi_s, \eta^1, \dots, \eta^s, g)$ be an almost $\mathcal{S}$–structure and R its Riemannian curvature tensor.

Proposition 3.1. *For each $i, k = 1, \dots, s$ and for each $X \in \mathcal{X}(M)$ the following formulas hold*

$$(\nabla_{\xi_i} h_k)X = \varphi(R_{\xi_i X}\xi_k) + \varphi X + \varphi h_i X - \varphi h_k X - \varphi((h_k \circ h_i)X) \tag{3.1}$$

$$(\nabla_{\xi_i} h_i)X = \varphi(R_{\xi_i X}\xi_i) + \varphi X - \varphi h_i^2 X. \tag{3.2}$$

Proof. From (2.2) and (2.8) we get

$$\begin{aligned} R_{\xi_i X}\xi_k &= -\nabla_{\xi_i}(\varphi X) - \nabla_{\xi_i}(\varphi h_k X) + \varphi([\xi_i, X]) + \varphi h_k([\xi_i, X]) \\ &= -\varphi(\nabla_X \xi_i) - \varphi((\nabla_{\xi_i} h_k)X) - \varphi h_k(\nabla_X \xi_i). \end{aligned} \tag{3.3}$$

Then we apply φ to both sides of the above equation and get

$$\begin{aligned} \varphi(R_{\xi_i X}\xi_k) &= \nabla_X \xi_i + (\nabla_{\xi_i} h_k)X + h_k(\nabla_X \xi_i) \\ &= -\varphi X - \varphi h_i X + (\nabla_{\xi_i} h_k)X + \varphi h_k X + \varphi((h_k \circ h_i)X), \end{aligned}$$

due to (2.2), (2.3), (2.9) and (2.10). Then (3.1) follows immediately. When $k = i$ equation (3.1) becomes (3.2). ∎

Corollary 3.1. *For each $i, k = 1, \dots, s$ and for each $X \in \mathcal{X}(M)$ we have*

$$R_{\xi_i X}\xi_k - \varphi(R_{\xi_i \varphi X}\xi_k) = 2\Big((h_k \circ h_i)X + \varphi^2 X\Big) \qquad (i \neq k) \tag{3.4}$$

$$R_{\xi_i X}\xi_i - \varphi(R_{\xi_i \varphi X}\xi_i) = 2\Big(h_i^2 X + \varphi^2 X\Big) \qquad (i = k). \tag{3.5}$$

Proof. Equation (3.3) can be rewritten as

$$R_{\xi_i X}\xi_k = -\varphi((\nabla_{\xi_i} h_k)X) + \varphi^2 X - h_i X + h_k X + (h_k \circ h_i)X.$$

On the other hand, from (3.1), (2.3) and (2.7), we have

$$-\varphi(R_{\xi_i \varphi X}\xi_k) = \varphi((\nabla_{\xi_i} h_k)X) + \varphi^2 X + h_i X - h_k X + (h_k \circ h_i)X.$$

Finally, the last two equations give (3.4). Putting $k = i$ we obtain (3.5). ∎

Corollary 3.2. *If M is flat, then for each $i, k = 1, \dots, s$ we have:*

$$h_k \circ h_i = -\varphi^2 = I - \sum_{j=1}^{s} \eta^j \otimes \xi_j. \tag{3.6}$$

Corollary 3.3. *If $l \in \{1, \dots, s\}$ and if $R_{\xi_l X}\xi_l = 0$ for each $X \in \mathcal{D}$, then*

$$h_l^2 = -\varphi^2 = I - \sum_{j=1}^{s} \eta^j \otimes \xi_j. \tag{3.7}$$

Proposition 3.2. *Let $l \in \{1, \dots, s\}$ and suppose that the sectional curvature of each 2–plane containing ξ_l vanishes. Then h_l has rank $2n$ and $\mathcal{D}$ is decomposable in two eigenspaces of h_l. Moreover (3.7) holds.*

Proof. Let $X \in \mathcal{D}$. Taking the scalar product of (3.5) with X we obtain $g(h_l^2 X + \varphi^2 X, X) = 0$. This implies $\|h_l X\| = \|\varphi X\| = \|X\|$. If X is an eigenvector with eigenvalue λ, then $|\lambda|\|X\| = \|X\|$ so that $\lambda = \pm 1$ and hence, due to (2.7), φX is an eigenvector of eigenvalue $-\lambda$. Then the eigenvalues of h_l are 0 with multiplicity s and ± 1 with multiplicity n. Consequently $TM = V_0 \oplus V_{-1} \oplus V_1$ where $V_0 = \mathtt{ker}\varphi$ and $V_{-1} \oplus V_1 = \mathcal{D}$. Obviously $h_l^2 = -\varphi^2$. ∎

We shall denote by V_+^l and V_-^l the eigenspaces of h_l relative to the eigenvalues 1 and -1 where $l = 1, \dots, s$.

