Equivariant group cohomology and Brauer group [*]

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Abstract

In this paper we prove that, for any Galois finite field extension F/K on which a separated group of operators Γ is acting, there is an isomorphism between the group of equivariant isomorphism classes of finite dimensional central simple K-algebras endowed with a Γ-action and containing F as an equivariant strictly maximal subfield and the second equivariant cohomology group of the Galois group of the extension.

Introduction

The 2nd cohomology group of a group has a long history in several branches of Mathematics. In Algebra, the two most classic topics on these cohomology groups deal with the study of group extensions and Brauer groups. Thus, since the 1930s, it has been well-known, first that for any group G and any G-module M there is a one-to-one correspondence (isomorphism) between elements of $H^2(G, M)$ and equivalence classes of group extensions $M \rightarrowtail E \twoheadrightarrow G$, which are compatible with the given action of G on M. Second, it has been long known that, for any finite Galois extension of fields F/K with Galois group G, there is a bijection, actually an isomorphism, between the 2nd cohomology group of G with coefficients in the G-module $F^\times$, the multiplicative group of F, and the group of isomorphism classes

of finite dimensional central simple K-algebras containing F as a strictly maximal subfield, that is, $H^2(G, F^\times) \cong Br(F/K)$.

This paper is concerned with the equivariant 2nd cohomology group of a group G on which a certain group of operators Γ is acting by automorphisms. We refer to [4] for the background about these equivariant group cohomology groups $H^n_\Gamma(G, M)$ whose origin can be found in the article by Whitehead [9]. It is known [4, 5] that, for any Γ-group G and any Γ-equivariant G-module M, the abelian group $H^2_\Gamma(G, M)$ classifies equivariant extensions of G by M, that is, extensions $M \rightarrowtail E \twoheadrightarrow G$ of G by the G-module M in which E is a Γ-group and the inclusion and projection maps are compatible with the Γ-action. The purpose of this paper is to prove Theorem 1 below, which states a suitable counterpart for an equivariant situation of the classic Brauer-Hasse-Noether result.

Theorem 1. *Let F/K be a Galois finite field extension with Galois group G and suppose that an action of a group Γ on F by K-automorphisms is given. Then, there is an isomorphism,*

$$H^2_\Gamma(G, F^\times) \cong Br_\Gamma(F/K),$$

between the 2nd equivariant cohomology group of the Γ-group G (where the Γ-action is by conjugation) with coefficients in the Γ-equivariant G-module $F^\times$ (with the natural actions of Γ and G) and the group of equivariant isomorphism classes of finite dimensional central simple K-algebras endowed with a Γ-action by K-automorphisms and containing F as a Γ-equivariant strictly maximal subfield.

This theorem is obtained in a slight more generalized form in Section 2. In the first section we recall briefly the facts required on cohomology of groups with operators.

1   Quick review of the equivariant group cohomology $H^2_\Gamma(G, A)$

Hereafter, Γ is a fixed group of operators. Let us recall that a Γ-group G means a group G enriched with a left Γ-action by automorphisms, $(\sigma, x) \mapsto {}^\sigma x$, and that a Γ-equivariant module over a Γ-group G is a Γ-module A, that is, an abelian Γ-group, endowed with a G-module structure such that ${}^\sigma({}^x a) = {}^{(\sigma x)}({}^\sigma a)$ for all $\sigma \in \Gamma$, $x \in G$ and $a \in A$ [4, Definition 2.1], which means that the G-action map $G \times A \to A$ is Γ-equivariant.

For any Γ-equivariant G-module A, a Γ-*equivariant derivation* (or *crossed Γ-homomorphism*) from G into A is a map $d: G \to A$ with the properties

i) $d(xy) = d(x) + {}^x d(y)$, $\quad x, y \in G$,

ii) $d({}^\sigma x) = {}^\sigma d(x)$, $\quad \sigma \in \Gamma$, $x \in G$.

