Greguš type fixed points for weakly compatible maps

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Abstract

In this paper we prove a common fixed point theorem of Greguš contraction type by using minimal type commutativity without continuity requirements. The theorem extends known results on compatible continuous maps of [1]-[3], [7], [8] and [9], to a larger class of mappings.

1 Introduction

The Banach contraction theorem is useful, but the hypothesis of that theorem are very strong and may be difficult to satisfy. The literature tells us that there are examples of functions which are not continuous but have **fixed points**. May be the most simple and surprising function among them is the monster of Dirichlet defined on $\mathbb{R}$ by,

$$f(x) = \begin{cases} 1 & if \quad x \in \mathbb{Q} \,(: the\, rational\ numbers) \\ 0 & if \quad x \notin \mathbb{Q} \end{cases}$$

Clearly, this function is nowhere continuous, and hence cannot be contraction, but have $x_0 = 1$ as a fixed point. Note here that $f^2(x) = 1$(: f^2 the composition of f with itself), hence f^2 is trivially a contraction mapping on $\mathbb{R}$. This shows that a function need not be continuous for Banach contraction theorem to apply. Going by the spirit of this observation, several authors proved fixed point theorems for contractive conditions without continuity requirements, and the result was further generalized and extended by other authors.

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In [4], G. Jungck introduced the concept of compatible maps, which is a generalization of commuting maps. Self maps $\mathcal{A}$ and $\mathcal{S}$ of a normed space $(\mathcal{X},\|.\|)$ are said to be compatible ([4]) iff $\|\mathcal{AS}x_n - \mathcal{SA}x_n\| \to 0$ whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\mathcal{A}x_n,\ \mathcal{S}x_n \to t \in \mathcal{X}$. Many authors proved various fixed point theorems using this definition. The concept of compatible maps of type (A) is defined by G. Jungck, P. P. Murthy and Y. J. Cho [5] .$\mathcal{A}$ and $\mathcal{S}$ above are compatible of type (A) if we have $\|\mathcal{AS}x_n - \mathcal{SS}x_n\| \to 0$, $\|\mathcal{SA}x_n - \mathcal{AA}x_n\| \to 0$ whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\mathcal{A}x_n,\ \mathcal{S}x_n \to t \in \mathcal{X}$. This definition is equivalent to the concept of compatible maps under some conditions and examples are given to show that the two notions are independent. Fixed point theorems of Greguš type in Banach space are proved by P. P. Murthy, Y. J. Cho and B. Fisher [7] for compatible maps of type (A).

$\mathcal{A}$ and $\mathcal{S}$ above are said to be compatible of type (B) ([9]), if we have

$$\begin{aligned}
\lim_n \|\mathcal{SA}x_n - \mathcal{AA}x_n\| &\le \frac{1}{2}\lim_n \left[\|\mathcal{SA}x_n - \mathcal{S}t\| + \ \|\mathcal{S}t - \mathcal{SS}x_n\|\right] \\
&\quad \textit{and} \\
\lim_n \|\mathcal{AS}x_n - \mathcal{SS}x_n\| &\le \frac{1}{2}\lim_n \left[\|\mathcal{AS}x_n - \mathcal{A}t\| + \|\mathcal{A}t - \mathcal{AA}x_n\|\right]
\end{aligned}$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\mathcal{A}x_n,\ \mathcal{S}x_n \to t \in \mathcal{X}$. We say that $\mathcal{A}$, $\mathcal{S}$ above are compatible of type (C) ([8]), if we have

