Large deviations for hitting times of some decreasing sets

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Abstract

In this paper we consider a suitable $\mathbb{R}^d$-valued process (Z_t) and a suitable family of nonempty subsets $(A(b) : b > 0)$ of $\mathbb{R}^d$ which, in some sense, decrease to empty set as $b \to \infty$. In general let T_b be the first hitting time of $A(b)$ for the process (Z_t). The main result relates the large deviations principle of $(\frac{T_b}{b})$ as $b \to \infty$ with a large deviations principle concerning (Z_t) which agrees with a generalized version of Mogulskii Theorem. The proof has some analogies with the proof presented in [4] for a similar result concerning nondecreasing univariate processes and their inverses with general scaling function.

1 Introduction

Throughout this paper we consider a suitable $\mathbb{R}^d$-valued process (Z_t) starting at zero and a suitable family of nonempty subsets $(A(b) : b > 0)$ of $\mathbb{R}^d$ which, in some sense, decrease to empty set as $b \to \infty$ (for a precise statement see (**Z**) and (**A**) below in section 2 devoted to some preliminaries). Moreover let us consider the random variables $(T_b : b > 0)$ where, in general, T_b is the first time t at which $Z_t \in A(b)$.

The aim of this paper is to relate the large deviations principle of $(\frac{T_b}{b})$ as $b \to \infty$ with a large deviations principle concerning (Z_t) which agrees with a generalized version of Mogulskii Theorem with a continuous parameter (see (**Z**) with some related remarks in section 2).

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Some propositions and the main result (Theorem 3.3) are presented in section 3. Section 4 is devoted to present some examples which can be related to some queueing problems. In section 5 we present the proofs of the results in section 3. Finally a conjecture on first hitting places $(Z_{T_b} : b > 0)$ is presented in section 6.

The proofs presented in section 5 have some analogies with the proofs presented in [4] (section 6) for similar results concerning nondecreasing univariate processes and their inverses with general scaling function; on the other hand a work with the linear scaling function used in this paper is [6].

2 Preliminaries

In this paper we deal with a $\mathbb{R}^d$-valued process (Z_t) with cadlag path such that $Z_0 = 0$ and the following condition (**Z**) holds. In view of presenting (**Z**) we introduce some notation: let $D[0,1]$ be the family of all $\mathbb{R}^d$-valued functions with cadlag paths defined on $[0,1]$, let $AC[0,1]$ be the family of all $\mathbb{R}^d$-valued absolutely continuous functions defined on $[0,1]$ and let $AC_0[0,1]$ be the set

$$AC_0[0,1] = \{\phi \in AC[0,1] : \phi(0) = 0\}.$$

Moreover we use the notation $\frac{Z_{\alpha\cdot}}{\alpha}$ for the (random) function

$$t \in [0,1] \mapsto \frac{Z_{\alpha t}}{\alpha}$$

in $D[0,1]$.
(**Z**) Let $\xi : \mathbb{R}^d \to [0,\infty]$ be a convex and good rate function which has a unique zero $\underline{m}^{(0)}$ (namely $\xi(\underline{y}) = 0$ if and only if $\underline{y} = \underline{m}^{(0)}$) with $\underline{m}^{(0)} \neq 0$; then $(\frac{Z_{\alpha\cdot}}{\alpha})$ satisfies the large deviations principle in $D[0,1]$ (as $\alpha \to \infty$) with the good rate function

$$\phi \in D[0,1] \mapsto I(\phi) = \begin{cases} \int_0^1 \xi(\dot{\phi}(t))dt & \text{if } \phi \in AC_0[0,1] \\ \infty & \text{if } \phi \notin AC_0[0,1] \end{cases}; \tag{1}$$

indeed we recall that $\phi \in AC[0,1]$ implies that ϕ is differentiable almost everywhere in $[0,1]$.

It is useful to point out some consequences of (**Z**).
First of all, if we consider the function ϕ_0 defined by

$$t \in [0,1] \mapsto \phi_0(t) = t\underline{m}^{(0)},$$

we have $I(\phi) = 0$ if and only if $\phi = \phi_0$. Moreover ξ is the large deviations rate function for $(\frac{Z_\alpha}{\alpha})$ (as $\alpha \to \infty$) as a consequence of contraction principle (see [3], Theorem 4.2.1, page 110).

