The Capacity of some Sets in the Complex Plane

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Abstract

For a positive integer s and for $0 \leq a < b$, let

$$K = K^s_{a,b} = \bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}}[a, b].$$

We find that the *capacity* $\mathrm{Cap}(K)$ of K is

$$\mathrm{Cap}(K) = \sqrt[s]{\frac{b^s - a^s}{4}}. \tag{1}$$

From this relation we derive several classical results, due to Akhiezer, Henrici, and Bartolomeo and He, on capacities of some sets in the complex plane.

An extension relation (1) to more general sets in the complex plane, together with potential theoretic techniques, is then used to obtain *saturation* theorems pertaining to approximation by polynomials with integer coefficients.

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1 Introduction and Preliminaries from Approximation Theory in the Complex Plane.

Let E be a compact simply connected set of the complex plane containing more than one point and let $\omega = \phi(z)$ map conformally $\mathrm{Ext}(E)$ into $|\omega| > 1$ and with $\phi(\infty) = \infty$. The map $\phi(z)$ has the form

$$\phi(z) = \frac{z}{c} + a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \cdots$$

The number $c > 0$ is the *capacity* $\mathrm{Cap}(E)$ of the set E.

The capacity is a concept of fundamental importance in complex analysis and potential theory where it plays an important role in the theory of *removable* sets. It also plays an increasingly important role in partial differential equation of elliptic type. See [5], [11].

In some few cases the $\mathrm{Cap}(E)$ of the set E can be computed explicitly.

As well known

$$\mathrm{Cap}\left([a,b]\right) = \frac{b-a}{4}\cdot \tag{1.1}$$

Another important result (see [8] and [2]) is

$$\mathrm{Cap}\left(\bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}} \left[0, \sqrt[s]{4}\right]\right) = 1. \tag{1.2}$$

This result has been used by Henrici, and Bartolomeo and He (see [2], [9]) in their study of the Faber polynomials associated with certain regions of the complex plane.

Another standard result states that

$$\mathrm{Cap}\left([-1,-a] \cup [a,1]\right) = \sqrt{\frac{1-a^2}{4}}\cdot \tag{1.3}$$

This result has been obtained by Akhiezer (see [1] where the term transfinite diameter is used.)

Another classical result states that

$$\mathrm{Cap}\left(|z| = b\right) = \mathrm{Cap}\left(a \le |z| \le b\right) = \mathrm{Cap}\left(|z| \le b\right) = b. \tag{1.4}$$

Then using standard conformal mapping arguments, we obtain another proof of the well known fact that the capacity of a closed Jordan curve equals that of the set enclosed by this curve.

It is the purpose of the first part of this work to show that the four mentioned results are all particular cases of a more general result, namely

Theorem. *For a positive integer s and for $0 \le a < b$, let*

$$K = K_{a,b}^s = \bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}} [a,b].$$

Then the capacity $\mathrm{Cap}(K)$ of K is

$$\mathrm{Cap}(K) = \sqrt[s]{\frac{b^s - a^s}{4}}\cdot \tag{1.5}$$

It is clear that relation (1.5) implies (1.1), (1.2) and (1.3).

Note that

$$\bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}}[a,b] = \bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}}[0,b] \; - \; \bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}}[0,a).$$

However this relation together with (1.2) does not imply (1.5) because the set function $\mathrm{Cap}(\,\cdot\,)$ is not an additive set function.

The fact that (1.5) implies (1.4) can be heuristically explained as follows. Let $0 \le a < b$. Then

$$\lim_{s\to\infty} \sqrt[s]{\frac{b^s - a^s}{4}} = b. \tag{1.6}$$

On the other hand

$$\lim_{s\to\infty} \bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}}[a,b] = \{a \le |z| \le b\}\,.$$

This statement must be understood as follows. $\forall \epsilon > 0 \;\; \exists S > 0$ such that $s \ge S$ implies $\bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}}[a,b]$ is ϵ dense in $\{a \le |z| \le b\}$. Hence, if we can prove that

$$\lim_{s\to\infty} \mathrm{Cap}\left(\bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}}[a,b]\right) = \mathrm{Cap}\left(a \le |z| \le b\right), \tag{1.7}$$

then this relation, together with (1.5) and (1.6), would yield

$$\mathrm{Cap}\left(a \le |z| \le b\right) = b.$$

Now, letting first $a \to 0$ then letting $a \to b$, we should recover the classical result (1.4).

However it is possible to construct sets S_n with $\lim_{n\to\infty} S_n = S$ and $\lim_{n\to\infty} \mathrm{Cap}(S_n) \neq \mathrm{Cap}(S)$. This shows that the above heuristic argument will have to be refined in order to show that (1.5) implies (1.4). In fact relation (1.7) *is* true and part of the work to show that (1.5) implies (1.4) will precisely reduce to proving the validity of (1.7).