$$\begin{aligned}
\lim_n \|\mathcal{AS}x_n - \mathcal{SS}x_n\| &\le \frac{1}{3}\lim_n \left[\ \|\mathcal{AS}x_n - \mathcal{A}t\| + \|\mathcal{A}t - \mathcal{SS}x_n\| + \right. \\
&\quad \left. \|\mathcal{A}t - \mathcal{AA}x_n\|\ \right] \\
&\quad \textit{and} \\
\lim_n \|\mathcal{SA}x_n - \mathcal{AA}x_n\| &\le \frac{1}{3}\lim_n \left[\ \|\mathcal{SA}x_n - \mathcal{S}t\| + \|\mathcal{S}t - \mathcal{AA}x_n\| + \right. \\
&\quad \left. \|\mathcal{S}t - \mathcal{SS}x_n\|\ \right]
\end{aligned}$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\mathcal{A}x_n,\ \mathcal{S}x_n \to t \in \mathcal{X}$. As obvious from the definitions, compatible maps of type (B) and (C) generalize those of compatible maps of type (A). Fixed point theorems of Greguš type in Banach space are proved in [9] for compatible maps of type (B) and in [8] for compatible maps of type (C).

In a recent paper Jungck and Rhoades [6] defined weakly compatible maps and showed that compatible maps are weakly compatible but converse need not be true. The purpose of the present paper is to prove fixed point theorem of Greguš type contraction for weakly compatible maps without need of continuity, thus we extend the results of [1]-[3], Murthy et al. [7], Pathak et al. [8] and Pathak et al [9] to wider class of mappings.

2 Preliminaries.

Definition 1. *([6])The self maps $\mathcal{A}$ and $\mathcal{S}$ of a normed space $(\mathcal{X},\|.\|)$ are called* **weakly compatible** *if $\mathcal{A}x = \mathcal{S}x$ implies $\mathcal{A}\mathcal{S}x = \mathcal{S}\mathcal{A}x$.*

Thus, two self maps $\mathcal{A}$ and $\mathcal{S}$ can fail to be weakly compatible only if there is some x in $\mathcal{X}$ such that $\mathcal{A}x = \mathcal{S}x$ but $\mathcal{A}\mathcal{S}x \neq \mathcal{S}\mathcal{A}x$, that is only if they possess a coincidence point at which they do not commute. But since a common fixed point is also a coincidence point, this means that weakly compatible maps is the minimal condition for contractive maps to have common fixed point.

Lemma 2. *([4]-[7], [8], [9])Let $\mathcal{A}$ and $\mathcal{S}$ be self maps of a normed space $(\mathcal{X},\|.\|)$. If $\mathcal{A}$ and $\mathcal{S}$ are compatible, compatible of type (A) (resp. type (B) or type (C)), then $\mathcal{A}$ and $\mathcal{S}$ are weakly compatible.*

However, as we shall show in the following example, there exist weakly compatible maps which are neither compatible nor compatible of type (A) (resp. compatible of type (B), compatible of type (C)).

Example 3. *Let $\mathcal{X} = [2, 20]$ with the usual metric. Define*

$$\mathcal{A}(x) = \begin{cases} 2 & x = 2 \\ 13 + x & 2 < x \leq 5 \\ x - 3 & x > 5 \end{cases} ; \mathcal{S}(x) = \begin{cases} 2 & x \in 2 \cup (5, 20] \\ 8 & 2 < x \leq 5 \end{cases}$$

Let (x_n) be the sequence defined by $x_n = 5 + \frac{1}{n}$, $n \geq 1$. Then

$$\mathcal{A}x_n = x_n - 3 \to 2; \quad \mathcal{S}x_n = 2\,; \ \mathcal{S}2 = \mathcal{A}2 = 2 = t; \ \mathcal{S}\mathcal{A}2 = \mathcal{A}\mathcal{S}2$$

Clearly $\mathcal{A}$ and $\mathcal{S}$ are weakly compatible maps, since they commute at coincidence point at $x = 2$. On the other hand

$$\begin{array}{ll} \mathcal{S}\mathcal{A}x_n = \mathcal{S}(x_n - 3) \to 8; & \mathcal{A}\mathcal{S}x_n = 2; \\ \mathcal{S}\mathcal{S}x_n = 2; & \mathcal{A}\mathcal{A}x_n = \mathcal{A}(x_n - 3) = 13 + x_n - 3 \to 15 \end{array}$$

