Polarities of Symplectic Quadrangles

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Abstract

We give a simple proof of the known fact that the symplectic quadrangle is self-dual if and only if the ground field is perfect of characteristic 2, and that a polarity exists exactly if there is a root of the Frobenius automorphism. Moreover, we determine all polarities, characterize the conjugacy classes of polarities, and use the results to give a simple proof that the centralizer of any polarity acts two-transitively on the ovoid of absolute points. The proofs use elementary calculations in solvable subgroups of the symplectic group.

In this note, we give an elementary proof of the fact that the symplectic quadrangle $\mathrm{W}(F)$ is self-dual if, and only if, the field F has characteristic 2 and the Frobenius endomorphism $\phi : F \to F : a \mapsto a^2$ is surjective. Moreover, we show that the existence of a polarity is equivalent to the existence of a square root σ of ϕ. In a deeper context, these results have been obtained by J. Tits [10] who also studies the ovoids corresponding to these polarities; cf. 6.3 below. The proof in [11] 7.3.2 uses coordinatization via quadratic quaternary rings. One can also view the symplectic quadrangle as the Bruhat–Tits building corresponding to PSp_4F, a simple group of Lie type $B_2 = C_2$. In that context, the dualities treated here occur as group automorphisms corresponding to graph automorphism interchanging long and short root subgroups, see [1] 12.3.3. Our treatment avoids the use of the simple (and thus complicated) group and concentrates on elation and translation groups (in fact, the unipotent radicals of maximal parabolic subgroups): these groups are nilpotent, and actually abelian in the interesting case where the ground field has characteristic 2.

Finally, we give a system of representatives for the conjugacy classes of polarities (under the action of the symplectic group, see 5.4 below), and show that the centralizer of a polarity acts 2-transitively on the corresponding ovoid (cf. 6.3).

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1 The Quadrangles, and the Groups.

Let F be a commutative field. Writing

$$\boldsymbol{i} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \boldsymbol{j} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\text{and} \quad S := \begin{pmatrix} 0 & \boldsymbol{j} \\ -\boldsymbol{j} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{i} & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

we obtain an alternating form $\langle v|w\rangle = vSw'$ on the space F^4 of row vectors, where w' denotes the transpose of $w \in F^4$. This form describes a symplectic polarity $\perp$ of the projective space $\mathbb{P}F^4$ via $U^\perp := \left\{v \in V \mid \forall u \in U : \langle u|v\rangle = 0\right\}$. The corresponding polar space, consisting of all nontrivial subspaces U of F^4 satisfying $U \le U^\perp$, is a generalized quadrangle, known as the symplectic quadrangle $\mathrm{W}(F)$ over F. We will write P and L for the set of all points and lines of $\mathrm{W}(F)$, respectively. For $(x,y) \in P \times L$, we have the line pencil $L_x := \left\{z \in L \mid x \le z\right\}$ and the point row $P_y := \left\{z \in P \mid z \le y\right\}$.

Clearly, the symplectic group $\mathrm{Sp}_4F := \left\{A \in F^{4\times 4} \mid ASA' = S\right\}$ acts by automorphisms on $\mathrm{W}(F)$, inducing the group $\mathrm{PSp}_4F = \mathrm{Sp}_4F/Z$ where $Z = \{\mathbf{1}, -\mathbf{1}\}$. Moreover, every automorphism of $\mathrm{W}(F)$ extends to an automorphism of the ambient projective space, see [11] 4.6.1. The full group of all automorphisms of $\mathrm{W}(F)$ is the product of PSp_4F with the multiplicative group $F^\times$ of F and the group of all automorphisms of F, cf. [2] p.83f or [3] 2.1. To be precise:

Every automorphism of $\mathrm{W}(F)$ is induced by a semilinear map

$$(v_0, v_1, v_2, v_3) \mapsto (v_0^\alpha, v_1^\alpha, v_2^\alpha, v_3^\alpha)A\,,$$

where $\alpha \in \mathrm{Aut}(F)$, and $A \in F^{4\times 4}$ satisfies $ASA' = aS$ for some $a \in F \setminus \{0\}$.

1.1 Definitions. Let $p := F(1,0,0,0)$, and let $\ell := p + F(0,1,0,0)$. Then $\{p, \ell\}$ is a flag in $\mathrm{W}(F)$. We consider two special subgroups of $\mathrm{Aut}(\mathrm{W}(F))$, namely

$$\begin{aligned} \Gamma_p &:= \left\{\gamma \in \mathrm{Aut}(\mathrm{W}(F)) \mid \forall z \in L_p : z^\gamma = z\right\} \\ \text{and, dually,} \quad \Gamma_\ell &:= \left\{\gamma \in \mathrm{Aut}(\mathrm{W}(F)) \mid \forall z \in P_\ell : z^\gamma = z\right\}. \end{aligned}$$

In fact, these groups are contained in PSp_4F; easy computations yield:

1.2 Lemma. *1. The elements of Γ_p are represented by the elements of the subgroup of Sp_4F consisting of all block matrices of the form*

$$\begin{pmatrix} c & 0 & 0 \\ -c\boldsymbol{i}v' & \mathbf{1} & 0 \\ z & v & c^{-1} \end{pmatrix}, \qquad \textit{where } c \in F^\times, v \in F^2, z \in F.$$

2. *The elements of Γ_ℓ are represented by the elements of the subgroup of Sp_4F consisting of all block matrices of the form*

$$\begin{pmatrix} c\mathbf{1} & 0 \\ X & c^{-1}\mathbf{1} \end{pmatrix}, \qquad \textit{where } c \in F^\times, X \in F^{2\times 2} \textit{ such that } (X\boldsymbol{j})' = X\boldsymbol{j}.$$

The condition $(X\boldsymbol{j})' = X\boldsymbol{j}$ means $X = \begin{pmatrix} x & u \\ z & x \end{pmatrix}$ with $x, u, z \in F$.