Remark 3.1. *The above result holds replacing, for a fixed $l \in \{1, \dots, s\}$, the condition on the sectional curvature with the condition $R_{\xi_l X}\xi_l = 0$ for each $X \in \mathcal{D}$.*

Proposition 3.3. *If M is flat, then all the operators h_i coincide.*

Proof. Let $i, j \in \{1, \dots, s\}$. Since (3.6) implies $h_i^2 = h_j^2 = h_i \circ h_j = h_j \circ h_i = -\varphi^2$ then $h_i = h_j$ on $\mathcal{D}$. On the other hand $h_i \xi_k = 0 = h_j \xi_k$ for each $k = 1, \dots, s$ and then $h_i = h_j$. ∎

In the flat case we shall denote by V_+ and V_- the eigenspaces with the eigenvalues 1 and -1 of all the h_i's.

Proposition 3.4. *Let $l \in \{1,\dots,s\}$ be such that for each 2–plane containing ξ_l, the sectional curvature is zero and $R_{XY}\xi_l = 0$ for all $X, Y \in V_-^l$; then the distribution V_-^l is integrable.*

Proof. If $X, Y \in V_-^l$, then from (2.8) it follows that $\nabla_X \xi_l = \nabla_Y \xi_l = 0$. Hence

$$0 = R_{XY}\xi_l = -\nabla_{[X,Y]}\xi_l = -\varphi([X,Y]) - \varphi h_l([X,Y])$$

and then applying φ, $h_l([X,Y]) = -[X,Y]$. Namely, from (2.6) we have that $\eta^k h_l([X,Y]) = 0$ and moreover $\eta^k([X,Y]) = -2d\eta^k(X,Y) = -2F(X,Y) = -2g(X,\varphi Y) = 0$, for all $k = 1,\dots,s$, since $\varphi X \in V_+^l$. ∎

Proposition 3.5. *Let $l \in \{1,\dots,s\}$. Suppose that for each 2–plane containing ξ_l the sectional curvature vanishes and $R_{XY}\xi_l = 0$ for all $X, Y \in V_-^l$. Then, under the supplementary assumption that $R_{\xi_k X}\xi_l = 0$, for each $k = 1,\dots,s$ and each $X \in V_-^l$, the distribution $V_-^l \oplus < \xi_1,\dots,\xi_s >$ is integrable.*

Proof. In this case $\nabla_{[\xi_k,X]}\xi_l = -R_{\xi_k X}\xi_l = 0$, for $X \in V_-^l$. From (2.8) it follows that $\varphi([\xi_k,X]) + \varphi h_l([\xi_k,X]) = 0$ and then $h_l([\xi_k,X]) = -[\xi_k,X]$ which means that $[\xi_k,X] \in V_-^l$. Hence together with Proposition 3.4 we get the claim. ∎

Under the hypotheses of Proposition 3.5, since $V_-^l \oplus < \xi_1,\dots,\xi_s >$ is an integrable distribution, there exist local coordinates $u_1,\dots,u_{2n+s}$ such that

$$\left\{\frac{\partial}{\partial u_{n+1}},\dots,\frac{\partial}{\partial u_{2n}},\frac{\partial}{\partial u_{2n+1}},\dots,\frac{\partial}{\partial u_{2n+s}}\right\}$$

is a local basis of $V_-^l \oplus < \xi_1,\dots,\xi_s >$. Then we consider local functions ρ_α^j, $\alpha = 1,\dots,n$, $j = n+1,\dots,2n+s$ such that the local vector fields

$$X_\alpha = \frac{\partial}{\partial u_\alpha} + \sum_{j=n+1}^{2n+s} \rho_\alpha^j \frac{\partial}{\partial u_j} \tag{3.8}$$

belong to V_+^l. $X_1,\dots,X_n$ are linearly independent and hence they give a basis of V_+^l. From Proposition 3.5 we obtain $[\frac{\partial}{\partial u_j}, X_\alpha] \in V_-^l \oplus < \xi_1,\dots,\xi_s >$ for $\alpha = 1,\dots,n$ and $j = n+1,\dots,2n+s$. It follows that ξ_l is parallel along $[\frac{\partial}{\partial u_j}, X_\alpha]$. In fact, locally $[\frac{\partial}{\partial u_j}, X_\alpha] = X + \sum_{j=1}^s \sigma^j \xi_j$ where $X \in V_-^l$ and $\sigma^1,\dots,\sigma^s$ are differentiable functions. Then, due to (2.2), we get

$$\nabla_{[\frac{\partial}{\partial u_j}, X_\alpha]}\xi_l = \nabla_X \xi_l + \sum_{j=1}^s \sigma^j \nabla_{\xi_j}\xi_l = 0.$$

4 Flat almost $\mathcal{S}$–manifolds

In this section we fix an almost $\mathcal{S}$–manifold $(M, \varphi, \xi_1, \dots, \xi_s, \eta^1, \dots, \eta^s, g)$ and we suppose that M is flat. Hence we may apply the results of the previous section.