The abelian groups of Γ-equivariant derivations from a Γ-group G into equivariant G-modules define a left-exact functor $\mathrm{Der}_\Gamma(G, -)$ from the category of equivariant G-modules to the category of abelian groups, whose right derived functors yield to the equivariant cohomology functors $H^*_\Gamma(G, -)$. More precisely, the cohomology groups of a Γ-group G with coefficients in an equivariant G-module A are [4, (3)]

$$H^n_\Gamma(G, A) = \left(R^{n-1} \mathrm{Der}_\Gamma(G, -) \right)(A), \quad n \geq 1.$$

We shall recall from [4, (7)] that the cohomology groups $H^2_\Gamma(G, A)$ can be described in terms of *equivariant normalized 2-cocycles*. Thus,

$$H^2_\Gamma(G, A) = Z^2_\Gamma(G, A)/B^2_\Gamma(G, A),$$

where $Z^2_\Gamma(G, A)$ denotes the set of all functions

$$\varphi : (G \times G) \cup (G \times \Gamma) \longrightarrow A,$$

which are normalized in the sense that $\varphi(x, 1_G) = 0 = \varphi(1_G, x) = \varphi(x, 1_\Gamma)$, for all $x \in G$, and satisfying the cocycle conditions

$$^x\varphi(y, z) + \varphi(x, yz) = \varphi(xy, z) + \varphi(x, y), \tag{1}$$

$$^\sigma\varphi(x, y) + \varphi(xy, \sigma) = \varphi(^\sigma x, {}^\sigma y) + {}^{(^\sigma x)}\varphi(y, \sigma) + \varphi(x, \sigma), \tag{2}$$

$$\varphi(x, \sigma\tau) = {}^\sigma\varphi(x, \tau) + \varphi(^\tau x, \sigma), \tag{3}$$

for all $x, y, z \in G$ and $\sigma, \tau \in \Gamma$. This set is an abelian group under the termwise addition, and the subgroup of equivariant 2-coboundaries $B^2_\Gamma(G, A) \subseteq Z^2_\Gamma(G, A)$ consists of all functions φ of the form $\varphi = \partial\psi$, where $\partial\psi$ is defined by

$$(\partial\psi)(x, y) = {}^x\psi(y) - \psi(xy) + \psi(x), \tag{4}$$

$$(\partial\psi)(x, \sigma) = {}^\sigma\psi(x) - \psi(^\sigma x), \tag{5}$$

from any function $\psi : G \to A$ with $\psi(1) = 0$.

Like that in the ordinary cohomology of groups, several explicit computations of equivariant cohomology groups can be done (see [4]). An elementary, although illustrative, example in our context is given below.

Let us consider the Galois extension $\mathbb{C}/\mathbb{R}$, whose Galois group is the cyclic group of order two $G(\mathbb{C}/\mathbb{R}) = \langle x \mid x^2 = 1 \rangle$, where $x : \mathbb{C} \to \mathbb{C}$ is the complex conjugation automorphism. Let $\Gamma = C_2 = \langle \sigma \mid \sigma^2 = 1 \rangle$ be a separated cyclic group of order two acting trivially on $\mathbb{C}$. In this case, an equivariant normalized 2-cochain φ of the trivial C_2-group $G(\mathbb{C}/\mathbb{R})$ consists of a pair of complexes $\left(\varphi(x, x), \varphi(x, \sigma)\right) \in \mathbb{C}^\times \times \mathbb{C}^\times$. Condition (1), in order for φ to be an equivariant 2-cocycle, merely says that $\varphi(f, f) \in \mathbb{R}^\times$. Condition (2) says that $\varphi(x, \sigma)$ has a module equal to 1, whereas condition (3) holds if and only if $\varphi(x, \sigma) = \pm 1$. Hence, $Z^2_{C_2}\left(G(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times\right) = \mathbb{R}^\times \times \{\pm 1\}$. Since it is plain to see that an equivariant 2-cocycle φ is a coboundary if and only if $\varphi(x, x) > 0$ and $\varphi(x, \sigma) = 1$, we conclude that

$$H^2_{C_2}\left(G(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times\right) \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Hence, by Theorem 1, $Br_{C_2}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, in contrast with the ordinary $Br(\mathbb{C}/\mathbb{R}) = Br(\mathbb{R}) \cong \mathbb{Z}_2$.