As far as condition (**Z**) is concerned, we point out that Mogulskii Theorem is a well known result on large deviations for sequences of processes with cadlag path (see e.g. [3], Theorem 5.1.2, page 152; another reference is [5], section 3); another result on large deviations for sequences of processes is Theorem 5.1 in [7] (page 71) which provides a large deviations principle for multidimensional jump Markov processes and the corresponding rate function is, in some sense, more general than I in (1). Furthermore we can say that (**Z**) can be seen as a generalized version of Mogulskii Theorem where α is a continuous parameter; this generalized version holds true when (Z_t) is a quite general Lévy process (see Theorem 1.2 in [2] which deals with a general separable Banach space instead of $\mathbb{R}^d$).

Now we introduce a quite general class of nonempty subsets $(A(b) : b > 0)$ of $\mathbb{R}^d$; more precisely, given a function $\psi : \mathbb{R}^d \to \mathbb{R}$ such that condition (**A**) presented below holds, we set

$$A(b) = \{\underline{y} \in \mathbb{R}^d : \psi(\underline{y}) \geq b\} \quad (\forall b > 0). \tag{2}$$

In particular we set $A = A(1)$ and, for all $b > 0$, the boundary of $A(b)$ is

$$\partial A(b) = \{\underline{y} \in \mathbb{R}^d : \psi(\underline{y}) = b\} \quad (\forall b > 0);$$

moreover we point out that, in some sense, $A(b)$ decreases to emptyset as $b \to \infty$.

Condition (**A**) consists of the following three conditions:
(**A1**) the function ψ is homogeneous of degree 1, namely

$$\psi(\gamma \underline{y}) = \gamma \psi(\underline{y}) \quad (\forall \gamma > 0 \text{ and } \forall \underline{y} \in \mathbb{R}^d),$$

and continuous.
(**A2**) $\psi(\underline{m}^{(0)}) > 0$.
(**A3**) for all $b > 0$

$$\psi(\underline{z}) = b, \ \gamma \in (0,1) \Rightarrow \psi(\underline{z} + \gamma \underline{m}^{(0)}) \geq b$$

which is equivalent to

$$\underline{z} \in \partial A(b), \ \gamma \in (0,1) \Rightarrow \underline{z} + \gamma \underline{m}^{(0)} \in A(b). \tag{3}$$

It is useful to point out some consequences of (**A**).
First of all $\psi(\underline{0}) = 0$ follows from (**A1**). Moreover (**A2**) means that, in some sense, (Z_t) is directed to A; indeed (**A2**) is equivalent to

$$\{\gamma \underline{m}^{(0)} : \gamma > 0\} \cap A(b) \neq \emptyset \quad (\forall b > 0).$$

Finally, as far as (**A3**) is concerned, (3) gives a condition on the shape of the sets $(A(b) : b > 0)$; some examples are presented in section 4. On the other hand, for instance, the function ψ defined by

$$\underline{y} \in \mathbb{R}^d \mapsto \psi(\underline{y}) = \|\underline{y}\|$$

(where $\|\cdot\|$ is the usual norm) satisfies (**A1**) and (**A2**) but (**A3**) fails.

A reference concerning this topic is [1] and (**A2**) can be considered as the opposite of (H2) in [1]; indeed in [1] we have a general sequence of random variables (Y_n) in place of (Z_t) and, in some sense, (Y_n) is directed away from A. Furthermore we point out that the set A in [1] is more general than A in this paper and (**Z**) is not considered in [1] for the two following reasons: it does not allow to handle general sequences of random variables; the approximations derived from (**Z**) on finite time intervals are not sufficient because, in some sense, we have the opposite of (**A2**) for (Y_n).

In this paper we deal with a family of random variables $(T_b : b > 0)$ defined by

$$T_b = \inf\{t \geq 0 : Z_t \in A(b)\} \quad (\forall b > 0);$$

thus, in general, T_b is the first hitting time of $A(b)$ for the process (Z_t).

We remark that each T_b is almost surely finite by (**A2**); moreover we also have

$$\frac{b}{\psi(\underline{m}^{(0)})}\underline{m}^{(0)} \in \partial A(b) \quad (\forall b > 0)$$

so that

$$\ell \underline{m}^{(0)} \in \partial A, \quad \text{where} \quad \ell = \frac{1}{\psi(\underline{m}^{(0)})}. \tag{4}$$

The value ℓ plays a crucial role in what follows; indeed we shall see below that ℓ is the limit of $\frac{T_b}{b}$ as $b \to \infty$ because is the unique base for the corresponding large deviations rate function J (namely we have $J(x) = 0$ if and only if $x = \ell$; for this terminology concerning a rate function, see [4], section 2).