1.3 Definition. We will write $\langle c; X\rangle$ and $[c; z, v]$ for the elements of PSp_4F represented by the block matrices

$$\begin{pmatrix} c\mathbf{1} & 0 \\ X & c^{-1}\mathbf{1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c & 0 & 0 \\ -c\boldsymbol{i}v' & \mathbf{1} & 0 \\ z & v & c^{-1} \end{pmatrix}, \quad \text{respectively.}$$

While the products and inverses

$$\langle c; X\rangle \cdot \langle d; Y\rangle = \left\langle cd; Xd + c^{-1}Y \right\rangle \qquad \text{and} \qquad \langle c; X\rangle^{-1} = \left\langle c^{-1}; -X \right\rangle$$

are simple to compute, the formulae are more complicated for elements of Γ_p:

$$\begin{aligned} [c; x, v] \cdot [d; y, w] &= \left[cd; xd - dv\boldsymbol{i}w' + c^{-1}y, v + c^{-1}w\right] \\ \text{and} \qquad [c; x, v]^{-1} &= \left[c^{-1}; -x, -cv\right] \quad . \end{aligned}$$

The actions by conjugation will be needed:

$$\begin{aligned} \langle c; X\rangle^{-1} \langle 1; Y\rangle \langle c; X\rangle &= \left\langle 1; c^2Y \right\rangle \\ [c; x, v]^{-1} [1; y, w] [c; x, v] &= \left[1; c^2y + 2c^2v\boldsymbol{i}w', cw\right] . \end{aligned}$$

If the field F has characteristic 2, the latter formula simplifies to

$$[c; x, v]^{-1} [1; y, w] [c; x, v] = \left[1; c^2y, cw\right] .$$

1.4 Definition. We need another pair of subgroups:

$$\begin{aligned} \mathrm{E}_p &:= \left\{[1; x, v] \,\middle|\, v \in F^2, x \in F\right\} , \\ \mathrm{E}_\ell &:= \left\{\langle 1; X\rangle \,\middle|\, X \in F^{2\times 2}, (X\boldsymbol{j})' = X\boldsymbol{j}\right\} . \end{aligned}$$

1.5 Lemma. *Let $x \in \{p, \ell\}$.*

1. *The commutator subgroup of Γ_x is E_x, unless F is a field with 2 or 3 elements.*

2. *For $|F| = 2$, we have that $\Gamma_x = \mathrm{E}_x$ is (elementary) abelian of order 2^3.*

3. *For $|F| = 3$, the group $\Gamma_\ell = \mathrm{E}_\ell$ is (elementary) abelian of order 3^3, while E_p is not abelian, and equals the commutator group of Γ_p.*

In any case, every isomorphism from Γ_ℓ onto Γ_p maps E_ℓ onto E_p.

Proof: Clearly, the subgroup E_x is a normal subgroup with abelian quotient Γ_x/E_x. Thus it suffices to show that E_x consists of commutators. This is easily done by computing $[c;0,0]^{-1}\cdot[1;z,v]^{-1}\cdot[c;0,0]\cdot[1;z,v]=[1;(1-c^2)z,(1-c)v]$ and $\langle c;0\rangle^{-1}\cdot\langle 1;X\rangle^{-1}\cdot\langle c;0\rangle\cdot\langle 1;X\rangle=\langle 1;(1-c^2)X\rangle$. Existence of an element $c\in F\smallsetminus\{0,1,-1\}$ secures the equality, as claimed in 1.

For the case where $|F|=3$, our computation of commutators shows that the commutator subgroup of Γ_p contains the set $\left\{[1;0,v]\mid v\in F^2\right\}$, which generates E_p. The rest is obvious. ∎

Vector Space Structures.

We assume that F has characteristic 2. For $x\in\{p,\ell\}$, the group E_x is then elementary abelian (that is, a vector space over $\mathbb{F}_2$), and Γ_x acts via conjugation on E_x, inducing group automorphisms. We use these actions to turn E_x into a vector space over F. First of all, we note that the centralizer of E_x in Γ_x is just E_x, and the group $\Gamma_x/\mathrm{E}_x\cong F^\times$ acts effectively. Moreover, the subgroups $\Phi_p:=\left\{[c;0,0]\mid c\in F^\times\right\}$ and $\Phi_\ell=\left\{\langle c;0\rangle\mid c\in F^\times\right\}$ form natural sets of representatives for Γ_x/E_x.

1.6 Definition. For $[1;z,v]\in\mathrm{E}_p$, $\langle 1;X\rangle\in\mathrm{E}_\ell$ and $a\in F^\times$, we put

$$\begin{aligned} a*\langle 1;X\rangle &:= \langle a;0\rangle^{-1}\langle 1;X\rangle\langle a;0\rangle &&= \left\langle 1;a^2X\right\rangle\\ a\odot[1;z,v] &:= [a;0,0]^{-1}[1;z,v][a;0,0] &&= \left[1;a^2z,av\right]\end{aligned}$$

Moreover, we set $0*\langle 1;X\rangle:=\langle 1;0\rangle$ and $0\odot[1;z,v]:=[1;0,0]$.

Simple computations, using the fact that the Frobenius endomorphism $\phi=(f\mapsto f^2)$ is an isomorphism from F onto $F^\phi:=\left\{f^2\mid f\in F\right\}$, suffice to check that, with multiplication by scalars defined as in 1.6, the groups E_p and E_ℓ are vector spaces over F.