2 Proof of Theorem 1

We shall actually prove a more general result than the one stated in Theorem 1, namely Theorem 2 below.

Throughout this section, S/R is a Galois extension of commutative rings with finite Galois group $G \subseteq Aut_R(S)$ and, in addition, we assume that a separated group of operators Γ is acting on S by a given homomorphism $\Gamma \to G$.

Following Hattory [7], an S/R-*Azumaya algebra* is a pair (A, j) consisting of an Azumaya R-algebra (i.e. central separable) A and a maximal commutative embedding $j : S \hookrightarrow A$. When A is endowed with a Γ-action by R-automorphisms such that j is Γ-equivariant, then the pair (A, j) is called a Γ-*equivariant S/R-Azumaya algebra*. Two such algebras, (A, j) and (A', j'), are isomorphic whenever there exists a Γ-equivariant isomorphism of R-algebras $\phi : A \cong A'$ which respects the embeddings of S, that is, $\phi j = j'$. Let

$$\widehat{Br}_\Gamma(S/R)$$

denote the corresponding set of isomorphism classes.

Theorem 2. *Let S/R be a Galois extension of commutative rings with finite Galois group $G \subseteq \mathrm{Aut}_R(S)$ and suppose that an action of a group Γ on S by automorphisms in G is given. Then, there exists an exact sequence of abelian groups*

$$0 \longrightarrow H_\Gamma^2\big(G, U(S)\big) \overset{\Delta}{\longrightarrow} \widehat{Br}_\Gamma(S/R) \overset{\Theta}{\longrightarrow} \mathrm{Der}_\Gamma(G, Pic_S(S))\,. \tag{6}$$

Proof. This is inspired by the proof by Kanzaki [8] of the Chasse-Harrison-Rosenberg seven-term exact sequence, and it falls naturally into eight parts.

1. The abelian groups $H_\Gamma^2\big(G, U(S)\big)$. The Galois group G is a Γ-group by the diagonal action $(\sigma, f) \mapsto {}^\sigma f$, where $({}^\sigma f)(s) = {}^\sigma(f({}^{\sigma^{-1}}s))$ and, further, the group of units of S, $U(S)$, is a Γ-equivariant G-module with the natural actions of Γ and G on it. Thus, the equivariant cohomology group $H_\Gamma^2\big(G, U(S)\big)$ is defined.

2. Definition of Δ. The map Δ is defined by means of the *equivariant crossed product construction* $[\varphi] \mapsto [\Delta(\varphi; S, G), j]$ where $\Delta(\varphi; S, G) = \bigoplus_{f \in G} S \times \{f\}$ is the crossed product R-algebra, whose multiplication is given by $(s, f)(s', f') = (s\, f(s')\, \varphi(f, f'), ff')$ and on which Γ-acts by ${}^\sigma(s, f) = ({}^\sigma s\, \varphi(f, \sigma), {}^\sigma f)$, and $j : S \cong S \times \{1\}$ is the obvious isomorphism. Since $\varphi|_{G^2} : G^2 \to U(S)$ is an ordinary normalized 2-cocycle of G in $U(S)$, it is known [1, Theorem A.12] that $(\Delta(\varphi; S, G), j)$ is an S/R-Azumaya algebra. Moreover, Γ actually acts on $\Delta(\varphi; S, G)$ since:

$$
\begin{aligned}
{}^\sigma\big((s, f)(s', f')\big) &= {}^\sigma\big(s\, f(s')\, \varphi(f, f'), ff'\big) \\
&= \big({}^\sigma s\; {}^\sigma(f(s'))\; {}^\sigma\varphi(f, f')\, \varphi(ff', \sigma), {}^\sigma f\; {}^\sigma f'\big) \\
&\overset{(2)}{=} \big({}^\sigma s\; ({}^\sigma f)({}^\sigma s')\; \varphi(f, \sigma)\; ({}^\sigma f)(\varphi(f', \sigma))\, \varphi({}^\sigma f, {}^\sigma f'), {}^\sigma f\; {}^\sigma f'\big) \\
&= \big({}^\sigma s\, \varphi(f, \sigma), {}^\sigma f\big)({}^\sigma s'\, \varphi(f', \sigma), {}^\sigma f') = {}^\sigma(s, f)\; {}^\sigma(s', f')\,,
\end{aligned}
$$

$${}^{\sigma\tau}(s, f) = \big({}^{\sigma\tau}s\, \varphi(f, \sigma\tau), {}^{\sigma\tau}f\big) \overset{(3)}{=} \big({}^{\sigma\tau}s\; {}^\sigma\varphi(f, \tau)\, \varphi({}^\tau f, \sigma), {}^{\sigma\tau}f\big) = {}^\sigma({}^\tau(s, f)).$$

The equalities $j(^\sigma s) = (^\sigma s, 1) = {}^\sigma(s, 1) = {}^\sigma j(s)$ ensure that j is an equivariant embedding and therefore $(\Delta(\varphi; S, G), j)$ is an equivariant S/R-Azumaya algebra.

Note that if $\varphi, \varphi' \in Z^2_\Gamma(G, U(S))$ are two equivariant cohomologous 2-cocycles, by means of a map $\psi : G \to U(S)$, that is, $\varphi = \varphi' \cdot \partial \psi$, then the associated equivariant S/R-Azumaya algebras $(\Delta(\varphi; S, G), j)$ and $(\Delta(\varphi'; S, G), j)$ are made isomorphic by the map $\Psi : (s, f) \mapsto (s\,\psi(f), f)$. Indeed, it is known that Ψ is an isomorphism of algebras since, by (4), ψ establishes a cohomology between the ordinary 2-cocycles $\varphi|_{G^2}$ and $\varphi'|_{G^2}$. Furthermore, Ψ is Γ-equivariant since:

$$
\begin{aligned}
\Psi\big(^\sigma(s, f)\big) &= \Psi\big(^\sigma s\, \varphi(f, \sigma), {}^\sigma f\big) = \big(^\sigma s\, \varphi(f, \sigma)\, \psi(^\sigma f), {}^\sigma f\big) \\
&\overset{(5)}{=} \big(^\sigma s\, {}^\sigma\psi(f)\, \varphi'(f, \sigma), {}^\sigma f\big) = {}^\sigma\Psi(s, f).
\end{aligned}
$$

3. The abelian group $\mathrm{Der}_\Gamma\big(G, Pic_S(S)\big)$. Let us recall Bass' split exact sequence of groups [2, II, (5.4)]

$$
1 \to Pic_S(S) \xrightarrow{\;in\;} Pic_R(S) \underset{\delta}{\overset{h}{\rightleftarrows}} \mathrm{Aut}_R(S) \to 1 \;, \tag{7}
$$

in which h maps the class of any invertible (S, S)-bimodule $[P] \in Pic_R(S)$ to the automorphism $h([P]) = h_P \in \mathrm{Aut}_R(S)$ such that $h_P(s)\, p = p\, s$, for all $s \in S$ and $p \in P$, and $\delta : f \mapsto [S_f]$ where S_f denotes the invertible (S, S)-bimodule which is the underlying left S-module S with right S-action via f, that is, $s \cdot s' = s\, f(s')$.