3 A variational formula for J in terms of ξ

The main result in this section is Theorem 3.3 which provides a variational formula for large deviations rate function J of $(\frac{T_b}{b})$ as $b \to \infty$ in terms of ξ.

In order to prove this variational formula it is useful to consider the following subsets of $D[0,1]$ varying $x \in (0, \infty)$:

$$H_x = \{\phi : \phi(s) \notin A(x)^\circ, \forall s \leq 1\}$$

and

$$H^x = \{\phi : \exists s \leq 1 \text{ such that } \phi(s) \in A(x)\}.$$

Proposition 3.1. *Assume that* (**Z**) *and* (**A**) *hold.*
(i) We have

$$-f_+(x^+) \leq \liminf_{\alpha \to \infty} \frac{1}{\alpha} \log P(\frac{Z_{\alpha \cdot}}{\alpha} \in H^x) \leq \limsup_{\alpha \to \infty} \frac{1}{\alpha} \log P(\frac{Z_{\alpha \cdot}}{\alpha} \in H^x) \leq -f_+(x) \tag{5}$$

for all $x \in (0, \infty)$ *and some nondecreasing and lower semicontinuous function* f_+ *on* $(0, \infty)$ *if and only if*

$$-g_-(x^-) \leq \liminf_{b \to \infty} \frac{1}{b} \log P(\frac{T_b}{b} \leq x) \leq \limsup_{b \to \infty} \frac{1}{b} \log P(\frac{T_b}{b} \leq x) \leq -g_-(x) \tag{6}$$

for all $x \in (0, \infty)$ *and some nonincreasing and lower semicontinuous function* g_- *on* $(0, \infty)$*; in such a case*

$$g_-(x) \equiv x f_+(\frac{1}{x}). \tag{7}$$

(ii) We have

$$-f_-(x^-) \leq \liminf_{\alpha \to \infty} \frac{1}{\alpha} \log P(\frac{Z_{\alpha \cdot}}{\alpha} \notin H^x) \leq \limsup_{\alpha \to \infty} \frac{1}{\alpha} \log P(\frac{Z_{\alpha \cdot}}{\alpha} \notin H^x) \leq -f_-(x) \tag{8}$$

for all $x \in (0,\infty)$ *and some nonincreasing and lower semicontinuous function* f_- *on* $(0,\infty)$ *if and only if*

$$-g_+(x^+) \leq \liminf_{b\to\infty} \frac{1}{b} \log P(\frac{T_b}{b} > x) \leq \limsup_{b\to\infty} \frac{1}{b} \log P(\frac{T_b}{b} > x) \leq -g_+(x) \qquad (9)$$

for all $x \in (0,\infty)$ *and some nondecreasing and lower semicontinuous function* g_+ *on* $(0,\infty)$*; in such a case*

$$g_+(x) \equiv x f_-(\frac{1}{x}).$$

(iii) If, with f_- *as in (ii), (8) holds for all* $x \in (0,\infty)$*, then for all* $x \in (0,\infty)$

$$-g_+(x^+) \leq \liminf_{b\to\infty} \frac{1}{b} \log P(\frac{T_b}{b} \geq x) \leq \limsup_{b\to\infty} \frac{1}{b} \log P(\frac{T_b}{b} \geq x) \leq -g_+(x).$$

(iv) If, with g_+ *as in (ii), (9) holds for all* $x \in (0,\infty)$*, then for all* $x \in (0,\infty)$

$$-f_-(x^-) \leq \liminf_{\alpha\to\infty} \frac{1}{\alpha} \log P(\frac{Z_{\alpha\cdot}}{\alpha} \in H_x) \leq \limsup_{\alpha\to\infty} \frac{1}{\alpha} \log P(\frac{Z_{\alpha\cdot}}{\alpha} \in H_x) \leq -f_-(x).$$

By referring to the functions f_- and f_+ in Proposition 3.1, as in [4] (see eq. (100) in the proof of Theorem 1) for all $x \in (0,\infty)$ we can set

$$f_-(x) \equiv \inf\{I(\phi) : \phi \in H_x\} \qquad (10)$$

and

$$f_+(x) \equiv \inf\{I(\phi) : \phi \in H^x\}. \qquad (11)$$

Then the next result gives an alternative expression for the functions f_- and f_+.