The subspaces $\mathrm{K}_p:=\left\{[1;z,0]\in\mathrm{E}_p\mid z\in F\right\}$ and

$$\mathrm{K}_\ell:=\left\{\left\langle 1;\begin{pmatrix} b & 0\\ 0 & b\end{pmatrix}\right\rangle\in\mathrm{E}_\ell\;\middle|\; b\in F\right\}$$

are defined *geometrically*; in fact, standard arguments from linear algebra show:

1.7 Lemma.

1. *The subgroup $\mathrm{K}_p\le\mathrm{E}_p$ is the intersection of all stabilizers of points on lines in L_p.*
2. *Dually, the subgroup $\mathrm{K}_\ell\le\mathrm{E}_\ell$ is the intersection of all stabilizers of lines in L meeting ℓ.*
3. *The respective dimensions are*

$$\begin{aligned}\dim_F\mathrm{E}_p &= 2+\dim_{F^\phi}F, & \dim_F(\mathrm{E}_p/\mathrm{K}_p) &= 2,\\ \dim_F\mathrm{E}_\ell &= 3\dim_{F^\phi}F, & \dim_F(\mathrm{E}_\ell/\mathrm{K}_\ell) &= 2\dim_{F^\phi}F\,.\end{aligned}$$

∎

2 Dualities and Elation Groups.

Now let δ be a duality of $\mathrm{W}(F)$. Replacing δ by $\delta\alpha$ for a suitable element $\alpha \in \mathrm{PSp}_4F$, we may assume $p^\delta = \ell$ and $\ell^\delta = p$ (recall that PSp_4F acts transitively on the set of flags in $\mathrm{W}(F)$). Since Γ_p and Γ_ℓ are defined geometrically, we obtain ${\Gamma_x}^\delta := \delta^{-1}\Gamma_x\delta = \Gamma_{x^\delta}$ for $x \in \{p, \ell\}$. We also find ${\mathrm{E}_x}^\delta = \mathrm{E}_{x^\delta}$ because E_x is characteristic in Γ_x, see 1.5. In particular, the group E_p is abelian (since E_ℓ is abelian, in any case). This implies that F has characteristic 2.

Conjugation with δ induces a group isomorphism from E_ℓ onto E_p, and a multiplicative bijection $\sigma\colon F \to F$ such that $\mathrm{E}_\ell \cdot \langle a; 0\rangle^\delta \in \mathrm{E}_p \cdot [a^\sigma; 0, 0]$.

2.1 Lemma. *For each $a \in F$ and every $v \in \mathrm{E}_\ell$, we have $(a * v)^\delta = a^\sigma \odot v^\delta$.* ∎

Since conjugation by $\langle a; 0\rangle$ is translated into conjugation by $[a^\sigma; 0, 0]$, the isomorphism from E_ℓ onto E_p is in fact a σ-semilinear bijection, and σ is an automorphism of the field F.

2.2 Theorem. *Every duality δ of a symplectic quadrangle $\mathrm{W}(F)$ with a fixed flag gives rise to a semilinear bijection from E_p onto E_ℓ.* ∎

The intersection $\mathrm{E}_\ell \cap \mathrm{E}_p$ is invariant under δ, while K_ℓ and K_p are interchanged. Comparing dimensions of the spaces $\mathrm{E}_p/\mathrm{K}_p$ and $\mathrm{E}_\ell/\mathrm{K}_\ell$ (see 1.7), we obtain:

2.3 Corollary. *If $\mathrm{W}(F)$ admits a duality then F is a perfect field, of characteristic 2.* ∎

Conversely, we will construct a duality using the Frobenius automorphism, whenever the field F is perfect; see 4.7.

3 Polarities.

Assume now that δ is a polarity, that is, a duality with $\delta^2 = \mathbf{1}$. Our reduction at the beginning of Section 2 is, in general, not possible without destroying the property $\delta^2 = \mathbf{1}$. Therefore, we need the following, cf. [10] 3.2.

3.1 Lemma. *Every polarity of a generalized quadrangle fixes two flags at maximal distance (that is, flags such that the lines do not have a common point and the points do not have a joining line in the generalized quadrangle).*

Proof: Let δ be a polarity, and let q be a point. If q^δ contains q then $\{q, q^\delta\}$ is a fixed flag. In the remaining case, there is a unique flag $\{r, m\}$ such that $\{q, m\}$ and $\{r, q^\delta\}$ are flags. Now $\{m^\delta, r^\delta\}$ is a flag such that $\{q, r^\delta\} = \{q^{\delta\delta}, r^\delta\}$ and $\{m^\delta, q^\delta\}$ are flags. Uniqueness yields $\{m^\delta, r^\delta\} = \{r, m\}$.

If $\{r, m\}$ is a fixed flag under the polarity δ, choose any point x on m and any line $k \neq m$ through x. As before, we find that the unique flag connecting k with k^δ is fixed. ∎

Without loss, we may thus assume $p^\delta = \ell$ and $\ell^\delta = p$ again, and use the results and the notation from Section 2. We are going to have a closer look at the semilinear bijection from E_ℓ onto E_p. In order to take advantage of our assumption $\delta^2 = \mathbf{1}$, we consider the δ-invariant intersection $\mathrm{E}_\ell \cap \mathrm{E}_p$.

The field automorphism $\sigma\colon F \to F$ has a very special property; it is a square root of the Frobenius endomorphism ϕ:

3.2 Theorem. *If* $\mathrm{W}(F)$ *admits a polarity then there exists an automorphism* σ *of* F *such that* $\sigma^2 = \phi$.