Sequence (7) is clearly of $\mathrm{Aut}_R(S)$-groups, where $\mathrm{Aut}_R(S)$ acts on $Pic_R(S)$ via δ by conjugation, that is, $^f[P] = [S_f \otimes_S P \otimes_S S_{f^{-1}}] = [^f P]$, where for any invertible (S, S)-bimodule P, $^f P$ is the same R-module as P with $s \cdot p = f^{-1}(s)\, p$ and $p \cdot s = p\, f^{-1}(s)$, for $s \in S$ and $p \in P$. Hence, sequence (7) is also, by restriction, of G-groups as well as of Γ-groups via the given homomorphism $\Gamma \to G \subseteq \mathrm{Aut}_R(S)$. In particular, $Pic_S(S)$ is both a G-module and a Γ-module and it is actually a Γ-equivariant G-module. Thus, the abelian group of Γ-equivariant derivations $\mathrm{Der}_\Gamma\big(G, Pic_S(S)\big)$ is defined.

4. Definition of Θ . Let

$$
Pic_R(S) \longrightarrow Pic_S(S), \quad [P] \mapsto [\xi P], \tag{8}
$$

be the equivariant derivation that assigns to each class of an invertible (S, S)-bimodule P the class of the (S, S)-bimodule ξP, which is the same P as left S-module but with right S-action $p \cdot s = sp$, that is, $[\xi P] = [P \otimes_S S_{h_P^{-1}}] = [P][S_{h_P}]^{-1} = [P]\big(\delta h([P])\big)^{-1}$. Then, for any invertible (S, S)-bimodules P and Q, we have canonical isomorphisms of (S, S)-bimodules

$$
P \otimes_S \xi Q \cong {}^{h_P}(\xi Q) \otimes_S P, \quad p \otimes q \mapsto q \otimes p, \tag{9}
$$

$$
\xi(P \otimes_S Q) \cong \xi P \otimes_S {}^{h_P}(\xi Q), \quad p \otimes q \mapsto p \otimes q, \tag{10}
$$

$$
\xi(^f P) = {}^f(\xi P). \tag{11}
$$

If (A, j) is any equivariant S/R-Azumaya algebra and, for each $f \in G$, we let

$$
A_f = \{a \in A \mid f(s)\, a = a\, s \text{ for all } s \in S\}. \tag{12}
$$

Then, by [8, Proposition 3] or [3, Theorem I 2.15], we have that $A = \bigoplus_{f \in G} A_f$, $A_f A_g = A_{fg}$ for all $f, g \in G$ and $A_1 = j(S) \cong S$, so that A is a strongly graded R-algebra [6] whose 1-component is equivariantly isomorphic to S. Then, there are canonical isomorphisms of (S, S)-bimodules

$$A_f \otimes_S A_g \cong A_{fg}, \quad a_f \otimes a_g \mapsto a_f a_g, \tag{13}$$

and each component A_f is an invertible (S, S)-bimodule. Moreover, for any $\sigma \in \Gamma$ and $f \in G$, the map $a \mapsto {}^\sigma a$ establishes an (S, S)-bimodule isomorphism

$$
\begin{aligned}
{}^\sigma A_f &\cong \{ {}^\sigma a \mid f(s)\, a = a\, s, \ s \in S \} = \{ a \mid f({}^{\sigma^{-1}} s)\, {}^{\sigma^{-1}} a = {}^{\sigma^{-1}} a\, {}^{\sigma^{-1}} s, \ s \in S \} \tag{14} \\
&= \{ a \mid {}^\sigma \big(f({}^{\sigma^{-1}} s) \big)\, a = a\, s, \ s \in S \} = \{ a \mid ({}^\sigma f)(s)\, a = a\, s, \ s \in S \} = A_{\sigma f} .
\end{aligned}
$$

All in all, the map $\Theta : \widehat{Br}_\Gamma(S/R) \to \mathrm{Der}_\Gamma(G, Pic_S(S))$ is given by:

$$[A, j] \mapsto \Theta_{[A,j]}, \text{ where } \Theta_{[A,j]}(f) = [\xi A_f] \text{ for each } f \in G.$$