Relation (1.7) will be proved in the Appendix.

We end this introduction with the following celebrated theorem of Walsh and Bernstein [16] that will play a fundamental role in the sequel.

Theorem 1.1. *(Walsh-Bernstein) Let E be as above and let $P_n(z)$ be a polynomial of degree (at most) n with*

$$\|P_n(z)\|_E \le 1.$$

Then, for $\rho > 1$, one has

$$\|P_n(z)\|_{\Gamma_\rho} \le \rho^n.$$

Because of the importance of this theorem in our work, we give a simplified proof of it made possible by the additional assumption that $\mathrm{B}_r(E)$ is a Jordan curve. In that case the maximum principle extends to the extended mapping function.

Proof. Because the Laurent expansion at $\omega = \infty$ of $z = \psi(\omega)$ is of the form $\psi(\omega) = c\,\omega + b_0 + \frac{b_{-1}}{\omega} + \frac{b_{-2}}{\omega^2} + \cdots$, the function $P_n(\psi(\omega))$ has a pole of order n at ∞. It follows that

$$f(\omega) := \frac{P_n(\psi(\omega))}{\omega^n}$$

is analytic at ∞. Now the maximum principle yields, for $\rho \geq 1$,

$$\sup_{|\omega|=1} \left| \frac{P_n(\psi(\omega))}{\omega^n} \right| \geq \sup_{|\omega|=\rho} \left| \frac{P_n(\psi(\omega))}{\omega^n} \right|.$$

Walsh's theorem follows if we remark that

$$\sup_{|\omega|=\rho} |P_n(\psi(\omega))| = \sup_{z\in\Gamma_\rho} |P_n(z)|.$$

∎

This paper is organized as follows: In the next section we construct the fundamental conformal mapping – and Green's function – which will allow us to find the capacity of K. The study of the level curves of this mapping function together with the Walsh-Bernstein theorem will be the principal tools for proving, in Section 4, our saturation results pertaining to approximation by polynomials with integer coefficients.

2 Construction of the Fundamental Conformal Mapping and the Capacity of K.

Recall that the set K was defined by

$$K = \bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}}[a,b],$$

with $0 \leq a < b$.

Lemma 2.1. *The s-to-one mapping function*

$$\omega = g(z) = \frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s} + \sqrt{\left(\frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s}\right)^2 - 1}$$

transforms $\mathrm{Ext}(K)$ *into* $|\omega| > 1$. *Its (continuous) extension to* $\mathbb{C}$ *maps each of the intervals* $e^{2\pi i \frac{k}{s}}[a,b]$, *traversed twice, into* $|\omega| = 1$.

Proof. The function

$$\omega = z + \sqrt{z^2 - 1}$$

transforms conformally $\mathrm{Ext}[-1,1]$ into $|\omega| > 1$. Hence the function

$$\omega = \frac{2z}{b-a} - \frac{b+a}{b-a} + \sqrt{\left(\frac{2z}{b-a} - \frac{b+a}{b-a}\right)^2 - 1}$$

transforms conformally $\mathrm{Ext}[a,b]$ into $|\omega| > 1$. Here the square root in the function $\omega = z + \sqrt{z^2-1}$ is uniquely chosen in such a way that its branch cut is the interval $[-1,1]$. Its (continuous) extension to $\mathbb{C}$ maps the interval $[a,b]$, traversed twice, into $|\omega| = 1$. Hence the continuous extension of $g(z)$ to $\mathbb{C}$ maps each of the intervals $e^{2\pi i \frac{k}{s}}[a,b]$, traversed twice, into $|\omega| = 1$. The lemma follows now by direct computation. ∎

Corollary 2.1. *The harmonic function*

$$G(z) = \frac{1}{s}\log\left|\frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s} + \sqrt{\left(\frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s}\right)^2 - 1}\right|$$

is the Green function with logarithmic pole at ∞ *of* $\mathrm{Ext}(K)$ *with boundary values*

$$\lim_{z \to z_0 \in K} G(z) = 0.$$

2.1 Study of the Level Curve $|h(z)| = C$.

Theorem 2.1. *With* $0 < a < b$, *the level curve*

$$|h(z)| = \left|\sqrt[s]{\frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s} + \sqrt{\left(\frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s}\right)^2 - 1}}\right| = \sqrt[s]{\frac{b^{\frac{s}{2}} + a^{\frac{s}{2}}}{b^{\frac{s}{2}} - a^{\frac{s}{2}}}}$$

passes through the point 0.