Proof: For the sake of readability, we write the elements of $\mathrm{E}_\ell \cap \mathrm{E}_p$ as

$$(z,x) := [1; z, (x,0)] = \left\langle 1; \begin{pmatrix} x & 0 \\ z & x \end{pmatrix} \right\rangle .$$

Since δ interchanges K_ℓ with K_p, there are $s,t \in F$ such that $(0,1)^\delta = (t,0)$ and $(1,0)^\delta = (0,s)$. Using 2.1, we compute

$$\begin{aligned} (a^2,0) &= (a^2,0)^{\delta^2} = (a*(1,0))^{\delta^2} = (a^\sigma \odot (0,s))^\delta = (0,a^\sigma s)^\delta \\ &= \left((a^\sigma s)^{\phi^{-1}} * (0,1)\right)^\delta = \left((a^\sigma s)^{\phi^{-1}}\right)^\sigma \odot (t,0) = ((a^\sigma s)^\sigma t, 0) \ , \end{aligned}$$

note that ϕ commutes with every endomorphism of F. This gives $a^{\sigma^2} s^\sigma t = a^\phi$ for each $a \in F$. Specializing $a = 1$ yields $s^\sigma t = 1$, and we obtain $\sigma^2 = \phi$ as claimed. ∎

Section 4 of [10] discusses existence and uniqueness of square roots of the Frobenius endomorphism. We just note that an algebraic extension A of $\mathbb{F}_2$ possesses a (unique) square root of ϕ if, and only if, the field $\mathbb{F}_4$ is not contained in A. In particular, the order of A has to be an odd power 2^{2m+1} of 2 if A is finite. Perfect hulls of purely transcendental extensions of such fields, with a transcendency basis of even (or infinite) order, allow (lots of conjugacy classes of) square roots of ϕ.

4 Constructing Dualities and Polarities.

Let $\mathcal{Q}$ be a generalized quadrangle, and let s be a point of $\mathcal{Q}$. Then $\mathcal{Q}$ is, up to isomorphism, determined by the *point-affine derivation* having $P \setminus \bigcup_{k\in L_s} P_k$ as point set and $L \setminus L_s$ as line set, compare [7]. For the sake of completeness, we describe the extension of an isomorphism between affine derivations explicitly for the case at hand, in 4.6 below; in fact, we just give explicitly the steps of Kantor's description of an elation generalized quadrangle [5] using a Kantor family, for the special cases at hand.

For $\mathcal{Q} = \mathrm{W}(F)$ and $s = p$, the affine derivation can quite easily be described in terms of the group E_p: we identify $P \setminus \bigcup_{k\in L_p} P_k$ with E_p (thanks to a sharply transitive action). The line set is obtained as $\bigcup_{\mathrm{X}\in S} \mathrm{E}_p/\mathrm{X}$, where S is the set of stabilizers of lines through some fixed point q in $P \setminus \bigcup_{k\in L_s} P_k$. We can describe S explicitly:

4.1 Lemma. *Let F be any field, and consider the point $q := F(0,0,0,1)$ of the symplectic quadrangle* $\mathrm{W}(F)$.

1. *Each line through q in $\mathrm{W}(F)$ is of the form*

$$\ell_{(a,b)} := F(0,0,0,1) + F(0,a,b,0),$$

where (a,b) is a nonzero element of F^2.

2. *The stabilizer of $\ell_{(a,b)}$ in E_p is $S_{(a,b)} := \Big\{[1;0,f(a,b)] \mid f \in F\Big\}$.*

3. *The stabilizer of the unique point $d \in \ell_{(a,b)} \cap p^\perp$ is $\mathrm{K}_p S_{(a,b)}$. This stabilizer acts transitively on $L_d \smallsetminus \{p+d\}$.*

Note that $S_{(a,b)} = F \odot [1;0,(a,b)]$ is a subspace of E_p if F has characteristic 2.

Proof: It suffices to observe that each line through q in $\mathrm{W}(F)$ is of the form $q+x$ for some one-dimensional F-subspace $x \le q^\perp$, and that $q^\perp = F(0,0,0,1)+F(0,1,0,0)+F(0,0,1,0)$. The rest of the assertion is verified by simple computations. ∎

We return to the characteristic 2 case now. Although, at a spurious glimpse, the set $S = \Big\{S_v \mid v \in F^2 \smallsetminus \{(0,0)\}\Big\}$ looks like a line in the projective plane $\mathbb{P}\mathrm{E}_p$, it is a nondegenerate quadric if F is a perfect field. In order to see this, we translate the situation into the more familiar vector space E_ℓ.

4.2 Definitions. We define maps δ and π from E_p to E_ℓ as follows.
For $[1;z,(x,y)] \in \mathrm{E}_p$, we put

$$[1;z,(x,y)]^\delta := \left\langle 1; \begin{pmatrix} z+xy & y^2 \\ x^2 & z+xy \end{pmatrix} \right\rangle .$$

If a square root σ of the Frobenius endomorphism ϕ exists in $\mathrm{Aut}(F)$, we put

$$[1;z,(x,y)]^\pi := \left\langle 1; \begin{pmatrix} (z+xy)^{\sigma^{-1}} & y^\sigma \\ x^\sigma & (z+xy)^{\sigma^{-1}} \end{pmatrix} \right\rangle .$$

Note that π is obtained by modifying the images of elements of E_p under δ by an application of σ^{-1} to every entry of the matrix describing an element of E_ℓ.