Since any isomorphism $\Psi : (A, j) \cong (A', j)$ of equivariant S/R-Azumaya algebras restricts, for any $f \in G$, to an (S, S)-bimodule isomorphism $A_f \cong A'_f$, $\Theta_{[A,j]} : G \to Pic_S(S)$ is a well-defined map which is actually an equivariant derivation because the isomorphisms of (S, S)-bimodules

$$
\begin{aligned}
\xi A_{fg} &\overset{(13)}{\cong} \xi(A_f \otimes_S A_g) \overset{(10)}{\cong} \xi A_f \otimes_S {}^{h_{A_f}}(\xi A_g) = \xi A_f \otimes_S {}^f(\xi A_g) , \\
\xi A_{\sigma f} &\overset{(14)}{\cong} \xi\big({}^\sigma A_f \big) \overset{(11)}{=} {}^\sigma(\xi A_f) .
\end{aligned}
$$

5. The abelian group structure of $\widehat{Br}_\Gamma(S/R)$. The operation on $\widehat{Br}_\Gamma(S/R)$ is given by

$$[A, j][A', j'] = [A \otimes A', j \otimes j'] ,$$

where $A \otimes A' = \bigoplus_{f \in G} \xi A_f \otimes_S A'_f$, in which, for each $f \in G$, A_f and A'_f are defined by (12) and ξA_f by (8), with multiplication given by the chain of isomorphisms:

$$
\begin{aligned}
\xi A_f \otimes_S A'_f \otimes_S \xi A_g \otimes_S A'_g &\overset{(9)}{\cong} \xi A_f \otimes_S {}^f(\xi A_g) \otimes_S A'_f \otimes_S A'_g \\
&\overset{(10)}{\cong} \xi(A_f \otimes_S A_g) \otimes_S A'_f \otimes_S A'_g \overset{(13)}{\cong} \xi A_{fg} \otimes_S A'_{fg} ,
\end{aligned}
$$

that is, $(a_f \otimes a'_f)(a_g \otimes a'_g) = a_f a_g \otimes a'_f a'_g$, and $j \otimes j'$ is the composition $S \cong S \otimes_S S \cong A_1 \otimes_S A'_1 = \xi A_1 \otimes_S A'_1$. Since, for any $f \in G$, $a \otimes a' \in \xi A_f \otimes_S A'_f$ and $s \in S$, the equality $(a \otimes a')s = f(s)(a \otimes a')$ holds, it follows from [8, Proposition 2] that $(A \otimes A', j \otimes j')$ is an S/R-Azumaya algebra that is actually Γ-equivariant with the Γ-action given by ${}^\sigma(a_f \otimes a'_f) = {}^\sigma a_f \otimes {}^\sigma a'_f$.

The class of the skew group algebra $\Delta(S, G) = \Delta(0; S, G) = \bigoplus_{f \in G} S \times \{f\}$, whose multiplication is given by $(s, f)(s', g) = (s\, f(s'), fg)$ and on which Γ-acts by ${}^\sigma(s, f) = ({}^\sigma s, {}^\sigma f)$, gives the identity element of $\widehat{Br}_\Gamma(S/R)$, and the inverse is given by

$$[A, j]^{-1} = [A^{op}, j] ,$$

where A^{op} is the opposite R-algebra of A with the same Γ-action (observe that, for any $f \in G$, $A_f^{op} = A_{f^{-1}}$, and then, $A \otimes A^{op} \cong \Delta(S, G)$ by the map $\sum_f a_f \otimes b_{f^{-1}} \mapsto \sum_f (a_f b_{f^{-1}}, f)$.