Lemma 4.1. *Let $X_1, \dots, X_n$ be the local frame of V_+ defined in (3.8) then the following formulas hold for $\alpha, \beta, \gamma = 1, \dots, n$*

$$\nabla_{\varphi X_\alpha} \varphi X_\beta = 0 \tag{4.1}$$
$$\nabla_{X_\alpha}(\varphi X_\beta) = \nabla_{X_\beta}(\varphi X_\alpha) \tag{4.2}$$
$$g([X_\alpha, \varphi X_\beta], X_\gamma) = 0 \tag{4.3}$$

Proof. We have already proved that for each $i = 1, \dots, s$, $j = n+1, \dots, n+s$ and $\beta = 1, \dots, n$, ξ_i is parallel along $[\frac{\partial}{\partial u_j}, X_\beta]$. Using (2.8) and $R = 0$ we get

$$0 = \nabla_{[\frac{\partial}{\partial u_j}, X_\beta]}\xi_i = \nabla_{\frac{\partial}{\partial u_j}}(\nabla_{X_\beta}\xi_i) - \nabla_{X_\beta}(\nabla_{\frac{\partial}{\partial u_j}}\xi_i) = -2\nabla_{\frac{\partial}{\partial u_j}}(\varphi X_\beta).$$

Since $\varphi X_\alpha \in V_-$ we immediately get (4.1). Using (3.8) we have $[X_\alpha, X_\beta] \in V_- \oplus < \xi_1, \dots, \xi_s >$. Furthermore:

$$\begin{aligned} g([X_\alpha, X_\beta], \xi_i) &= -g(X_\beta, \nabla_{X_\alpha}\xi_i) + g(X_\alpha, \nabla_{X_\beta}\xi_i) \\ &= -g(X_\beta, -\varphi X_\alpha - \varphi h_i X_\alpha) \\ &\quad + g(X_\alpha, -\varphi X_\beta - \varphi h_i X_\beta) \\ &= 4g(X_\beta, \varphi X_\alpha) = 0 \end{aligned}$$

so that

$$[X_\alpha, X_\beta] \in V_-. \tag{4.4}$$

It follows that $0 = R_{X_\alpha X_\beta}\xi_i = -2(\nabla_{X_\alpha}(\varphi X_\beta) - \nabla_{X_\beta}(\varphi X_\alpha))$ from which we get (4.2). Finally from (4.1), (2.8):

$$0 = R_{X_\alpha \varphi X_\beta}\xi_i = -\nabla_{[X_\alpha, \varphi X_\beta]}\xi_i = \varphi([X_\alpha, \varphi X_\beta]) + \varphi h_i([X_\alpha, \varphi X_\beta])$$

and then $[X_\alpha, \varphi X_\beta] - \sum_{j=1}^s \eta^j([X_\alpha, \varphi X_\beta])\xi_j + h_i([X_\alpha, \varphi X_\beta]) = 0$. Taking the scalar product with X_γ we get

$$g([X_\alpha, \varphi X_\beta], X_\gamma) = -g(h_i([X_\alpha, \varphi X_\beta]), X_\gamma) = -g([X_\alpha, \varphi X_\beta], X_\gamma)$$

from which we obtain (4.3). ∎

Lemma 4.2. *For each $X, Y \in V_+$ the following formula holds:*

$$(\nabla_X \varphi)Y = 2g(X, Y)\overline{\xi}. \tag{4.5}$$

Proof. We observe first that $\varphi X_1, \dots, \varphi X_n$ are linearly independent and thus give a basis of V_-. From (2.11) we have that for each $\alpha, \beta = 1, \dots, n$

$$(\nabla_{X_\alpha}\varphi)X_\beta + (\nabla_{\varphi X_\alpha}\varphi)(\varphi X_\beta) = 2g(X_\alpha, X_\beta)\overline{\xi}.$$

Taking the scalar product with φX_γ, $\gamma = 1, \ldots, n$, and using (4.1) we obtain

$$\begin{aligned} g((\nabla_{X_\alpha}\varphi)X_\beta, \varphi X_\gamma) &= -g((\nabla_{\varphi X_\alpha}\varphi)(\varphi X_\beta), \varphi X_\gamma) \\ &= g(\nabla_{\varphi X_\alpha} X_\beta, \varphi X_\gamma) \\ &= -g(X_\beta, \nabla_{\varphi X_\alpha}\varphi X_\gamma) = 0. \end{aligned}$$