Straightforward calculations, using $(f^{\sigma^{-1}})^2 = f^{\sigma^{-1}\sigma^2} = f^\sigma$, yield:

4.3 Lemma. *Let F be a field of characteristic* 2.

1. *The map δ is an injective linear map from E_p to E_ℓ.*

2. *The map δ is surjective if, and only if, the field F is perfect.*

3. *If F is perfect and σ is a square root of the Frobenius automorphism, then π is a σ^{-1}-semilinear bijection from E_p onto E_ℓ.*

4. *If F is perfect then the linear bijection δ maps S onto the quadric*

$$R := \left\{ F * \left\langle 1; \begin{pmatrix} z & x \\ y & z \end{pmatrix} \right\rangle \,\middle|\, x,y,z \in F, xy = z^2 \right\} .$$

5. *If σ is a square root of ϕ then π also maps S onto R.* ∎

We are going to translate the description of the affine derivation

$$\left(P \setminus \bigcup_{k\in L_p} P_k, L \setminus L_p\right) \cong \left(\mathrm{E}_p, \bigcup_{\mathrm{X}\in S} \mathrm{E}_p/\mathrm{X}\right)$$

via δ or via π, respectively. To this end, we need a sharply transitive action of E_ℓ, and certain point stabilizers in E_ℓ.

As before, we write $\ell := F(1,0,0,0) + F(0,1,0,0)$. Let k be a line of the projective space $\mathbb{P}F^4$. If k does not intersect ℓ, we have $F^4 = \ell + k$ and $k = F(r,s,1,0) + F(t,u,0,1)$ for suitable elements $r,s,t,u \in F$. The line k belongs to $\mathrm{W}(F)$ if $r = u$. Now a straightforward computation shows that E_ℓ acts sharply transitively on the set of lines of $\mathrm{W}(F)$ that do not intersect ℓ. Thus we are in a situation dual to the one considered above: the symplectic quadrangle is also determined by its *line-affine derivation* consisting of the set of lines that do not meet ℓ and the set of points that do not lie on ℓ. We choose the line

$$m := \ell_{(0,1)} = F(0,0,1,0) + F(0,0,0,1)\,,$$

identify its orbit with E_ℓ, and identify the point set $P \setminus P_\ell$ with $\bigcup_{\Xi\in R} \mathrm{E}_\ell/\Xi$, where R is the set of stabilizers of points on m. Again, easy computations yield the required stabilizers:

4.4 Proposition. *Let F be a field. The stabilizer of $q := F(0,0,0,1)$ in E_ℓ is*

$$R_{(0,1)} = \left\{ \left\langle 1; \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \right\rangle \,\middle|\, x \in F \right\},$$

and the stabilizer of $F(0,0,1,u)$ is

$$R_{(1,u)} = \left\{ \left\langle 1; \begin{pmatrix} -uy & u^2y \\ y & -uy \end{pmatrix} \right\rangle \,\middle|\, y \in F \right\}.$$

If F is a perfect field of characteristic 2, this means that the set of all these stabilizers is just the quadric R obtained as the image under the (semi-)linear bijections δ and π in 4.3.

If F has characteristic 2, we also note that $\mathrm{K}_\ell R_{(u,v)}$ is the stabilizer of the unique line j joining $F(0,0,u,v)$ to a point on ℓ, and that this group acts transitively on $P_j \setminus P_\ell$. ∎

4.5 Remark. The result obtained in 4.4 implies that, irrespective of the characteristic of F, the dual of $\mathrm{W}(F)$ is a translation quadrangle of Tits type (corresponding to the ovoid R).

In fact, the dual of $\mathrm{W}(F)$ is an orthogonal quadrangle, because the ovoid is a quadric.

4.6 Lemma. *Let $\gamma : \mathrm{E}_p \to \mathrm{E}_\ell$ be a group isomorphism mapping K_p to K_ℓ and S onto R. Then γ induces an isomorphism of affine derivations that extends uniquely to an isomorphism from the symplectic quadrangle onto its dual.*

Proof: It is quite obvious that such a group isomorphism $\gamma : \mathrm{E}_p \to \mathrm{E}_\ell$ yields an isomorphism from $(\mathrm{E}_p, \bigcup_{\mathrm{X}\in S} \mathrm{E}_p/\mathrm{X})$ onto $(\mathrm{E}_\ell, \bigcup_{\Xi\in R} \mathrm{E}_\ell/\Xi)$; see [8] and [9] for a systematic treatment. This isomorphism translates to an isomorphism $\hat\gamma$ from the affine derivation of $\mathrm{W}(F)$ at p onto the affine derivation of the dual quadrangle at ℓ.

In order to extend $\hat\gamma$ to the points on lines through p, note that such a point $d \neq p$ is determined by $L_d \setminus \{p+d\}$. Let d_0 be the unique point on $p+d$ that is joined to q, and choose v such that S_v is the stabilizer of d_0+q. Then $L_{d_0} \setminus \{p+d_0\}$ is the orbit $(q+d_0)^{\mathrm{K}_p}$. There exists $\alpha \in \mathrm{E}_p$ with $d_0^\alpha = d$, and $L_d \setminus \{p+d\} = (q+d_0)^{\mathrm{K}_p\alpha}$. Translating this into the group theoretic description, we obtain that d corresponds to $S_v\mathrm{K}_p\alpha$, which is mapped to a set of the form $R_w\mathrm{K}_\ell\alpha^\gamma$ by our group isomorphism. This corresponds to the set of points on a line $d^{\hat\gamma} = d_0^{\hat\gamma\alpha^\gamma}$ meeting ℓ.

The line $p+d = p+d_0$ now consists of p and the orbit of d_0 under E_p, and the images of the points in this orbit are the lines through the intersection point of ℓ with the line $d^{\hat\gamma}$: note that $\hat\gamma$ translates the action of E_p into that of E_ℓ.