6. Δ and Θ are homomorphisms. Let $\varphi, \varphi' \in Z_\Gamma^2(G, U(S))$ be any two equivariant 2-cocycles. Then, we have a canonical isomorphism of (S, S)-bimodules $\Upsilon : \Delta(\varphi; S, G) \otimes \Delta(\varphi'; S, G) \cong \Delta(\varphi \varphi'; S, G)$ given by $\Upsilon\big((s, f) \otimes (s', f)\big) = (ss', f)$, for each $s, s' \in S$ and $f \in G$, which is in fact an equivariant isomorphism of algebras since:

$$
\begin{aligned}
\Upsilon\Big[\big((s, f) \otimes (s', f)\big)\big((t, g) \otimes (t', g)\big)\Big] &= \Upsilon\big((s\,f(t)\,\varphi(f, g), fg) \otimes (s'f(t')\,\varphi'(f, g), fg)\big) \\
&= \big(s\,f(t)\,\varphi(f, g)\,s'f(t')\,\varphi'(f, g), fg\big) = \big(s\,s'f(t)\,f(t')\,(\varphi \cdot \varphi')(f, g), fg\big) \\
&= (ss', f)(tt', g) = \Upsilon\big((s, f) \otimes (s', f)\big)\Upsilon\big((t, g) \otimes (t', g)\big),
\end{aligned}
$$

and

$$
\begin{aligned}
\Upsilon\Big[{}^\sigma\big((s, f) \otimes (s', f)\big)\Big] &= \Upsilon\big({}^\sigma(s, f) \otimes {}^\sigma(s', f)\big) \\
&= \Upsilon\big(({}^\sigma s\,\varphi(f, \sigma), {}^\sigma f) \otimes ({}^\sigma s'\,\varphi'(f, \sigma), {}^\sigma f)\big) = ({}^\sigma s\,\varphi(f, \sigma)\,{}^\sigma s'\,\varphi'(f, \sigma), {}^\sigma f) \\
&= ({}^\sigma s\,{}^\sigma s'\,(\varphi \cdot \varphi')(f, \sigma), {}^\sigma f) = {}^\sigma(ss', f) = {}^\sigma\Big[\Upsilon\big((s, f) \otimes (s', f)\big)\Big].
\end{aligned}
$$

Hence, Δ is a homomorphism.

Suppose now that $[A, j], [A', j] \in \widehat{Br}_\Gamma(S/R)$. Then, for any $f \in G$, we have isomorphisms of (S, S)-bimodules

$$
\xi(A \otimes A')_f \overset{def.}{=} \xi\big(\xi A_f \otimes_S A'_f\big) \overset{(10)}{\cong} \xi A_f \otimes_S \xi A'_f
$$

which shows that $\Theta_{[A,j][A',j]}(f) = \Theta_{[A,j]}(f)\Theta_{[A',j]}(f)$, whence Θ is a homomorphism.

To prove the remaining two parts, note that, for any ordinary normalized 2-cocycle $\varphi \in Z^2\big(G, U(S)\big)$,

$$
\Delta(\varphi; S, G)_f = S \times \{f\}, \tag{15}
$$

for any $f \in G$. Indeed, an element $z = \sum_g (s_g, g) \in \Delta(\varphi; S, G)_f$ if and only if the equality $f(s)s_g = g(s)s_g$ holds for all $g \in G$ and $s \in S$. Since S/R is a Galois extension, there exist $s_1, \cdots, s_n$ and $s'_1, \cdots, s'_n$ in S such that $\sum_i f(s_i)g(s'_i) = \delta_{f,g}$, and therefore, for $g \neq f$, $s_g = \sum_i f(s_i)f(s'_i)s_g = \sum_i f(s_i)g(s'_i)s_g = 0$ and (15) follows.