Proof. Recall that the square root in the function

$$f(z) = \frac{2z}{b-a} - \frac{b+a}{b-a} + \sqrt{\left(\frac{2z}{b-a} - \frac{b+a}{b-a}\right)^2 - 1}$$

is uniquely chosen in such a way that its branch cut is the interval $[a, b]$. Now if $x \in \mathbb{R}$, $x < a < b$, then

$$\left(\frac{2x}{b-a} - \frac{b+a}{b-a}\right)^2 - 1 > 0.$$

It follows that

$$f(x) = \frac{2x}{b-a} - \frac{b+a}{b-a} - \sqrt{\left(\frac{2x}{b-a} - \frac{b+a}{b-a}\right)^2 - 1}, \qquad \text{if } \ x < a < b.$$

Here $\sqrt{\left(\frac{2x}{b-a} - \frac{b+a}{b-a}\right)^2 - 1}$ must be understood as the positive square root of the positive number $\left(\frac{2x}{b-a} - \frac{b+a}{b-a}\right)^2 - 1$. It follows in a similar manner that, if $0 < a < b$, then

$$h(0) = \sqrt[s]{-\frac{b^s + a^s}{b^s - a^s} - \sqrt{\left(\frac{b^s + a^s}{b^s - a^s}\right)^2 - 1}},$$

where, again, $\sqrt{\left(\frac{b^s+a^s}{b^s-a^s}\right)^2 - 1}$ is the positive square root of the positive number $\left(\frac{b^s+a^s}{b^s-a^s}\right)^2 - 1$. Hence

$$|h(0)| = \sqrt[s]{\frac{b^{\frac{s}{2}} + a^{\frac{s}{2}}}{b^{\frac{s}{2}} - a^{\frac{s}{2}}}}.$$

■

Remark. *If $x \in \mathbb{R}$, $a < b < x$, then*

$$f(x) = \frac{2x}{b-a} - \frac{b+a}{b-a} + \sqrt{\left(\frac{2x}{b-a} - \frac{b+a}{b-a}\right)^2 - 1}, \qquad a < b < x,$$

with an interpretation of the square root similar to that of Proposition 2.1.

Corollary 2.2. *The one-to-one mapping function*

$$\omega = h(z) = \sqrt[s]{\frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s} + \sqrt{\left(\frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s}\right)^2 - 1}},$$

with inverse map

$$z = \psi(\omega) = \sqrt[s]{\frac{1}{4}\left(b^s - a^s\right)\left(\omega^s + \frac{1}{\omega^s}\right) + \frac{b^s + a^s}{2}},$$

maps conformally

$$\left\{ z \ ; \ |h(z)| > \sqrt[s]{\frac{b^{\frac{s}{2}} + a^{\frac{s}{2}}}{b^{\frac{s}{2}} - a^{\frac{s}{2}}}} \right\}$$

into

$$|\omega| > \sqrt[s]{\frac{b^{\frac{s}{2}} + a^{\frac{s}{2}}}{b^{\frac{s}{2}} - a^{\frac{s}{2}}}}.$$

Proof. The s root above is uniquely defined by imposing

$$\sqrt[s]{z} = \sqrt[s]{\rho e^{i\theta}} = \sqrt[s]{\rho} e^{i\frac{\theta}{s}}.$$

In view of the previous Theorem 2.1, the level curve

$$|h(z)| = M$$

with

$$M > \sqrt[s]{\frac{b^{\frac{s}{2}} + a^{\frac{s}{2}}}{b^{\frac{s}{2}} - a^{\frac{s}{2}}}}$$

consists of a single analytic curve. Hence with the s root so defined, $h(z) = \sqrt[s]{g(z)}$ transforms the s-to-one function $g(z)$ into the one-to-one function $h(z)$ and maps

$$\text{Ext}\left(|h(z)| = \sqrt[s]{\frac{b^{\frac{s}{2}} + a^{\frac{s}{2}}}{b^{\frac{s}{2}} - a^{\frac{s}{2}}}} \right)$$

into

$$|\omega| > \sqrt[s]{\frac{b^{\frac{s}{2}} + a^{\frac{s}{2}}}{b^{\frac{s}{2}} - a^{\frac{s}{2}}}}.$$

∎

Remark. *In the case $s = 1$, the above restriction $|\omega| > \frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}}$ is not needed and, clearly, in that case, $h(z)$ maps conformally* $\operatorname{Ext}[a, b]$ *into $|\omega| > 1$. However in the case $s \geq 2$, the restriction $|\omega| > \sqrt[s]{\frac{b^{\frac{s}{2}}+a^{\frac{s}{2}}}{b^{\frac{s}{2}}-a^{\frac{s}{2}}}}$ cannot be relaxed. This is due to the fact, as will be noticed as a consequence of Theorem 2.2 below, that the level curve $|h(z)| = C$, with $1 < C < \sqrt[s]{\frac{b^{\frac{s}{2}}+a^{\frac{s}{2}}}{b^{\frac{s}{2}}-a^{\frac{s}{2}}}}$ consists of s non intersecting closed curves.*

Our mapping function $\omega = h(z)$ allows us to recover, as a special case, the following result.