Finally, we put $p^{\hat\gamma} := \ell$. It is now easy to check that the extension $\hat\gamma$ preserves incidence. ■

In fact, every isomorphism of affine quadrangles extends to an isomorphism of the completions, see [7]. The proof uses the description of the sets $L_d \setminus \{p+d\}$ as maximal sets of lines at distance 6 in the affine quadrangle.

We define maps $\hat\delta$ and $\hat\pi$ (if a suitable σ exists) from the orbit of q under E_p onto the orbit of m under E_ℓ via $(q^\varepsilon)^{\hat\delta} := q^{(\varepsilon^\delta)}$ and $(q^\varepsilon)^{\hat\pi} := q^{(\varepsilon^\pi)}$, respectively, for each $\varepsilon \in \mathrm{E}_p$. Since δ and π map the set S of line stabilizers in E_p onto the set R of point stabilizers in E_ℓ, the maps δ and π induce isomorphisms of incidence structures (cf. [8]):

4.7 Theorem. *Let F be a perfect field of characteristic 2, and let E_p, S, E_ℓ and R be as above.*

1. *The affine quadrangles $(\mathrm{E}_p, \bigcup_{\mathrm{X}\in S} \mathrm{E}_p/\mathrm{X})$ and $(\mathrm{E}_\ell, \bigcup_{\Xi\in R} \mathrm{E}_\ell/\Xi)$ are isomorphic, an isomorphism being induced by δ. This isomorphism extends to a duality of $\mathrm{W}(F)$.*

2. *If a square root σ of the Frobenius automorphism ϕ exists, the map π induces an isomorphism of affine quadrangles that extends to a polarity of $\mathrm{W}(F)$.*

Proof: It remains to show that the duality induced by π is an involution. Because an isomorphism between generalized quadrangles is determined by its restriction to any affine derivation, it suffices to consider the points in the orbit of q under E_p. The point $q^{[1;z,(x,y)]}$ is mapped by $\hat\pi$ to the image of m under

$$[1;z,(x,y)]^\pi = \left\langle 1; \begin{pmatrix} (z+xy)^{\sigma^{-1}} & y^\sigma \\ x^\sigma & (z+xy)^{\sigma^{-1}} \end{pmatrix} \right\rangle ,$$

that is, to the line

$$\begin{aligned} & F\left((z+xy)^{\sigma^{-1}}, y^\sigma, 1, 0\right) &+& \; F\left(x^\sigma, (z+xy)^{\sigma^{-1}}, 0, 1\right) \\ =\; & F\left((z+xy)^{\sigma^{-1}} + x^\sigma, y^\sigma + (z+xy)^{\sigma^{-1}}, 1, 1\right) &+& \; F\left(x^\sigma, (z+xy)^{\sigma^{-1}}, 0, 1\right) \\ =\; & q^{\left[1;(z+xy)^{\sigma^{-1}}+x^\sigma,\left(y^\sigma+(z+xy)^{\sigma^{-1}},1\right)\right]} &+& \; q^{\left[1;x^\sigma,\left((z+xy)^{\sigma^{-1}},0\right)\right]} . \end{aligned}$$

Since $\hat\pi$ is a duality, we find that the image of $q^{[1;z,(x,y)]}$ under $\hat\pi^2$ is the intersection of the lines obtained as images of m under

$$\left[1;(z+xy)^{\sigma^{-1}}+x^\sigma,\left(y^\sigma+(z+xy)^{\sigma^{-1}},1\right)\right]^\pi \text{ and } \left[1;x^\sigma,\left((z+xy)^{\sigma^{-1}},0\right)\right]^\pi,$$

respectively. These are the lines

$$\begin{aligned} &F\left((x+y),1,1,0\right) &+& \;F\left(y^2+z+xy,x+y,0,1\right)\\ \text{and}\quad &F\left(x,0,1,0\right) &+& \;F\left(z+xy,x,0,1\right), \end{aligned}$$

whose intersection is $F(z,x,y,1)=q^{[1;z,(x,y)]}$, as claimed. ∎

4.8 Remark. If F has characteristic 2 but is not perfect then δ is a linear injection (see 4.3), and every stabilizer $S_{(a,b)} := F \odot [1;0,f(a,b)]$ from S is mapped to the intersection $R_{(0,1)} \cap \mathrm{E}_p^\delta$ or $R_{(1,ab^{-1})} \cap \mathrm{E}_p^\delta$, respectively, of the image E_p^δ with some stabilizer in R. Obviously, every element of R is needed here.

Thus δ induces an embedding of the affine quadrangle $(\mathrm{E}_p, \bigcup_{\mathrm{X}\in S}\mathrm{E}_p/\mathrm{X})$ into the affine quadrangle $(\mathrm{E}_\ell, \bigcup_{\Xi\in R}\mathrm{E}_\ell/\Xi)$; the image under this embedding is $\left(\mathrm{E}_p^\delta, \bigcup_{\Xi\in R}\mathrm{E}_p^\delta/(\Xi\cap\mathrm{E}_p^\delta)\right)$.

This yields an embedding of $\mathrm{W}(F)$ into its dual. However, one has to be careful: a simple quotation of the result [7] that every *isomorphism* of affine quadrangles extends to an isomorphism of completions does not suffice.

A first way to proceed is to check that the sub-quadrangle generated by the affine quadrangle $\left(\mathrm{E}_p^\delta, \bigcup_{\Xi\in R}\mathrm{E}_p^\delta/(\Xi\cap\mathrm{E}_p^\delta)\right)$ in the completion of $(\mathrm{E}_\ell, \bigcup_{\Xi\in R}\mathrm{E}_\ell/\Xi)$ is isomorphic to the completion of the smaller affine quadrangle; this is rather easy because the smaller structure contains full line pencils.