7. Δ is injective. Suppose that there is an equivariant isomorphism of S/R-Azumaya algebras $\Psi : \Delta(\varphi; S, G) \cong \Delta(S, G)$ where $\varphi \in Z_\Gamma^2(G, U(S))$. By (15), for any $f \in G$, Ψ restricts to an isomorphism of (S, S)-bimodules $\Psi : S \times \{f\} \cong S \times \{f\}$. Then, if we write $\Psi(1, f) = (\psi(f), f)$, we have a normalized map $\psi : G \to U(S)$ such that $\Psi(s, f) = (s\,\psi(f), f)$ for all $f \in G$ and $s \in S$. Because Ψ is a homomorphism of algebras, it follows that $\varphi(f, f')\,\psi(ff') = \psi(f)\,f(\psi(f'))$, for all $f, f' \in G$, whereas $\varphi(f, \sigma)\,\psi({}^\sigma f) = {}^\sigma\psi(f)$, for all $\sigma \in \Gamma$ and $f \in G$, also holds since Ψ is Γ-equivariant. Thus, $[\varphi] = [\partial \psi] = 0 \in H_\Gamma^2(G, U(S))$.

8. Exactness at $\widehat{Br}_{\Gamma}(S/R)$. It is clear that $\Theta\Delta$ is the zero map. Suppose now that (A, j) is any equivariant S/R-Azumaya algebra such that $\Theta[A, j] = 1$ is the trivial derivation, what means that, for any $f \in G$, there is an isomorphism of left S-modules $\Psi_f : S \cong A_f$. If we write $\Psi_f(1) = u_f$, then $A_f = Su_f = u_fS$, taking into account that $f(s)u_f = u_f s$ for all $s \in S$. Since $1 \in A_f A_{f^{-1}} = Su_f Su_{f^{-1}} = u_f Su_{f^{-1}}$, it follows that u_f is left invertible and, analogously, u_f is right invertible, that is, $u_f \in U(A)$ for all $f \in G$. Hence, each $f \in G$ extends to the inner automorphism of A of conjugation by u_f and therefore, by [1, Proposition A.13], we can assume, up to isomorphism, that (A, j) is a crossed product $(\Delta(\varphi; S, G), j)$ for an ordinary normalized two cocycle $\varphi : G^2 \to U(S)$.

Now, for any $\sigma \in \Gamma$ and $f \in G$, the map $(s, f) \mapsto {}^{\sigma}(s, f)$ gives an isomorphism

$$
{}^{\sigma}\Big(S \times \{f\}\Big) \overset{(15)}{=} {}^{\sigma}\Big(\Delta(\varphi; S, G)_f\Big) \overset{(14)}{\cong} \Delta(\varphi; S, G)_{\sigma f} \overset{(15)}{=} S \times \{{}^{\sigma}f\},
$$

so that we can write ${}^{\sigma}(1, f) = (\varphi(f, \sigma), {}^{\sigma}f)$ for a normalized map $\varphi : G \times \Gamma \to U(S)$. Observe that ${}^{\sigma}(s, f) = {}^{\sigma}\Big(j(s)(1, f)\Big) = {}^{\sigma}j(s){}^{\sigma}(1, f) = j({}^{\sigma}s)(\varphi(f, \sigma), {}^{\sigma}f) = \Big({}^{\sigma}s\,\varphi(f, \sigma), {}^{\sigma}f\Big)$ and then it is not hard to see that (2) follows from the equality ${}^{\sigma}\Big((1, f)(1, f')\Big) = {}^{\sigma}(1, f){}^{\sigma}(1, f')$, while (3) is a consequence of the equality ${}^{\sigma}({}^{\tau}(1, f)) = {}^{\sigma\tau}(1, f)$. Thus, $\varphi : G^2 \cup (G \times \Gamma) \to U(S)$ is an equivariant 2-cocycle whose cohomology class is mapped by Δ to $[A, j]$. Hence, $\mathrm{Ker}\Theta = Im\Delta$. ∎

When S is a field, but also when S is any commutative ring with $Pic_S(S) = 0$ (a principal ideal domain, a local ring,...), then $\widehat{Br}_{\Gamma}(S/R) = Br_{\Gamma}(S/R)$ and Theorem 1 is an immediate consequence of Theorem 2.

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