Proposition 3.2. *Assume that* $(\mathbf{Z})$ *and* $(\mathbf{A})$ *hold. Moreover let* f_- *and* f_+ *be the functions defined by (10) and (11) respectively. Then we have*

$$x \in (0,\infty) \mapsto f_-(x) = \begin{cases} \inf\{\xi(\underline{z}) : \underline{z} \in \partial A(x)\} & \text{if } x < \ell^{-1} \\ 0 & \text{if } x \geq \ell^{-1} \end{cases}$$

and

$$x \in (0,\infty) \mapsto f_+(x) = \begin{cases} 0 & \text{if } x \leq \ell^{-1} \\ \inf\{\xi(\underline{z}) : \underline{z} \in \partial A(x)\} & \text{if } x > \ell^{-1} \end{cases}.$$

Thus we immediately have the following

Theorem 3.3. *Assume that* $(\mathbf{Z})$ *and* $(\mathbf{A})$ *hold. Then the large deviations principle holds for* $(\frac{T_b}{b})$ *as* $b \to \infty$ *and, for the corresponding rate function* J *defined on* $(0,\infty)$*, we have*

$$J(x) \equiv x \inf\{\xi(\underline{y}) : \underline{y} \in \partial A(\frac{1}{x})\} \equiv \inf\{x\xi(\frac{\underline{y}}{x}) : \underline{y} \in \partial A\}.$$

The next result shows that the expressions of J in Theorem 3.3 are more explicit when $d = 1$; indeed ∂A is reduced to a single point (see Appendix for details) and, if in general we simply write down y instead of $\underline{y} \in \mathbb{R}^d = \mathbb{R}$, we have

$$\partial A = \{y \in \mathbb{R} : \psi(y) = 1\} = \{\ell m^{(0)}\}$$

by (4).

Corollary 3.4. *Assume that* (**Z**) *and* (**A**) *hold and let be* $d = 1$. *Then, for all* $x \in (0, \infty)$, *we have*

$$J(x) \equiv x\xi(\frac{\ell m^{(0)}}{x}).$$

In the section 5 we present the proofs of Proposition 3.1 and Proposition 3.2. The proof of Theorem 3.3 immediately follows from Proposition 3.2 together with some results in [4] (namely Theorem 13 and Lemma 4). Indeed we still point out that we adapt in a suitable way the content of section 6 in [4]; in particular Proposition 3.1 here plays the role of Theorem 12 in [4].

We conclude with some differences between [4] and this paper. In [4] (Z_t) is a nondecreasing (univariate) process while in this paper (Z_t) is a (possibly multivariate) process such that (**Z**) and (**A**) hold. We also remark that the hypotheses on (Z_t) in this paper are more general than monotonicity of (Z_t) (monotonicity with respect to each component of (Z_t), if (Z_t) is multivariate) and we only obtain the large deviations principle for $(\frac{T_b}{b})$ starting from the large deviations principle for $(\frac{Z_{\alpha\cdot}}{\alpha})$ but not vice versa as in [4].

4 Examples

The results in [6], which relate the large deviations behaviour of counting processes and their inverses, are motivated by applications to queues (see the references cited therein); in this section we motivate the results in this paper by applications to some similar queueing problems concerning d queues with $d > 1$.

In order to do that let us consider $d > 1$ and the notation

$$Z_t \equiv (Z_t^{(1)}, \ldots, Z_t^{(d)}) \quad (\forall t > 0)$$

for the random variables of the process (Z_t); moreover, for each $\underline{y} \in \mathbb{R}^d$, we use the notation

$$\underline{y} = (y_1, \ldots, y_d).$$

Example 1. Let $\underline{a} \in \mathbb{R}^d$ be such that $a_1, \ldots, a_d > 0$ and set

$$\underline{y} = (y_1, \ldots, y_d) \in \mathbb{R}^d \mapsto \psi(\underline{y}) = \max\{\frac{y_i}{a_i} : i \in \{1, \ldots, d\}\}.$$

Thus, for all $b > 0$, T_b is the first time t at which at least one among the events

$$(\{Z_t^{(i)} \geq ba_i\} : i \in \{1, \ldots, d\})$$

occurs. We point out that (**A1**) holds, (**A2**) is equivalent to $\max\{m_i^{(0)} : i \in \{1, \ldots, d\}\} > 0$ and, as far as (**A3**) is concerned, a further restriction on $\underline{m}^{(0)}$ is needed: $m_1^{(0)}, \ldots, m_d^{(0)} > 0$.