A second way would be to describe the relations "lines at distance 6": different elements $\Xi\xi, \Psi\psi \in \bigcup_{\Xi\in R}\mathrm{E}_\ell/\Xi$ have distance 6 in $(\mathrm{E}_\ell, \bigcup_{\Xi\in R}\mathrm{E}_\ell/\Xi)$ exactly if $\Xi=\Psi$ and $\xi\psi^{-1}\in \mathrm{Z}\Xi$, where $\mathrm{Z} := \left\{ \left\langle 1; \begin{pmatrix} z & 0\\ 0 & z\end{pmatrix} \right\rangle \,\middle|\, z\in F \right\}$ is the knot of the quadric R. On the other hand, two lines $(\Xi\cap\mathrm{E}_p^\delta)\xi$ and $(\Psi\cap\mathrm{E}_p^\delta)\psi$ have distance 6 in the small affine quadrangle exactly if $\Xi=\Psi$ and $\xi\psi^{-1}\in\mathrm{Z}\Xi\cap\mathrm{E}_p^\delta = (\mathrm{Z}\cap\mathrm{E}_p^\delta)(\Xi\cap\mathrm{E}_p^\delta)$. Thus points at infinity (i.e., equivalence classes of lines at distance 0 or 6; cf. [7]) are added consistently, and the embedding of affine quadrangles extends to an embedding of completions. ∎

Combining 2.3, 3.2, and 4.7, we obtain a conclusive result:

4.9 Theorem. *A symplectic quadrangle* $\mathrm{W}(F)$ *is self-dual if, and only if, the field* F *is a perfect field of characteristic* 2*. Such a quadrangle admits a polarity if, and only if, the Frobenius automorphism has a square root in* $\mathrm{Aut}(F)$*.* ∎

5 Classification of Polarities.

From now on, we assume that F is a perfect field of characteristic 2 such that the Frobenius automorphism ϕ has at least one square root in $\mathrm{Aut}(F)$. We want to classify the polarities of $\mathrm{W}(F)$, up to conjugation under $\mathrm{Aut}(\mathrm{W}(F))$, and up to conjugation under the smaller group PSp_4F.

5.1 Lemma. *Let p, q be points, and let ℓ, m be lines in* $\mathrm{W}(F)$ *such that* $\{p,\ell\}$ *and* $\{q,m\}$ *are flags, but ℓ, m do not have a common point and p, q do not have a joining line in* $\mathrm{W}(F)$.

Then every conjugacy class of polarities under the action of PSp_4F *has a representative fixing the flags* $\{p,\ell\}$ *and* $\{q,m\}$.

Proof: It suffices to note that PSp_4F acts transitively on the set of (ordinary) quadrangles in $\mathrm{W}(F)$, and then recall 3.1. ∎

From now on, we concentrate on the set Π of polarities fixing $\{p,\ell\}$ and $\{q,m\}$, where p, ℓ, q, and $m := \ell_{(0,1)}$ are chosen as in Section 4. For $\pi \in \Pi$, conjugation with π induces a σ^{-1}-semilinear bijection (denoted by π again) from E_p onto E_ℓ, where σ is a square root of ϕ, see 2.1. We have $\mathrm{K}_p^\pi = \mathrm{K}_\ell$ by 1.7, and $\mathrm{E}_p \cap \mathrm{E}_\ell$ is π-invariant. The vector space $\mathrm{E}_p \cap \mathrm{E}_\ell$ is the direct sum of K_p and $S_{(1,0)}$; and also the direct sum of K_ℓ and $R_{(1,0)}$. The stabilizer of m in E_p is $S_{(0,1)}$, while the stabilizer of q in E_ℓ is $R_{(0,1)}$. Since π fixes $\{q,m\}$ and maps the set $\left\{S_v \mid v \in F^2 \smallsetminus \{0\}\right\}$ of stabilizers of lines through p onto the set R of stabilizers of points on m, we obtain $S_{(1,0)}^\pi = R_{(1,0)}$, and $S_{(0,1)}^\pi = R_{(0,1)}$.

5.2 Proposition. *Every element $\pi \in \Pi$ is determined by the associated field automorphism of the semilinear bijection induced from* E_p *onto* E_ℓ.

Proof: The discussion above yields that there are $a, b, c \in F^\times$ with

$$\begin{aligned}
[1;1,(0,0)]^\pi &= \left\langle 1; \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right\rangle \\
[1;0,(1,0)]^\pi &= \left\langle 1; \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \right\rangle \\
[1;0,(0,1)]^\pi &= \left\langle 1; \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \right\rangle .
\end{aligned}$$

Via σ^{-1}-semilinear extension, this gives

$$[1;z,(x,y)]^\pi = \left\langle 1; \begin{pmatrix} (z+xy)^{\sigma^{-1}}a & y^\sigma c \\ x^\sigma b & (z+xy)^{\sigma^{-1}}a \end{pmatrix} \right\rangle ;$$

recall that $\sigma^2 = \phi$ is the Frobenius automorphism.