Corollary 2.3. *(Henrici [8]) The mapping function*

$$z = \psi(\omega) = \left(\omega^{\frac{s}{2}} + \frac{1}{\omega^{\frac{s}{2}}}\right)^{\frac{2}{s}}$$

maps conformally

$$|\omega| > 1$$

into

$$\operatorname{Ext}\left(\bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}} \left[0, \sqrt[s]{4}\right]\right).$$

Proof. A rather lengthy but, at the same time, elementary computation shows that the inverse map $z = \psi(\omega)$, $|\omega| > \sqrt[s]{\frac{b^{\frac{s}{2}}+a^{\frac{s}{2}}}{b^{\frac{s}{2}}-a^{\frac{s}{2}}}}$, of the conformal map

$$\omega = h(z) = \sqrt[s]{\frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s} + \sqrt{\left(\frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s}\right)^2 - 1}}$$

is

$$z = \psi(\omega) = \sqrt[s]{\frac{1}{4}\left(b^s - a^s\right)\left(\omega^s + \frac{1}{\omega^s}\right) + \frac{b^s + a^s}{2}}\,.$$

Letting $a = 0$ and $b = \sqrt[s]{4}$ in $z = \psi(\omega)$ gives $z = \psi(\omega) = \left(\omega^{\frac{s}{2}} + \frac{1}{\omega^{\frac{s}{2}}}\right)^{\frac{2}{s}}$, $|\omega| > 1$. The image by $z = \psi(\omega)$ of $|\omega| > 1$ is $\operatorname{Ext}(K)$ with $a = 0$ and $b = \sqrt[s]{4}$. ∎

Remark. *The above expression for $z = \psi(\omega)$ shows, once more, that* $\operatorname{Cap}\left(\bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}} \left[0, \sqrt[s]{4}\right]\right) = 1$. *Indeed, it suffices to notice that the first term of the Laurent expansion at ∞ of $z = \psi(\omega) = \left(\omega^{\frac{s}{2}} + \frac{1}{\omega^{\frac{s}{2}}}\right)^{\frac{2}{s}}$ is $1 \times \omega$.*

We summarize at this point those consequences of Theorem 2.1 and Corollary 2.2 that, in conjunction with the Walsh-Bernstein theorem, will be the principal tools in the proofs of our main results.

Theorem 2.2. *Consider the level curve*

$$|h(z)| = \left|\sqrt[s]{\frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s} + \sqrt{\left(\frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s}\right)^2 - 1}}\right| = C.$$

If $C = 1$, *then the level curve* $|h(z)| = C$ *consists of the set* K.

If $1 < C < \sqrt[s]{\frac{b^{\frac{s}{2}}+a^{\frac{s}{2}}}{b^{\frac{s}{2}}-a^{\frac{s}{2}}}}$ *then the level curve* $|h(z)| = C$ *consists of* s *pairwise disjoint closed curves each of which contains in its interior exactly one of the intervals* $e^{2\pi i\frac{k}{s}}[a, b]$ *making up* $K = \bigcup_{k=0}^{s-1} e^{2\pi i\frac{k}{s}}[a, b]$.

If $C = \sqrt[s]{\frac{b^{\frac{s}{2}}+a^{\frac{s}{2}}}{b^{\frac{s}{2}}-a^{\frac{s}{2}}}}$ *then the level curve* $|h(z)| = C$ *is a lemniscate-like curve which consists of* s *closed branches, whose only common point is the point 0 and all of which coalesce at the point 0. Moreover each of these branches contains in its interior exactly one of the intervals* $e^{2\pi i\frac{k}{s}}[a, b]$.