Example 2. Let $\underline{a} \in \mathbb{R}^d$ be such that $a_1, \ldots, a_d > 0$ and set

$$\underline{y} = (y_1, \ldots, y_d) \in \mathbb{R}^d \mapsto \psi(\underline{y}) = \min\{\frac{y_i}{a_i} : i \in \{1, \ldots, d\}\}.$$

Thus, for all $b > 0$, T_b is the first time t at which all the events

$$(\{Z_t^{(i)} \geq ba_i\} : i \in \{1, \dots, d\})$$

occur. We point out that (**A1**) holds, (**A2**) is equivalent to $m_1^{(0)}, \dots, m_d^{(0)} > 0$ which implies (**A3**).

Example 3. Let $\underline{a} \in \mathbb{R}^d \setminus \{\underline{0}\}$ be arbitrarily fixed (where $\underline{0}$ is the null vector) and set

$$\underline{y} = (y_1, \dots, y_d) \in \mathbb{R}^d \mapsto \psi(\underline{y}) = \langle \underline{a}, \underline{y} \rangle.$$

Thus, for all $b > 0$, T_b is the first time t at which the event $\langle \underline{a}, Z_t \rangle \geq b$ occurs. We point out that (**A1**) holds, (**A2**) is $\langle \underline{a}, \underline{m}^{(0)} \rangle > 0$ which implies (**A3**); indeed, for all $b > 0$, $\psi(\underline{z}) = \langle \underline{a}, \underline{z} \rangle = b$ and $\gamma \in (0, 1)$ implies

$$\psi(\underline{z} + \gamma \underline{m}^{(0)}) = \langle \underline{a}, \underline{z} + \gamma \underline{m}^{(0)} \rangle = \underbrace{\langle \underline{a}, \underline{z} \rangle}_{=b} + \gamma \underbrace{\langle \underline{a}, \underline{m}^{(0)} \rangle}_{>0} \geq b.$$

The hypothesis of monotonicity for the processes $(Z_t^{(1)}), \dots, (Z_t^{(d)})$ is not necessary for applying the results in this paper to (Z_t). On the other hand, in view of applications to queues which extends in some sense the applications of the results in [6], here we think $(Z_t^{(1)}), \dots, (Z_t^{(d)})$ as counting processes related to the d queues.

Then $(T_b : b > 0)$ in example 1 refer to some first level crossing times concerning at least one among the d queues; $(T_b : b > 0)$ in example 2 refer to some level crossing times concerning all the d queues; if $a_1, \dots, a_d > 0$ are weights associated with each queue, $(T_b : b > 0)$ in example 3 refer to some level crossing times concerning the process $(\tilde{Z}_t)$ defined by

$$\tilde{Z}_t \equiv \langle \underline{a}, Z_t \rangle \equiv \sum_{k=1}^{d} a_k Z_t^{(k)},$$

i.e. the weighted sum of the processes $(Z_t^{(1)}), \dots, (Z_t^{(d)})$.

5 The proofs

Proof of Proposition 3.1. We prove (i) in one direction; the reverse is similar. For all $\alpha, x > 0$ we have

$$\{\frac{Z_{\alpha\cdot}}{\alpha} \in H^{1/x}\} = \{\exists s \leq 1 : \frac{Z_{\alpha s}}{\alpha} \in A(\frac{1}{x})\} = \{\exists s \leq 1 : \psi(\frac{Z_{\alpha s}}{\alpha}) \geq \frac{1}{x}\} =$$

$$= \{\exists s \leq 1 : \psi(Z_{\alpha s}) \geq \frac{\alpha}{x}\} = \{\exists s \leq 1 : Z_{\alpha s} \in A(\frac{\alpha}{x})\} = \{T_{\alpha/x} \leq \alpha\} = \{\frac{T_{\alpha/x}}{\alpha/x} \leq x\}$$

whence we obtain

$$\frac{1}{\alpha} \log P(\frac{Z_{\alpha\cdot}}{\alpha} \in H^{1/x}) = \frac{1}{x} \frac{1}{\alpha/x} \log P(\frac{T_{\alpha/x}}{\alpha/x} \leq x)$$

and (6) and (7) follow upon taking $\alpha \to \infty$ (and hence $b \to \infty$, with $b = \alpha/x$).

Moreover g_- is lower semicontinuous because f_+ is lower semicontinuous and then it remains to be shown that g_- is nonincreasing. By its lower semicontinuity the function g is discontinuous at most on a dense set Δ in $\mathbb{R}_+$ (as motivated in the proof of Theorem 12 in [4]) and, for $x \in \Delta$, lim inf and lim sup in (6) are both equal to $-g_-(x)$. So, by construction, the restriction of g_- to Δ is also nonincreasing as a limit of nonincreasing functions $x \mapsto -\frac{1}{b}\log P(\frac{T_b}{b} \leq x)$.