Hence $(\nabla_{X_\alpha}\varphi)X_\beta$ is orthogonal to V_-. Now, from (4.2) and (4.3) we have

$$\begin{aligned} g(\nabla_{X_\alpha}\varphi X_\beta, X_\gamma) - g(\varphi(\nabla_{X_\alpha} X_\beta), X_\gamma) &= g((\nabla_{X_\alpha}\varphi)X_\beta, X_\gamma) \\ &= -g((\nabla_{\varphi X_\alpha}\varphi)\varphi X_\beta, X_\gamma) \\ &= g(\nabla_{\varphi X_\alpha} X_\beta, X_\gamma) \\ &= g(\nabla_{X_\beta}\varphi X_\alpha, X_\gamma) \\ &= g(\nabla_{X_\alpha}\varphi X_\beta, X_\gamma) \end{aligned}$$

so that $g(\varphi(\nabla_{X_\alpha} X_\beta), X_\gamma) = 0$. Then $g(\nabla_{X_\alpha} X_\beta, \varphi X_\gamma) = 0$, that is, $\nabla_{X_\alpha} X_\beta$ is orthogonal to V_-. It follows that

$$\begin{aligned} g((\nabla_{X_\alpha}\varphi)X_\beta, X_\gamma) =&\ g(\nabla_{X_\alpha}\varphi X_\beta, X_\gamma) \\ =&\ -g(\varphi X_\beta, \nabla_{X_\alpha} X_\gamma) = 0. \end{aligned} \tag{4.6}$$

Finally

$$\begin{aligned} g((\nabla_{X_\alpha}\varphi)X_\beta, \xi_i) &= g(\nabla_{X_\alpha}\varphi X_\beta, \xi_i) - g(\varphi(\nabla_{X_\alpha} X_\beta), \xi_i) \\ &= -g(\varphi X_\beta, \nabla_{X_\alpha}\xi_i) = g(\varphi X_\beta, \varphi X_\alpha + \varphi(h_i X_\alpha)) \\ &= 2g(\varphi X_\beta, \varphi X_\alpha) = 2g(X_\beta, X_\alpha). \end{aligned}$$

Hence equation (4.5) immediately follows. ∎

Lemma 4.3. *For each $\alpha, \beta = 1, \ldots, s$ we have $\nabla_{X_\alpha} X_\beta \in V_+$ and then $[X_\alpha, X_\beta] = 0$.*

Proof. From the proof of Lemma 4.2 we get that $\nabla_{X_\alpha} X_\beta$ is orthogonal to V_-. Due to (2.8) we have $g(\nabla_{X_\alpha} X_\beta, \xi_j) = -g(\nabla_{X_\alpha}\xi_j, X_\beta) = 0$ for each $j = 1, \ldots, s$. Finally from (4.4) $[X_\alpha, X_\beta] \in V_+ \cap V_-$ and therefore vanishes. ∎

Theorem 4.1. *If $(M^{2n+s}, \varphi, \xi_1, \ldots, \xi_s, \eta^1, \ldots, \eta^s, g)$ is a flat almost $\mathcal{S}$–manifold, then n cannot be greater than one.*

Proof. Suppose $n > 1$. Then there exist $\alpha, \gamma \in \{1, \ldots, n\}$ such that X_α, X_γ are linearly independent. Let $\beta \in \{1, \ldots, n\}$. We write (4.5) for X_α and X_β, then we take the covariant derivative with respect to X_γ and get

$$\begin{aligned} \nabla_{X_\gamma}\nabla_{X_\alpha}\varphi X_\beta - (\nabla_{X_\gamma}\varphi)(\nabla_{X_\alpha} X_\beta) - \varphi(\nabla_{X_\gamma}\nabla_{X_\alpha} X_\beta) \\ = 2X_\gamma(g(X_\alpha, X_\beta))\overline{\xi} - 4sg(X_\alpha, X_\beta)\varphi X_\gamma. \end{aligned}$$

Taking the scalar product with φX_δ, for any $\delta \in \{1, \ldots, n\}$ and using Lemmas 4.2 and 4.3 we get

$$\begin{aligned} g(\nabla_{X_\gamma}\nabla_{X_\alpha}\varphi X_\beta, \varphi X_\delta) - g(\varphi(\nabla_{X_\gamma}\nabla_{X_\alpha} X_\beta), \varphi X_\delta) \\ = -4sg(X_\alpha, X_\beta)g(\varphi X_\gamma, \varphi X_\delta) \end{aligned}$$

that is:

$$\begin{aligned} g(\nabla_{X_\gamma}\nabla_{X_\alpha}\varphi X_\beta, \varphi X_\delta) - g(\nabla_{X_\gamma}\nabla_{X_\alpha} X_\beta, X_\delta) \\ = -4s g(X_\alpha, X_\beta) g(X_\gamma, X_\delta). \end{aligned} \tag{4.7}$$

We interchange γ and α in equation (4.7) and then subtract one from the other. Since M is flat, using Lemma 4.3 we obtain

$$-4s\Big(g(X_\alpha, X_\beta)g(X_\gamma, X_\delta) - g(X_\gamma, X_\beta)g(X_\alpha, X_\delta)\Big) = 0$$

and in particular for $\alpha = \beta$ and $\gamma = \delta$ we have $g(X_\alpha, X_\alpha)g(X_\gamma, X_\gamma) - g(X_\alpha, X_\gamma)^2 = 0$ which contradicts the linear independence of X_α, X_γ. ∎