Consequently, $|\mathcal{S}\mathcal{A}x_n - \mathcal{A}\mathcal{S}x_n| \to 6 \neq 0$, that is, $\mathcal{A}$ and $\mathcal{S}$ are not compatible.

$$|\mathcal{A}\mathcal{S}x_n - \mathcal{S}\mathcal{S}x_n| = 0, \ |\mathcal{S}\mathcal{A}x_n - \mathcal{A}\mathcal{A}x_n| \to 7 \neq 0$$

Thus, $\mathcal{A}$ and $\mathcal{S}$ are not compatible of type (A). Furthermore,

$$7 = \lim_n |\mathcal{S}\mathcal{A}x_n - \mathcal{A}\mathcal{A}x_n| \nleq \frac{1}{2}\left[\lim_n |\mathcal{S}\mathcal{A}x_n - \mathcal{S}t| + \lim_n |\mathcal{S}t - \mathcal{S}\mathcal{S}x_n|\right] = 3$$

This tells that $\mathcal{A}$ and $\mathcal{S}$ are not compatible of type (B). Finally,

$$\begin{aligned} 7 &= \lim_n |\mathcal{S}\mathcal{A}x_n - \mathcal{A}\mathcal{A}x_n| \\ &\nleq \frac{1}{3}\left[\lim_n |\mathcal{S}\mathcal{A}x_n - \mathcal{S}t| + \lim_n |\mathcal{S}t - \mathcal{S}\mathcal{S}x_n| + \lim_n |\mathcal{S}t - \mathcal{A}\mathcal{A}x_n|\right] = \frac{19}{3} \end{aligned}$$

hence, the maps $\mathcal{A}$ and $\mathcal{S}$ are not compatible of type (C).

3 Fixed point theorem.

Let $\mathbb{R}^+$ be the set of non-negative real numbers and F the family of mappings φ from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that φ is upper semi-continuous, nondecreasing and $\varphi(t) < t$ for any $t > 0$.

Lemma 4. *(see [10]) For any $t > 0$, $\varphi(t) < t$ if and only if $\lim_n \varphi^n(t) = 0$, where φ^n denotes the n-times repeated composition of φ with itself.*

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$ be mappings from a normed space $\mathcal{X}$ into itself such that

$$(3.1)\quad \mathcal{A}(\mathcal{X}) \subset \mathcal{T}(\mathcal{X}) \text{ and } \mathcal{B}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$$

$$\begin{aligned}(3.2)\quad \|\mathcal{A}x - \mathcal{B}y\|^p \le \varphi(a\,\|\mathcal{S}x - \mathcal{T}y\|^p + (1-a)\max\{\, & \alpha\,\|\mathcal{S}x - \mathcal{A}x\|^p, \\ & \beta\,\|\mathcal{T}y - \mathcal{B}y\|^p, \|\mathcal{S}x - \mathcal{A}x\|^{\frac{p}{2}} . \|\mathcal{T}y - \mathcal{A}x\|^{\frac{p}{2}}, \\ & \|\mathcal{T}y - \mathcal{A}x\|^{\frac{p}{2}}\,\|\mathcal{S}x - \mathcal{B}y\|^{\frac{p}{2}}, \\ & \frac{1}{2}\,[\|\mathcal{S}x - \mathcal{A}x\|^p + \|\mathcal{T}y - \mathcal{B}y\|^p]\,\}).\end{aligned}$$

for all x, y in $\mathcal{X}$, where $0 < a \le 1$, $0 < \alpha, \beta \le 1$, $p \ge 1$, and $\varphi \in F$. Then, by (3.1) since $\mathcal{A}(\mathcal{X}) \subset \mathcal{T}(\mathcal{X})$, for an arbitrary $x_0 \in \mathcal{X}$ there exists a point $x_1 \in \mathcal{X}$ such that $\mathcal{A}(x_0) = \mathcal{T}(x_1)$. Furthermore, since $\mathcal{B}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$, for this point x_1 one can choose $x_2 \in \mathcal{X}$ such that $\mathcal{B}(x_1) = \mathcal{S}(x_2)$. Continuing in this way, we can define inductively the sequence