Now we show that g_- is nonincreasing on the whole $\mathbb{R}_+$. Since Δ is dense, then for all $x, x' \in \Delta^c$ with $x < x'$ there exists $y \in \Delta$ such that $x < y < x'$. The it is sufficies to show that, for all such x, x', y, we have

$$g_-(x) \geq g_-(y) \geq g_-(x'). \tag{12}$$

For the second inequality in (12) we remark that

$$g_-(y) \geq \liminf_{\Delta \ni a \nearrow x'} g_-(a) \geq g_-(x')$$

because g_- is nonincreasing on Δ (for the first inequality) and g_- is lower semicontinuous (for the second inequality).
For the first inequality in (12), we remark that f_+ is continuous from the left: indeed in general we have

$$f_+(b) \leq \liminf_{a \nearrow b} f_+(a) \leq \limsup_{a \nearrow b} f_+(a) \leq f_+(b)$$

because f_+ is lower semicontinuous (for the first inequality) and nondecreasing (for the third inequality); thus g_- is continuous from the right by (7) and then

$$g_-(x) = \lim_{\Delta \ni a \searrow x} g_-(a) \geq g_-(y)$$

where the inequality holds because g_- is nondecreasing on Δ.
(ii) The proof is similar to the proof of (i).
(iii) If (8) holds, (9) holds for (ii) and we have the trivial lower bound

$$\liminf_{b\to\infty} \frac{1}{b} \log P(\frac{T_b}{b} \geq x) \geq \liminf_{b\to\infty} \frac{1}{b} \log P(\frac{T_b}{b} > x) \leq -g_+(x^+).$$

To obtain the complementary upper bound, note that for all $\varepsilon > 0$ we have

$$\limsup_{b\to\infty} \frac{1}{b} \log P(\frac{T_b}{b} \geq x) \leq \limsup_{b\to\infty} \frac{1}{b} \log P(\frac{T_b}{b} > x - \varepsilon) \leq -g_+(x - \varepsilon)$$

by (9) and the result follows from the lower semicontinuity of g_+ by taking $\varepsilon \to 0$.
(iv) The proof is similar to the proof of (iii). $\diamondsuit$

Proof of Proposition 3.2. First of all we point out two consequences of (10) and (11) respectively:

$$x \geq \ell^{-1} \Rightarrow \phi_0 \in H_x \Rightarrow f_-(x) = \inf\{I(\phi) : \phi \in H_x\} = 0;$$

$$x \leq \ell^{-1} \Rightarrow \phi_0 \in H^x \Rightarrow f_+(x) = \inf\{I(\phi) : \phi \in H^x\} = 0.$$

Thus we have two remaining cases:
(a) $f_-(x) = \inf\{\xi(\underline{z}) : \underline{z} \in \partial A(x)\}$ for $x < \ell^{-1}$;

(b) $f_+(x) = \inf\{\xi(\underline{z}) : \underline{z} \in \partial A(x)\}$ for $x > \ell^{-1}$.

Case **(a)**.
Let $\phi \in H_x$ be arbitrarily fixed, with $x < \ell^{-1}$. In order to avoid the trivial case $I(\phi) = \infty$, we assume that $\phi \in AC_0[0,1]$. Then we have

$$I(\phi) = \int_0^1 \xi(\dot{\phi}(t))dt \geq \xi(\int_0^1 \dot{\phi}(t)dt) = \xi(\phi(1) - \phi(0)) = \xi(\phi(1));$$

indeed the inequality follows from Jensen inequality because ξ is convex. Then, if we consider the function g_ϕ is defined by

$$t \in [0,1] \mapsto g_\phi(t) = t\phi(1),$$

we have $I(\phi) \geq \xi(\phi(1)) = I(g_\phi)$ and $g_\phi \in H_x$ because $\phi(1) \notin A(x)^\circ$; thus

$$\inf\{I(\phi) : \phi \in H_x\} = \inf\{\xi(\underline{y}) : \underline{y} \notin A(x)^\circ\}. \tag{13}$$