Remark 4.1. *Theorem 4.1 holds a fortiori for $\mathcal{S}$–manifolds.*

Corollary 4.1. *Let $(M, \varphi, \xi_i, \eta^i, g)$ be a $\mathcal{K}$–manifold of dimension $2n+s$ with $n \geq 2$ and also $s \geq 2$. Suppose that r of the η^i's are closed, $1 \leq r \leq s$, whereas $d\eta^i = F$ for the remaining $s-r$. Then M cannot be flat.*

Proof. It is known that in this case M is locally the product of an $\mathcal{S}$–manifold M_1 and a flat manifold M_2, cf. [7, Remark 3]. Since from Theorem 4.1 M_1 cannot be flat we obtain the claim. ∎

5 On the pull–back of a Riemannian fibration

Let M', B' and B be smooth manifolds. Let $\pi' : M' \to B'$ be a smooth locally trivial, not necessarily vector, fibration and let $u : B \to B'$ be a smooth map. Then we can consider the pull-back bundle $\pi : M \to B$ such that the following diagram

$$\begin{array}{ccc} M & \xrightarrow{U} & M' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ B & \xrightarrow{u} & B' \end{array}$$

commutes; U is the canonical map from M to M' such that $U(a, b) = b$ where $(a, b) \in M$. We recall that $(a, b) \in M$ if and only if $u(a) = \pi'(b)$. The standard fibre of the pull–back bundle coincides with the standard fibre of $\pi' : M' \to B'$ and the map U is a diffeomorphism when restricted to the fibres. Actually M is an embedded submanifold of $B \times M'$.

On the other hand the tangent maps $d\pi' : TM' \to TB'$ and $du : TB \to TB'$ allow us to define the pull–back vector bundle $E \to TB$ such that the following diagram

$$\begin{array}{ccc} E & \xrightarrow{U_1} & TM' \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle d\pi'} \\ TB & \xrightarrow{du} & TB' \end{array}$$

commutes. The bundle E is defined explicitly in the following way:

$$E = \bigcup_{(a,b)\in M} \Big\{(v, X) \in T_aB \times T_bM' \mid du_a(v) = d\pi'_b(X)\Big\}.$$

We have a canonical projection Π from E to M, such that $\Pi(v, X) = (a, b)$ where $(v, X) \in T_aB \times T_bM'$. Then it is easy to observe that $\Pi : E \to M$ is a vector bundle with the fibre over $(a, b) \in M$ consisting of all pairs of vectors $(v, X) \in T_aB \times T_bM'$ such that $du_a(v) = d\pi'_b(X)$.

Proposition 5.1. *The vector bundle $\Pi : E \to M$ is canonically isomorphic to the tangent bundle of M.*

Proof. Let $(v, X) \in E_{(a,b)}$ then $v \in T_aB$ and $X \in T_bM'$. There exists a curve $\gamma_1 : (-\varepsilon, \varepsilon) \to B$ such that $\gamma_1(0) = a$ and $\dot\gamma_1(0) = v$. Since $\pi' : M' \to B'$ is also a submersion, there exists a curve $\gamma_2 : (-\varepsilon, \varepsilon) \to M'$ such that $\gamma_2(0) = b$ and $\dot\gamma_2(0) = X$ and $u(\gamma_1(t)) = \pi'(\gamma_2(t))$ for all $t \in (-\varepsilon, \varepsilon)$. Hence it follows that $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ is a curve in M which defines a tangent vector to M at the point (a, b). Then we put $\Phi(v, X) := \dot\gamma(0)$. This defines the canonical ismorphisms of bundles $\Phi : E \to TM$ over M. It is easy to obtain the inverse of Φ. If $\gamma(t) = (\gamma_1(t), \gamma_2(t)) : (-\varepsilon, \varepsilon) \to M$ is a curve in M, then $u(\gamma_1(t)) = \pi'(\gamma_2(t))$ for all $t \in (-\varepsilon, \varepsilon)$. Hence $du(\dot\gamma_1(0)) = d\pi'(\dot\gamma_2(0))$. Then the inverse of Φ is given by $\Phi^{-1}(\dot\gamma(0)) = (\dot\gamma_1(0), \dot\gamma_2(0))$. ■

In what follows we shall use E as the representation of TM having in mind the isomorphism Φ constructed above. Hence local sections of TM will be represented by the pairs (v, X) where v is a local vector field on B and X is a local vector field on M' such that for each $(a, b) \in M \cap \big(\mathrm{domain}(v) \times \mathrm{domain}(X)\big)$ we have that $du(v_a) = d\pi'(X_b)$.