If $C > \sqrt[s]{\frac{b^{\frac{s}{2}}+a^{\frac{s}{2}}}{b^{\frac{s}{2}}-a^{\frac{s}{2}}}}$ *then the level curve* $|h(z)| = C$ *consists of a single closed curve and contains 0* $\left(\text{as well as } \bigcup_{k=0}^{s-1} e^{2\pi i\frac{k}{s}}[a, b]\right)$ *in its interior.*

2.2 The Capacity of K and of the Associated Level Curve $|h(z)| = C$.

Theorem 2.3. *The capacity* $\mathrm{Cap}(K)$ *of* K *is*

$$\mathrm{Cap}(K) = \sqrt[s]{\frac{b^s - a^s}{4}}.$$

Proof. For $|u|$ large we have

$$\sqrt{u^2 - 1} = u - \frac{1}{2u} - \frac{1}{8u^3} + \cdots$$

It follows that the Laurent expansion at ∞ of

$$g(z) = \frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s} + \sqrt{\left(\frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s}\right)^2 - 1}$$

is

$$2\left(\frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s}\right) - \frac{1}{2\left(\frac{2z^s}{b^s-a^s} - \frac{b^s+a^s}{b^s-a^s}\right)} - \frac{1}{8\left(\frac{2z^s}{b^s-a^s} - \frac{b^s+a^s}{b^s-a^s}\right)^3} + \cdots$$

Hence the theorem follows from Corollary 2.1. ■

Theorem 2.2, in conjunction with Theorem 2.3, allows us to find the value of C for which the capacity of the level curve $|h(z)| = C$ equals 1. These result will help us prove *saturation* results next section.

Corollary 2.4. *The capacity of the level curve*

$$|h(z)| = \sqrt[s]{\frac{b^{\frac{s}{2}} + a^{\frac{s}{2}}}{b^{\frac{s}{2}} - a^{\frac{s}{2}}}}$$

is

$$\sqrt[s]{\frac{\left(b^{\frac{s}{2}} + a^{\frac{s}{2}}\right)^2}{4}}.$$

Proof. It suffices to remark that

$$\sqrt[s]{\frac{b^{\frac{s}{2}} + a^{\frac{s}{2}}}{b^{\frac{s}{2}} - a^{\frac{s}{2}}}} \times \sqrt[s]{\frac{b^s - a^s}{4}} = \sqrt[s]{\frac{\left(b^{\frac{s}{2}} + a^{\frac{s}{2}}\right)^2}{4}}.$$

The conclusion follows from Theorem 2.3. ∎

Corollary 2.5. *With $b^s - a^s < 4$, the level curve*

$$|h(z)| = \left|\sqrt[s]{\frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s} + \sqrt{\left(\frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s}\right)^2 - 1}}\right| = \sqrt[s]{\frac{4}{b^s - a^s}}$$

has capacity one.

Proof. Because $b^s - a^s < 4$, $\mathrm{Cap}(K) = \sqrt[s]{\frac{b^s - a^s}{4}} < 1$. On the other hand $h(z)$ maps $\mathrm{Ext}(K)$ into $|\omega| > 1$. It follows that the curve

$$|h(z)| = \sqrt[s]{\frac{4}{b^s - a^s}} > 1$$

is indeed a single level curve of $|h(z)|$. That the capacity of this level curve is one follows from an argument similar to that of Corollary 2.4 and again from Theorem 2.3. ∎

3 Saturation Properties of Polynomials with Integer Coefficients.

Let $f(x)$ be a continuous function defined on $[a, b]$ and let $E_n(f)$ be the degree of approximation of f, for the supremum norm on $[a, b]$, by polynomials of degree (at most) n. Thus

$$E_n(f) = \inf_{P \in \mathbb{P}_n} \|f - P\|.$$

Here $\mathbb{P}_n$ denotes the space of polynomials of degree at most n and $\|\cdot\|$ is the supremum norm on $[a, b]$. Let $\mathbb{P}_n^I$ denote the subset of $\mathbb{P}_n$ formed by imposing that all the coefficients of $P \in \mathbb{P}_n$ be integers and let

$$E_n^I(f) = \inf_{P \in \mathbb{P}_n^I} \|f - P\|$$

be the corresponding degree of approximation of f, for the supremum norm on $[a, b]$, by polynomials of degree (at most) n with integer coefficients.

It is a classical result (see [10]) that approximation by polynomials with integer coefficients is not possible if

$$\mathrm{Cap}([a, b] \geq 1) \text{ that is to say if } b - a \geq 4. \tag{3.1}$$

More precisely, if relation (3.1) is satisfied and

$$\lim_{n \to \infty} E_n^I(f) = 0,$$

then

$$f \in \mathbb{P}_n^I$$

for some n.