Moreover $\underline{m}^{(0)} \in A(x)^\circ$ when $x < \ell^{-1}$ and, for $\underline{y} \notin A(x)^\circ$, there exists $\alpha \in (0,1]$ such that

$$\underline{z} = \alpha\underline{y} + (1-\alpha)\underline{m}^{(0)} \in \partial A(x)$$

by the continuity of ψ in (**A1**); thus

$$\xi(\underline{z}) = \xi(\alpha\underline{y} + (1-\alpha)\underline{m}^{(0)}) \leq \alpha\xi(\underline{y}) + (1-\alpha)\underbrace{\xi(\underline{m}^{(0)})}_{=0} = \alpha\xi(\underline{y}) \leq \xi(\underline{y})$$

by the convexity of ξ. Then, since we have $\partial A(x) \subset (A(x)^\circ)^c$, we obtain

$$\inf\{\xi(\underline{y}) : \underline{y} \notin A(x)^\circ\} = \inf\{\xi(\underline{z}) : \underline{z} \in \partial A(x)\}. \tag{14}$$

In conclusion the statement **(a)** is proved by (10), (13) and (14).

Case **(b)**.
Let $\phi \in H^x$ be arbitrarily fixed, with $x > \ell^{-1}$. In order to avoid the trivial case $I(\phi) = \infty$, we assume that $\phi \in AC_0[0,1]$. For such a function ϕ let t_ϕ be defined by

$$t_\phi = \sup\{t \in [0,1] : \phi(t) \in A(x)\}.$$

We remark that $\phi(t_\phi) \in A(x)$ because of ϕ is continuous and $A(x)$ is a closed set; moreover

$$t_\phi < 1 \Rightarrow \phi(1) \notin A(x) \ \text{ and } \ \phi(t_\phi) \in \partial A(x). \tag{15}$$

Then we have

$$I(\phi) = \int_0^1 \xi(\dot{\phi}(t))dt \geq \int_0^{t_\phi} \xi(\dot{\phi}(t))dt$$

by the nonnegativeness of ξ and we obtain

$$I(\phi) \geq t_\phi \int_0^{t_\phi} \xi(\dot{\phi}(t))\frac{dt}{t_\phi} \geq t_\phi\xi(\int_0^{t_\phi} \dot{\phi}(t)\frac{dt}{t_\phi}) = t_\phi\xi(\frac{\phi(t_\phi) - \phi(0)}{t_\phi}) = t_\phi\xi(\frac{\phi(t_\phi)}{t_\phi});$$

the second inequality follows from Jensen inequality because ξ is convex.
By taking into account $\xi(\underline{m}^{(0)}) = 0$ and the convexity of ξ, we have another inequality:

$$I(\phi) \geq t_\phi\xi(\frac{\phi(t_\phi)}{t_\phi}) = t_\phi\xi(\frac{\phi(t_\phi)}{t_\phi}) + (1-t_\phi)\xi(\underline{m}^{(0)}) \geq$$

$$\geq \xi(t_\phi \frac{\phi(t_\phi)}{t_\phi} + (1-t_\phi)\underline{m}^{(0)}) = \xi(\phi(t_\phi) + (1-t_\phi)\underline{m}^{(0)}).$$

Now let us consider the function h_ϕ defined by

$$t \in [0,1] \mapsto h_\phi(t) = t[\phi(t_\phi) + (1-t_\phi)\underline{m}^{(0)}]$$

and let us prove that $h_\phi \in H^x$. This follows from $h_\phi(1) \in A(x)$ which can be proved by considering two distinct cases: when $t_\phi = 1$ we have

$$h_\phi(1) = \phi(1) = \phi(t_\phi) \in A(x)$$

because $\phi(t_\phi) \in A(x)$ as pointed out above; when $t_\phi < 1$ we have

$$h_\phi(1) = \phi(t_\phi) + (1-t_\phi)\underline{m}^{(0)} \in A(x)$$

by (15) and (**A3**) (in particular for (**A3**) we have to refer to (3)).
Thus we have $I(\phi) \geq \xi(\phi(t_\phi) + (1-t_\phi)\underline{m}^{(0)}) = \xi(h_\phi(1)) = I(h_\phi)$ and $h_\phi \in H^x$ because $h_\phi(1) \in A(x)$; then we have

$$\inf\{I(\phi) : \phi \in H^x\} = \inf\{\xi(\underline{y}) : \underline{y} \in A(x)\}. \tag{16}$$