Proposition 5.2. *Suppose that $(v, X), (w, Y)$ are local vector fields on M then, under the canonical identification $\Phi : E \cong TM$ we have the following formula for the Lie bracket*

$$[(v, X), (w, Y)] = ([v, w], [X, Y]).$$

Proof. We observe that M is naturally immersed in the product manifold $B \times M'$. Hence we can consider the induced monomorphism of bundles $TM \to TB \times TM'$. From the construction of the isomorphism Φ it follows that the following diagram

$$\begin{array}{ccccc} TM & \xrightarrow{\ \Phi\ } & E & \xrightarrow{\text{mono}} & TB \times TM' \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ M & = & M & \xrightarrow{\text{injection}} & B \times M' \end{array}$$

commutes. Moreover the map $E \hookrightarrow TB \times TM'$ is just the inclusion determined by the inclusion $M \hookrightarrow B \times M'$. Hence (v, X), (w, Y) are vector fields on $B \times M'$ such that their restrictions to M are vector fields tangent to M. Thus the restriction of the Lie bracket $[(v, X), (w, Y)]$ to M gives the Lie bracket of (v, X) and (w, Y) restricted to M. Hence our proposition follows. ■

The fibration $\pi : M \to B$ determines its vertical bundle $\mathcal{V}(M) := \ker d\pi$. Then we have the following straightforward characterization

$$\mathcal{V}(M) = \bigcup_{(a,b)\in M} \Big\{(0, X) \in T_aB \times T_bM' \mid d\pi'_b(X) = 0\Big\}. \tag{5.1}$$

From now on we suppose that: (M', h'), (B', g') are Riemannian manifolds; $\pi' : (M', h') \to (B', g')$ is a Riemannian fibration; $u : B \to B'$ is a totally geodesic immersion. Then we put $h := c \cdot U^* h'$, $g := c \cdot u^* g'$, c being a fixed positive constant. The tensors g, h are Riemannian metrics on B and M, respectively. The horizontal bundle of $\pi : M \to B$ is given by

$$\mathcal{H}(M) = \bigcup_{(a,b)\in M} \Big\{(v, X) \in T_{(a,b)}M | d\pi'_b(X) = du_a(v), X \text{ is horizontal in } T_bM'\Big\}.$$

According to the definition of h and to (5.1) the fibration $\pi : (M, h) \to (B, g)$ is Riemannian; namely if $(v, X), (w, Y) \in \mathcal{H}_{(a,b)}(M)$, then we have that

$$\begin{aligned} h((v, X), (w, Y)) &= ch'(X, Y) = cg'(d\pi'(X), d\pi'(Y)) \\ &= cg'(du(v), du(w)) = g(d\pi(v, X), d\pi(w, Y)). \end{aligned}$$

Proposition 5.3. *Let v, w be local vector fields on B and X, Y be local vector fields on M' such that $(v, X), (w, Y)$ are tangent to the manifold $M \subset B \times M'$, then we have the following formula for the Levi–Civita connection of h*

$$\nabla^h_{(v,X)}(w, Y) = \Big(\nabla^g_v w, \nabla^{h'}_X Y\Big) \tag{5.2}$$

where ∇^g, $\nabla^{h'}$ denote the Levi–Civita connections of (B, g) and (M', h'), respectively.

Proof. We assume that $\widetilde{v}$, $\widetilde{w}$ are local vector fields on B' such that their restrictions to the image of u give the vector fields $u_*(v)$ and $u_*(w)$. Since u is totally geodesic, $u_*(\nabla^g_v w) = \nabla^{g'}_{\widetilde{v}} \widetilde{w}$. Since π' is locally trivial, we may assume that $\pi'_*(X) = \widetilde{v}$ and $\pi'_*(Y) = \widetilde{w}$. Then we observe that

$$d\pi'(\nabla^{h'}_X Y) = \nabla^{g'}_{\widetilde{v}} \widetilde{w} = du(\nabla^g_v w)$$

which follows from the fact that π' is a Riemannian fibration. This implies that $\nabla^h_{(v,X)}(w, Y)$ is tangent to the manifold M and then ∇^h is a well defined connection operator on the vector fields on M. Moreover, we have that

$$\begin{aligned} \nabla^h_{(v,X)}(w, Y) - \nabla^h_{(w,Y)}(v, X) &= \Big(\nabla^{g'}_v w, \nabla^{h'}_X Y\Big) - \Big(\nabla^{g'}_w v, \nabla^{h'}_Y X\Big) \\ &= \Big([v, w], [X, Y]\Big) \\ &= [(v, X), (w, Y)] \end{aligned}$$

and then it follows that ∇^h is torsionless. Finally we observe that $U^*(\nabla^h h) = c \cdot U^*(\nabla^{h'} h') = 0$. Since U is an immersion, we get that ∇^h is the Riemannian connection of h. ∎

Corollary 5.1.

1. *The map $U : (M, h) \to (M', h')$ is a totally geodesic conformal immersion with scaling constant equal to c ;*

2. *$R = c \cdot U^* R'$ where R and R' are Riemannian curvature tensors of h and h', respectively;*

3. if (M', h') is of constant sectional curvature K', then (M, h) is of constant sectional curvature $\frac{1}{c}K'$; in particular if (M', h') is flat, then (M, h) is also flat.