A typical question addressed in this section is the following: Let $b - a < 4$. If $E_n^I(f)$ tends to zero fast enough, can we conclude that $f \in \mathbb{P}_n^I$? For instance it is not difficult to find a number $\rho > 1$ such that the relation

$$E_n^I(f) \leq \frac{C}{\rho^n} \qquad \forall n \tag{3.2}$$

implies $f \in \mathbb{P}_n^I$ for some n. As an example consider the case $[a, b] = [-1, 1]$. Then relation (3.2) together with

$$\rho > 1 + \sqrt{2}$$

implies $f \in \mathbb{P}_n^I$ for some n. Indeed in view of Bernstein's theorem (recalled earlier in the form of Walsh-Bernstein's theorem) the relation

$$E_n(f) \leq E_n^I(f) \leq \frac{C}{\rho^n}$$

implies that f is analytic in the interior of the ellipse whose foci are -1 and 1 and whose axes have lengths $\rho + \frac{1}{\rho}$ and $\rho - \frac{1}{\rho}$. Moreover the polynomials (with integer coefficients) P_n converge uniformly to f on the compact subsets of the interior of this ellipse. Hence $P_n^{(k)}$ converge uniformly to $f^{(k)}$ on the same compact subsets, so that $\frac{f^{(k)}(0)}{k!} \in \mathbb{Z}$. Now with $\rho > 1 + \sqrt{2}$, we have $\rho - \frac{1}{\rho} > 2$ (and $\rho + \frac{1}{\rho} > 2\sqrt{2}$.) It follows that this ellipse contains in its interior a disk centered at 0 and with radius $a > 1$. Hence for the function f we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with

$$a_n \in \mathbb{Z}$$

and

$$\frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} \geq a > 1.$$

It follows that

$$a_n = 0, \quad n \geq N$$

for some N, so that $f \in \mathbb{P}_{N-1}^I$.

We will see (again in the case $[a, b] = [-1, 1]$) that the condition $\rho > 1+\sqrt{2}$ is unnecessarily strong and that $\rho > 2$ is sufficient to ensure that $f \in \mathbb{P}_n^I$. More precisely we will see that the condition

$$\sum_{n=1}^{\infty} 2^n E_n^I(f) < \infty$$

guarantees that $f \in \mathbb{P}_n^I$.

More generally we will show: Let

$$K = \bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}}[a,b]$$

with $0 \le a < b$. Approximation by polynomials with integers coefficients is not possible if $b^s - a^s \ge 4$.

Theorem 3.1. *Let s be a positive integer and let $0 \le a < b$ with $b^s - a^s < 4$. Let K be as above. If*

$$\sum_{n=1}^{\infty} \left(\sqrt[s]{\frac{4}{b^s - a^s}} \right)^n E_n^I(f,K) < \infty$$

then

$$f \in \mathbb{P}_n^I.$$

Theorem 3.1 is a typical *saturation* result. Saturation-type results tell us that the rate of convergence cannot exceed a certain speed no matter how smooth the function is, unless it belongs to a very special class. A classical example is provided by the Bernstein polynomials, $B_n(f)$, of a given function f. $B_n(f)$ cannot tend to f faster than $\mathcal{O}\left(\frac{1}{n}\right)$ no matter how smooth f is unless f is a polynomial of degree 1. Saturation theory is extensively treated in the classical books of G. G. Lorentz and H. S. Shapiro, [10] and [13]. For a treatment of saturation theory from a slightly different angle, see [3].

3.1 Proof of Theorem 3.1.

We have now built the necessary tools to prove the main result of this section that we reproduce here for convenience.

Theorem. *Let s be a positive integer and let $0 \le a < b$ with $b^s - a^s < 4$. Let*

$$K = \bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}}[a,b].$$

If

$$\sum_{n=1}^{\infty} \left(\sqrt[s]{\frac{4}{b^s - a^s}} \right)^n E_n^I(f,K) < \infty \tag{3.3}$$

then

$$f \in \mathbb{P}_n^I.$$

We will prove this result in the case $0 < a < b$ and then indicate the necessary modifications pertaining to the case $0 = a < b$.

Proof. We first remark that, in view of Theorem 3.1,

$$\operatorname{Cap}\left(\bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}}[a,b] \right) \ge 1$$

if

$$b^s - a^s \geq 4.$$

It follows that, if $b^s - a^s \geq 4$, then $E_n^I(f, K)$ cannot tend to 0 as n tends to ∞ unless $f \in \mathbb{P}_n^I$ for some n. See [10].