$$(3.3)\quad y_{2n} = \mathcal{A}x_{2n} = \mathcal{T}x_{2n+1} \text{ and } \quad y_{2n+1} = \mathcal{B}x_{2n+1} = \mathcal{S}x_{2n+2},$$

for every $n = 0, 1, 2, ...$

Lemma 5. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$ be mappings from a normed space $\mathcal{X}$ into itself which satisfy the conditions (3.1) and (3.2). Then $\lim_n \|y_n - y_{n+1}\| = 0$, where $\{y_n\}$ is the sequence in $\mathcal{X}$ defined by (3.3).*

Proof. By (3.2) and (3.3) we have

$$\begin{aligned}\|y_{2n} - y_{2n+1}\|^p &= \|\mathcal{A}x_{2n} - \mathcal{B}x_{2n+1}\| \\ &\le \varphi(a\,\|y_{2n-1} - y_{2n}\|^p + (1-a)\max\{\,\alpha\,\|y_{2n-1} - y_{2n}\|^p, \\ &\quad \beta\,\|y_{2n} - y_{2n+1}\|^p, \|y_{2n-1} - y_{2n}\|^{\frac{p}{2}} . \|y_{2n} - y_{2n}\|^{\frac{p}{2}}, \\ &\quad \|y_{2n} - y_{2n}\|^{\frac{p}{2}} . \|y_{2n-1} - y_{2n}\|^{\frac{p}{2}}, \\ &\quad \frac{1}{2}\,[\|y_{2n-1} - y_{2n}\|^p + \|y_{2n} - y_{2n+1}\|^p]\,\})\end{aligned}$$

If $\|y_{2n} - y_{2n+1}\| > \|y_{2n-1} - y_{2n}\|$ in the above inequality, then we have

$$\begin{aligned}\|y_{2n} - y_{2n+1}\|^p &\le \varphi(a\,\|y_{2n} - y_{2n+1}\|^p + (1-a)\max\{\,\alpha\,\|y_{2n} - y_{2n+1}\|^p, \\ &\quad \beta\,\|y_{2n} - y_{2n+1}\|^p, 0, 0, \\ &\quad \frac{1}{2}\,[\|y_{2n} - y_{2n+1}\|^p + \|y_{2n} - y_{2n+1}\|^p]\,\}). \\ &\le \varphi(\|y_{2n} - y_{2n+1}\|^p) \\ &< \|y_{2n} - y_{2n+1}\|^p\end{aligned}$$

which is a contradiction. Thus, we have

$$\|y_{2n} - y_{2n+1}\|^p \le \varphi(\|y_{2n-1} - y_{2n}\|^p)$$

Similarly, we have

$$\|y_{2n+1} - y_{2n+2}\|^p \le \varphi(\|y_{2n} - y_{2n+1}\|^p)$$

So, we have by induction

$$\|y_{2n} - y_{2n+1}\|^p \le \varphi(\|y_{2n-1} - y_{2n}\|^p) \le \dots \le \varphi^{2n}(\|y_0 - y_1\|^p),$$

and so from Lemma 4, we have

$$\lim_n \|y_n - y_{n+1}\| = 0$$

This completes the proof ∎

Lemma 6. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$ be mappings from a normed space $\mathcal{X}$ into itself satisfying the conditions (3.1) and (3.2). Then the sequence $\{y_n\}$ defined by (3.3) is a Cauchy one.*