Proof. Property 1 follows from (5.2). Then 2 and 3 follow immediately from 1 and the definition of the metric tensor h. ∎

6 Examples

We shall discuss in more detail the following example.

Example 6.1 (cf. [2]). Let $M_0 = \mathbb{R}^3$ with its standard coordinates (x, y, z). Then on M_0 we consider the following geometric objects:

- the Riemannian metric tensor $h_0 := \frac{1}{4}g_{\text{can}}$ where g_{can} is the standard flat metric on M_0
- h_0–orthonormal vector fields

$$\xi_0 = 2\cos(z)\frac{\partial}{\partial x} + 2\sin(z)\frac{\partial}{\partial y},\ \zeta_0 = -2\sin(z)\frac{\partial}{\partial x} + 2\cos(z)\frac{\partial}{\partial y},\ 2\frac{\partial}{\partial z}$$

- the forms

$$\eta_0 := \frac{\cos(z)}{2}dx + \frac{\sin(z)}{2}dy,\ d\eta_0 = \frac{\sin(z)}{2}dx \wedge dz - \frac{\cos(z)}{2}dy \wedge dz$$

- the endomorphism $\varphi_0 \in \text{End}(TM_0)$ such that

$$\varphi_0(2\frac{\partial}{\partial z}) = -\zeta_0,\ \varphi_0(\zeta_0) = 2\frac{\partial}{\partial z},\quad \varphi_0(\xi_0) = 0.$$

Then $(M_0, \xi_0, \eta_0, F_0, h_0)$ is a metric contact manifold i.e. $F_0 = d\eta_0$ where F_0 is the 2–form associated to h_0 and φ_0. Clearly the sectional curvature of h_0 vanishes.

We consider the action $\phi : \mathbb{R} \times M_0 \to M_0$ of the Lie group $G := \mathbb{R}$ on M_0 given by

$$\phi(t, (x, y, z)) = \Big(x + 2t\cos(z), y + 2t\sin(z), z\Big).$$

The infinitesimal transformation determined by $1 \in \mathbb{R} \cong Lie(G)$ is just the vector field ξ_0. We observe that the action ϕ preserves all given structures on M_0 i.e. for each $t \in G$ we have that

$$(\phi_t)^*h_0 = h_0, (\phi_t)_*\xi_0 = \xi_0, (\phi_t)^*\eta_0 = \eta_0, (\phi_t)^*\varphi_0 = \varphi_0, (\phi_t)^*F_0 = F_0. \tag{6.1}$$

Moreover we have that $i_{\xi_0}F_0 = 0$. Since the action ϕ is free and proper, there exists a natural structure of a smooth manifold on the quotient $B_0 := M_0/G$ such that $\pi_0 : M_0 \to B_0$ is a principal fibre bundle with structure group G. Since ϕ acts by isometries, there exists a unique Riemannian metric g_0 on B_0 such that $\pi_0 : (M_0, h_0) \to (B_0, g_0)$ is a Riemannian fibration. From the invariance of the given structures on M_0 it follows that F_0 is a tensorial 2–form and that it is projectable to a form Ω_0 on B_0. Moreover, from equations (6.1) it follows that φ_0 is projectable to a unique almost complex structure J_0 on B_0. It is easy to observe that all these structures on B_0 are compatible. This means that J_0 is orthogonal with respect to g_0, and $\Omega_0(\overline{X},\overline{Y}) = g_0(\overline{X}, J_0(\overline{Y}))$ for each $\overline{X},\overline{Y}$ vector fields on B_0. Hence (B_0, g_0, J_0) is a Riemann surface and its Kähler form is just Ω_0. Moreover, we have that $\pi_0^*\Omega_0 = F_0$. We will use this in the next example.

Remark 6.1. We observe that (B_0, g_0) is isometric to a helicoid. In fact if (μ,ν) are global coordinates on a helicoid, then $\psi : \mathbb{R}^2 \to B_0$ given by $\psi(\mu,\nu) = [(-2\mu\sin(\nu), 2\mu\cos(\nu), \nu)]_G$ defines an isometry where the bracket $[\quad]_G$ denotes the equivalence class with respect to the action of G.

We extend the structures constructed in Example 6.1 to more general $f.pk$–structures. We keep in mind the structures and notation of the previous example.