Applying π twice to the point $F(z,x,y,1) = q^{[1;z,(x,y)]}$, we obtain (cp. the proof of 4.7) the intersection of the two lines

$$\begin{aligned}
&F\Big((xb^{\sigma^{-1}} + yc^{\sigma^{-1}})a, c, 1, 0\Big) \oplus F\Big((y^2c^\sigma + za^\sigma + xya^\sigma)b, (xb^{\sigma^{-1}} + yc^{\sigma^{-1}})a, 0, 1\Big) \\
&\quad\text{and}\quad F\Big(xb^{\sigma^{-1}}a, 0, 1, 0\Big) \oplus F\Big((z+xy)a^\sigma b, xb^{\sigma^{-1}}a, 0, 1\Big) .
\end{aligned}$$

However, these lines have a point in common only if $a^2 = bc$. In that case, we specialize $(z,x,y) = (1,1,1)$ and find $c^{\sigma^{-1}}a = c$, and $b^{\sigma^{-1}}a = 1$. These three conditions imply $a = b = c = 1$, and π is the polarity $\hat{\pi}$, as constructed in 4.7. ∎

5.3 Corollary. *Each polarity of* $\mathrm{W}(F)$ *is conjugate to at least one of the polarities constructed in 4.7.2.* ∎

Via 5.1, the uniqueness result 5.2 yields a complete description of the conjugacy classes of polarities:

5.4 Theorem. *Polarities of* $\mathrm{W}(F)$ *are conjugates under* $\mathrm{PSp}_4 F$ *if, and only if, their conjugates in* Π *coincide. In particular, the set* Π *forms a system of representatives for the conjugacy classes (under* $\mathrm{PSp}_4 F$*).*

Under $\mathrm{Aut}(\mathrm{W}(F))$*, polarities are conjugates if, and only if, the associate field automorphisms are conjugates under* $\mathrm{Aut}(F)$*.* ■

5.5 Corollary. *Let* p, q *be points, and let* ℓ, m *be lines in* $\mathrm{W}(F)$ *such that* $\{p, \ell\}$ *and* $\{q, m\}$ *are flags, but* ℓ, m *do not have a common point and* p, q *do not have a joining line in* $\mathrm{W}(F)$*. Then the stabilizer of* $\{\{p, \ell\}, \{q, m\}\}$ *in the group* $\mathrm{PSp}_4 F$ *centralizes every element of* Π*.* ■

6 A Polarity's Centralizer, and Its Ovoid.

Let π be a polarity of a generalized quadrangle $\mathcal{Q}$. A point x is called *absolute* if $\{x, x^\pi\}$ is a flag (which is, of course, fixed by π). A simple geometric argument (see [11] 7.2.5) shows that the absolute points of any polarity of any generalized quadrangle form an *ovoid*: that is, a set $\mathcal{O}$ of points such that every line contains exactly one point of $\mathcal{O}$. The centralizer of π in $\mathrm{Aut}(\mathcal{Q})$ acts on the ovoid $\mathcal{O}_\pi$ of absolute points. For a proof that this action is 2-transitive, we use the notion of an *ideal line* in a quadrangle: i.e., for any two points x, y not on a common line, the set $x^\perp \cap y^\perp$ of all points that are joined to both.

6.1 Lemma. *Let* π *be a polarity of a generalized quadrangle, let* a *and* b *be two absolute points. Let* x *be the (unique) point of* a^π *in* $b^\perp$*, and let* z *be the line joining* x *and* b*. Then the ideal line* $x^\perp \cap (z^\pi)^\perp$ *contains no absolute points except* a *and* b*.*

Proof: First of all, we note that z^π is a point on b^π, joined to $a = a^{\pi\pi}$ by x^π. Aiming at a contradiction, we assume that c is a third absolute point in the ideal line $x^\perp \cap (z^\pi)^\perp$. Then c^π meets x^π, and we infer that c^π passes through z^π. Applying π once more, we find that both c and b lie on z, and b, c, z^π is a triangle — which is impossible. ■

In a symplectic quadrangle, the ideal lines are just those lines of the ambient projective space that do not belong to the quadrangle. In particular, there is exactly one ideal line through any two points of the ovoid $\mathcal{O}_\pi$, and 6.1 yields:

6.2 Corollary. *The ovoid of absolute points of a polarity of a symplectic quadrangle meets any line of the ambient projective space in at most* 2 *points.* ■

In fact, the ovoid in the symplectic quadrangle is an ovoid in the ambient projective space, see [10] or [11] 7.6.14. We use our result 5.5 to give a simple proof for another known fact:

6.3 Theorem. *The centralizer of a polarity* π *of a symplectic quadrangle acts* 2*-transitively on the ovoid* $\mathcal{O}_\pi$ *of absolute points. In fact, the intersection of that centralizer with* $\mathrm{PSp}_4 F$ *acts* 2*-transitively.*

Proof: Let Fu, Fv, Fw be different points of $\mathcal{O}_\pi$. Then u, v, w are linearly independent in F^4, by 6.2. Without loss, we may assume $uSv' = 1 = uSw'$. Now (u, v, w) can be extended to a basis for F^4 by an element $b \in \{v, w\}^\perp$. Linear extension of $u^\gamma = u$, $v^\gamma = w$, $w^\gamma = v$ and $b^\gamma = b$ gives an involution $\gamma \in \mathrm{Sp}_4F$ that fixes Fu but interchanges Fv with Fw. Since this involution leaves $\{\{Fv, Fv^\pi\}, \{Fw, Fw^\pi\}\}$ invariant, it centralizes π by 5.5. This shows that the stabilizer of any point Fu in $\mathcal{O}_\pi$ still acts transitively on $\mathcal{O}_\pi \setminus \{Fu\}$. ∎

The proof of 6.3 shows that the centralizer of a polarity of the symplectic quadrangle is generated by involutions. In fact, the intersection of this centralizer with PSp_4F is a simple group (of twisted Lie type ${}^2B_2$, also known as Suzuki group), with a 2-fold transitive action on the ovoid (called a Suzuki–Tits–Ovoid); see [10]. See [4] XI §3 or Sections 21, 22, and 24 in [6] for an (elementary) discussion of these groups and their subgroups.

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