Moreover $\underline{m}^{(0)} \notin A(x)$ when $x > \ell^{-1}$ and, for $\underline{y} \in A(x)$, there exists $\alpha \in (0,1]$ such that

$$\underline{z} = \alpha\underline{y} + (1-\alpha)\underline{m}^{(0)} \in \partial A(x)$$

by the continuity of ψ in (**A1**); thus

$$\xi(\underline{z}) = \xi(\alpha\underline{y} + (1-\alpha)\underline{m}^{(0)}) \leq \alpha\xi(\underline{y}) + (1-\alpha)\underbrace{\xi(\underline{m}^{(0)})}_{=0} = \alpha\xi(\underline{y}) \leq \xi(\underline{y})$$

by the convexity of ξ. Then, since we have $\partial A(x) \subset A(x)$, we obtain

$$\inf\{\xi(\underline{y}) : \underline{y} \in A(x)\} = \inf\{\xi(\underline{z}) : \underline{z} \in \partial A(x)\}. \tag{17}$$

In conclusion the statement **(b)** is proved by (11), (16) and (17). $\diamondsuit$

6 A conjecture on first hitting places $(Z_{T_b} : b > 0)$

It is known that $\underline{m}^{(0)}$ in (**Z**) is the limit of $\frac{Z_t}{t}$ as $t \to \infty$. Moreover ℓ is the limit of $\frac{T_b}{b}$ as $b \to \infty$ because ℓ is the unique base for J; indeed by Theorem 3.3 and (4) we have

$$0 \leq J(\ell) = \inf\{\ell\xi(\frac{\underline{y}}{\ell}) : \underline{y} \in \partial A\} \leq \ell\xi(\frac{\ell\underline{m}^{(0)}}{\ell}) = \ell\xi(\underline{m}^{(0)}) = 0$$

and, as far as the uniqueness is concerned, by Theorem 3.3 we have

$$J(x) = 0 \Leftrightarrow \underline{m}^{(0)} \in \partial A(\frac{1}{x}) \Leftrightarrow x\underline{m}^{(0)} \in \partial A \Leftrightarrow x = \ell.$$

In conclusion we have

$$\lim_{b\to\infty}(\frac{Z_{T_b}}{b}, \frac{T_b}{b}) = (\ell\underline{m}^{(0)}, \ell).$$

The author thinks that, under some possible further hypotheses, $((\frac{Z_{T_b}}{b}, \frac{T_b}{b}))$ satisfies the large deviations principle (as $b \to \infty$) and the corresponding rate function L should be

$$L(\underline{y}, x) \equiv \begin{cases} x\xi(\frac{\underline{y}}{x}) & \text{if } x > 0 \text{ and } \underline{y} \in \partial A \\ \infty & \text{otherwise} \end{cases};$$

moreover the large deviations rate function K for $(\frac{Z_{T_b}}{b})$ should be

$$K(\underline{y}) \equiv \begin{cases} \inf\{x\xi(\frac{\underline{y}}{x}) : x > 0\} & \text{if } \underline{y} \in \partial A \\ \infty & \text{if } \underline{y} \notin \partial A \end{cases}$$

as a consequence of contraction principle (see e.g. [3], Theorem 4.2.1, page 110).

Appendix: ∂A reduced to a single point when $d = 1$

In general, when $d = 1$, we simply write down y instead of $\underline{y} \in \mathbb{R}^d = \mathbb{R}$. Then, by taking into account (**A1**), we have

$$y \in \mathbb{R} \mapsto \psi(y) = \begin{cases} y\psi(1) & \text{if } y > 0 \\ 0 & \text{if } y = 0 \\ -y\psi(-1) & \text{if } y < 0 \end{cases}.$$

We recall that the sets $(A(b) : b > 0)$ are nonempty; thus the case $\psi(1), \psi(-1) \leq 0$ is not allowed by (2). Moreover the case $\psi(1), \psi(-1) > 0$ is not allowed by (**A3**); indeed we would have

$$A = (-\infty, -\frac{1}{\psi(-1)}] \cup [\frac{1}{\psi(1)}, \infty)$$

and we can have $\underline{z} + \gamma m^{(0)} \notin A$ for suitable choices of $z \in \partial A$ and $\gamma \in (0, 1)$, so that (3) would fail for $b = 1$.

In conclusion we can have two cases:
$\psi(1) > 0$ and $\psi(-1) \leq 0$ when $m^{(0)} > 0$ and the unique point of ∂A is $\frac{1}{\psi(1)} = \ell m^{(0)}$;
$\psi(-1) > 0$ and $\psi(1) \leq 0$ when $m^{(0)} < 0$ and the unique point of ∂A is $-\frac{1}{\psi(-1)} = \ell m^{(0)}$.

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