Let then $b^s - a^s < 4$ and assume that relation (3.3) holds. With $P_n \in \mathbb{P}_n^I$ such that

$$\|P_n - f\|_K = E_n^I(f, K),$$

we have

$$P_n - f = \sum_{k=n}^{\infty} P_k - P_{k+1},$$

where the convergence takes place for the $\|.\|_K$ norm. Let Γ_ρ be the level curve

$$\Gamma_\rho = \left| \sqrt[s]{\frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s} + \sqrt{\left(\frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s}\right)^2 - 1}} \right| = \rho.$$

Hence, with

$$\rho = \sqrt[s]{\frac{4}{b^s - a^s}},$$

$$\|P_n - f\|_{\Gamma_\rho} \leq \sum_{k=n}^{\infty} \|P_k - P_{k+1}\|_{\Gamma_\rho}.$$

It follows from Walsh-Bernstein's theorem that

$$\|P_n - f\|_{\Gamma_\rho} \leq \sum_{k=n}^{\infty} \rho^k \|P_k - P_{k+1}\|_K.$$

Hence

$$\|P_n - f\|_{\Gamma_\rho} \leq 2 \sum_{k=n}^{\infty} \rho^k \; E_k^I(f, K).$$

Because

$$\sum_{n=1}^{\infty} \rho^n \; E_n^I(f, K) < \infty,$$

$$\lim_{n \to \infty} \|P_n - f\|_{\Gamma_\rho} = 0.$$

Thus the function f is analytic inside the level curve Γ_ρ and continuous up to the boundary Γ_ρ. Moreover the sequence P_n converges uniformly to f on this level curve (as well as on its interior.) But Corollary 2.5 tells us that the level curve

$$|h(z)| = \left| \sqrt[s]{\frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s} + \sqrt{\left(\frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s}\right)^2 - 1}} \right| = \rho$$

has capacity one. Hence by the fundamental result on approximation theory by polynomials with integer coefficients [10], $f \in \mathbb{P}_n^I$ for some n.

In the case $0 = a < b$, the level curve Γ_ρ consists always of a unique analytic curve (of capacity one) and the proof remains unchanged. ∎

A Proof of Relation (1.7).

We reproduce here for convenience Relation (1.7).

Theorem A.1. *Let* $0 \le a < b$. *Then*

$$\lim_{s\to\infty} \mathrm{Cap}\left(\bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}}[a,b]\right) = \mathrm{Cap}\left(a \le |z| \le b\right).$$

Proof. Recall that for a compact (infinite) set $E \subset \mathbb{C}$

$$\mathrm{Cap}(E) = \lim_{n\to\infty} \sup_{z_i,\, z_j \,\in\, E} \left(\prod_{1\le i<j\le n} |z_i - z_j|\right)^{\frac{1}{\binom{n}{2}}}.$$

See [15]. Clearly

$$\mathrm{Cap}\left(\bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}}[a,b]\right) \le \mathrm{Cap}\left(a \le |z| \le b\right).$$

Hence

$$\limsup_{s\to\infty} \mathrm{Cap}\left(\bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}}[a,b]\right) \le \mathrm{Cap}\left(a \le |z| \le b\right). \qquad (a.1)$$

As well known

$$\sup_{z_i,\, z_j \,\in\, \{a\le|z|\le b\}} \left(\prod_{1\le i<j\le n} |z_i - z_j|\right)^{\frac{1}{\binom{n}{2}}}$$

is a decreasing function of n. (See [15].) Hence

$$\sup_{z_i,\, z_j \,\in\, E} \left(\prod_{1\le i<j\le n} |z_i - z_j|\right)^{\frac{1}{\binom{n}{2}}} \ge \mathrm{Cap}\left(a \le |z| \le b\right).$$

Let $z_1,\ z_2,\ \cdots\ z_n \ \in\ \{a \le |z| \le b\}$ designate the complex numbers for which the above supremum is reached. So, with those values for $z_1,\ z_2,\ \cdots\ z_n$, we have

$$\left(\prod_{1\le i<j\le n} |z_i - z_j|\right)^{\frac{1}{\binom{n}{2}}} \ge \mathrm{Cap}\left(a \le |z| \le b\right). \qquad (a.2)$$

Observe now that the function

$$f(z_1,\ z_2,\ \cdots\ z_n) := \left(\prod_{1\le i<j\le n} |z_i - z_j|\right)^{\frac{1}{\binom{n}{2}}}$$

is uniformly continuous for $z_1,\ z_2,\ \cdots\ z_n \ \in\ \{a \le |z| \le b\}$. Let now $n \in \mathbb{N}$ be an integer whose value will be determined later in such a way that relation (a.5) below holds true. Hence there exists $\delta > 0$ small enough such that

$$\left|z_i - z_i^{'}\right| \le \delta, \quad i = 1,\ 2,\ \cdots\ n$$

implies

$$\left|f(z_1,\ z_2,\ \cdots\ z_n)\ -\ f(z_1^{'},\ z_2^{'},\ \cdots\ z_n^{'})\right|\ \le\ \frac{1}{2}\epsilon$$