Example 6.2. Let s be a positive natural number. Then we put

$$M' := \overbrace{M_0 \times \cdots \times M_0}^{s},\; B' := \overbrace{B_0 \times \cdots \times B_0}^{s},\; \pi' := \overbrace{\pi_0 \times \cdots \times \pi_0}^{s}.$$

The manifolds M' and B' carry natural product Riemannian metric structures h' and g' respectively. Then the projection $\pi' : (M', h') \to (B', g')$ is a Riemannian fibration. We denote by $P_k : M' \to M_0$ and $p_k : B' \to B_0$ the projections on the k–th component, for $k = 1,\ldots,s$. We put $B := B_0$ and $u : B \to B'$ the diagonal map i.e. for each $x \in B$ we have $u(x) = (x,\ldots,x) \in B'$. Since u is an immersion, we can apply the construction of Section 5. In such a way we obtain the pull–back bundle $\pi : M \to B$ of $\pi' : M' \to B'$ via the map $u : B \to B'$. We also have the immersion bundle map $U : M \to M'$ such that the following diagram

$$\begin{array}{ccc} M & \xrightarrow{U} & M' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ B & \xrightarrow{u} & B' \end{array}$$

commutes. It is clear that M may be described as

$$M = \Big\{(a, b_1, \ldots, b_s) \in B \times (M')^s \mid a = \pi'(b_1) = \cdots = \pi'(b_s)\Big\}$$

and $U(a, b_1, \ldots, b_s) = (b_1, \ldots, b_s)$. For any $(a, b_1, \ldots, b_s) \in M$ the tangent fiber to M is

$$\Big\{(v, X_1, \ldots, X_s) \in T_aB \times T_{b_1}M' \times \cdots \times T_{b_s}M' | v = d\pi'(X_l),\; l = 1, \ldots, s\Big\}.$$

We consider the constant $c = \frac{1}{s^2}$ and define the Riemannian metric tensors $h := \frac{1}{s^2}U^*h'$ and $g := \frac{1}{s^2}u^*g'$ on M and B. From the general construction of Section 5 it follows that $\pi : (M, h) \to (B, g)$ is a Riemannian fibration. It is easy to observe that $g = \frac{1}{s}g_0$. On the other hand

$$h((v, X_1, \ldots, X_s), (w, Y_1, \ldots, Y_s)) = \frac{1}{s^2}\Big(h'(X_1, Y_1) + \cdots + h'(X_s, Y_s)\Big).$$

We define the supplementary structures on M and B which derive from the almost $\mathcal{S}$–structure on M_0. We put

- $\eta^k := \frac{1}{s}(P_k \circ U)^*\eta_0$ for $k = 1, \dots, s$; these are 1–forms on M.
- $\varphi \in \mathrm{End}(TM)$ defined by
$$\varphi(v, (X_1, \dots, X_s)) := (J_0(v), (\varphi_0(X_1), \dots, \varphi_0(X_s))).$$
- the actions $\phi_k : \mathbb{R} \times M \to M$, $k = 1, \dots, s$, such that
$$\phi_k(t, (a, b_1, \dots, b_s)) := (a, b_1, \dots, \phi(st, b_k), \dots, b_s)$$
for each $t \in \mathbb{R}$ and $(a, b_1, \dots, b_s) \in M$; these actions define global vector fields $\xi_1, \dots, \xi_s$ which are vertical with respect to the fibration $\pi : M \to B$;
- $\Omega := \frac{1}{s}\Omega_0$ is a 2–form on B.

We observe that $dU(\xi_k) = (0, 0, \dots, s\xi_k, \dots, 0)$ for $k = 1, \dots, s$. Moreover $\xi_1, \dots, \xi_s$ are h–orthonormal and ${}^\flat\xi_k = \eta^k$ for $k = 1, \dots, s$. Then for each $k = 1, \dots, s$ we have

$$\begin{aligned} d\eta^k &= \frac{1}{s}d(P_k \circ U)^*\eta_0 = \frac{1}{s}(P_k \circ U)^* d\eta_0 \\ &= \frac{1}{s}(\overbrace{p_k \circ u}^{=\mathrm{Id}_B} \circ \pi)^*\Omega_0 = \pi^*\Omega. \end{aligned}$$

On the other hand for given vectors $(v, X_1, \dots, X_s)$, $(w, Y_1, \dots, Y_s)$ tangent to M at a point we have that

$$\begin{aligned} h((v, X_1, \dots, X_s), \varphi(w, Y_1, \dots, Y_s)) &= \frac{1}{s^2}\sum_{k=1}^{s} h_0(X_k, \varphi(Y_k)) \\ &= \sum_{k=1}^{s} d\eta_0(X_k, Y_k) \\ &= \frac{1}{s^2}\sum_{k=1}^{s} \pi_0^*\Omega_0(X_k, Y_k) \\ &= \frac{1}{s}\Omega_0(v, w) \\ &= \pi^*\Omega((v, X_1, \dots, X_s), (w, Y_1, \dots, Y_s)). \end{aligned}$$

Hence it follows that $d\eta^k = \pi^*\Omega = h(-, \varphi(-))$ for each $k = 1, \dots, s$ and $(M, \varphi, \xi_1, \dots, \xi_s, \eta^1, \dots, \eta^s, h)$ is an almost $\mathcal{S}$–manifold. Since the map $u : (B, g) \to (B', g')$ is totally geodesic and (M', h') is flat, then from Corollary 5.1 we get that (M, h) is also flat.

Since $\dim M = 2 + s$, this example shows that the condition $n > 1$ in Theorem 4.1 is necessary.

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