Pick now s so large that

$$x \in \{a \le |z| \le b\}$$

implies

$$\text{dist}\left(\ x\ ,\ \bigcup_{k=0}^{s-1} e^{2\pi i\frac{k}{s}}[a,b]\right) \le \delta.$$

Now perturb $z_1,\ z_2,\ \cdots\ z_n\ \in\ \{a \le |z| \le b\}$ for which

$$\left(\prod_{1\le i<j\le n} |z_i - z_j|\right)^{\frac{1}{\binom{n}{2}}}\ \ge\ \text{Cap}\,(a \le |z| \le b)$$

into

$$z_1^{'},\ z_2^{'},\ \cdots\ z_n^{'}$$

in such a way that the following properties hold:

1. $$z_1^{'},\ z_2^{'},\ \cdots\ z_n^{'}\ \in\ \bigcup_{k=0}^{s-1} e^{2\pi i\frac{k}{s}}[a,b]$$

and

2. $$\left|z_i - z_i^{'}\right| \le \delta,\quad i = 1,\ 2,\ \cdots\ n.$$

Because of our choice for s, such a perturbation is possible.
Remark now that

$$\left(\prod_{1\le i<j\le n} |z_i - z_j|\right)^{\frac{1}{\binom{n}{2}}}\ \ge\ \left(\prod_{1\le i<j\le n} |z_i^{'} - z_j^{'}|\right)^{\frac{1}{\binom{n}{2}}}.$$

It then follows that

$$\left(\prod_{1\le i<j\le n} |z_i - z_j|\right)^{\frac{1}{\binom{n}{2}}}\ -\ \left(\prod_{1\le i<j\le n} |z_i^{'} - z_j^{'}|\right)^{\frac{1}{\binom{n}{2}}}\ \le\ \frac{\epsilon}{2}. \qquad (a.3)$$

Hence, from relations (a.2) and (a.3),

$$\left(\prod_{1\le i<j\le n} |z_i^{'} - z_j^{'}|\right)^{\frac{1}{\binom{n}{2}}}\ \ge\ \text{Cap}\,(a \le |z| \le b)\ -\ \frac{1}{2}\epsilon. \qquad (a.4)$$

But because $z_1^{'},\ z_2^{'},\ \cdots\ z_n^{'}\ \in\ \bigcup_{k=0}^{s-1} e^{2\pi i\frac{k}{s}}[a,b]$,

$$\text{Cap}\left(\bigcup_{k=0}^{s-1} e^{2\pi i\frac{k}{s}}[a,b]\right)\ \ge\ \left(\prod_{1\le i<j\le n} |z_i^{'} - z_j^{'}|\right)^{\frac{1}{\binom{n}{2}}} - \frac{1}{2}\epsilon \qquad (a.5)$$

if n is choosen big enough. This is the value of n referred to above after the observation that the function $f(z_1,\ z_2,\ \cdots\ z_n) := \left(\prod_{1\leq i<j\leq n}|z_i - z_j|\right)^{\frac{1}{\binom{n}{2}}}$ is uniformly continuous for $z_1,\ z_2,\ \cdots\ z_n\ \in\ \{a \leq |z| \leq b\}$. The term $-\frac{1}{2}\epsilon$ above in (a.5) is needed in general because, as previously noted,

$\sup\limits_{z_i,\, z_j\, \in\, E}\left(\prod\limits_{1\leq i<j\leq n}|z_i - z_j|\right)^{\frac{1}{\binom{n}{2}}}$ is a decreasing function of n. Hence (a.4) and (a.5) yield

$$\mathrm{Cap}\left(\bigcup_{k=0}^{s-1} e^{2\pi i\frac{k}{s}}[a,b]\right)\ \geq\ \mathrm{Cap}\,(a \leq |z| \leq b)\ -\ \epsilon.$$

Because this inequality holds for all s big enough,

$$\liminf_{s\to\infty} \mathrm{Cap}\left(\bigcup_{k=0}^{s-1} e^{2\pi i\frac{k}{s}}[a,b]\right)\ \geq\ \mathrm{Cap}\,(a \leq |z| \leq b)\ -\ \epsilon. \qquad (a.6)$$

Because $\epsilon > 0$ can be chosen arbitrarily small, it follows from relations (a.1) and (a.6) that, for $0 < a < b$,

$$\lim_{s\to\infty} \mathrm{Cap}\left(\bigcup_{k=0}^{s-1} e^{2\pi i\frac{k}{s}}[a,b]\right) = \mathrm{Cap}\,(a \leq |z| \leq b)\,,$$

as was to be proved. ■

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