Proof. By virtue of Lemma 5, it suffices to show that $\{y_{2n}\}$ is Cauchy. Suppose not. Then, there is an $\varepsilon > 0$ such that for any integer $2k$, there exist even integers $2m(k)$ and $2n(k)$ with $2m(k) > 2n(k) \ge 2k$ such that

$$\left\|y_{2m(k)} - y_{2n(k)}\right\| > \varepsilon \tag{3.4}$$

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (3.4), that is

$$\left\|y_{2m(k)-2} - y_{2n(k)}\right\| \le \varepsilon \text{ and } \left\|y_{2m(k)} - y_{2n(k)}\right\| > \varepsilon. \tag{3.5}$$

Then, for each integer $2k$, we have

$$\begin{aligned}\varepsilon &< \left\|y_{2n(k)} - y_{2m(k)}\right\| \\ &\le \left\|y_{2n(k)} - y_{2m(k)-2}\right\| + \left\|y_{2m(k)-2} - y_{2m(k)-1}\right\| + \left\|y_{2m(k)-1} - y_{2m(k)}\right\|.\end{aligned}$$

It follows from Lemma 5 and (3.5) that, as $k \to \infty$

$$\left\|y_{2m(k)} - y_{2n(k)}\right\| \to \varepsilon \tag{3.6}$$

By the triangle inequality, we have

$$\left\|y_{2n(k)} - y_{2m(k)-1}\right\| - \left\|y_{2n(k)} - y_{2m(k)}\right\| \le \left\|y_{2m(k)-1} - y_{2m(k)}\right\|$$

Therefore, $\limsup \left\|y_{2n(k)} - y_{2m(k)-1}\right\| \le \varepsilon$, by (3.6) and Lemma 5. Similarly, $\left\|y_{2n(k)} - y_{2m(k)}\right\| \le \left\|y_{2n(k)} - y_{2m(k)-1}\right\| + \left\|y_{2m(k)-1} - y_{2m(k)}\right\| \Rightarrow$

$\liminf \left\| y_{2n(k)} - y_{2m(k)-1} \right\| \geq \varepsilon$. Thus, $\lim_{k\to\infty} \left\| y_{2n(k)} - y_{2m(k)-1} \right\| = \varepsilon$.

In like manner, Lemma 5, (3.6), and the inequality

$$\begin{aligned} \left\| y_{2n(k)+1} - y_{2m(k)-1} \right\| - \left\| y_{2n(k)} - y_{2m(k)} \right\| &\leq \left\| y_{2m(k)-1} - y_{2m(k)} \right\| + \\ & \left\| y_{2n(k)} - y_{2n(k)+1} \right\|. \end{aligned}$$

imply $\left\| y_{2n(k)} - y_{2m(k)-1} \right\| \to \varepsilon$.

We thus have,

$$\left\| y_{2n(k)} - y_{2m(k)-1} \right\| \to \varepsilon \text{ and } \left\| y_{2n(k)+1} - y_{2m(k)-1} \right\| \to \varepsilon \tag{3.7}$$

Therefore, by (3.2) and (3.3), we have

$$\begin{aligned} \left\| y_{2n(k)} - y_{2m(k)} \right\| &\leq \left\| y_{2n(k)} - y_{2n(k)+1} \right\| + \left\| \mathcal{A}x_{2m(k)} - \mathcal{B}x_{2n(k)+1} \right\| \\ &\leq \left\| y_{2n(k)} - y_{2n(k)+1} \right\| + [\ \varphi(a \left\| y_{2m(k)-1} - y_{2n(k)} \right\|^p + \\ & (1-a)\max\{\ \alpha \left\| y_{2m(k)-1} - y_{2m(k)} \right\|^p, \\ & \beta \left\| y_{2n(k)} - y_{2n(k)+1} \right\|^p, \\ & \left\| y_{2m(k)-1} - y_{2m(k)} \right\|^{\frac{p}{2}} . \left\| y_{2m(k)} - y_{2n(k)} \right\|^{\frac{p}{2}}, \\ & \left\| y_{2m(k)} - y_{2n(k)} \right\|^{\frac{p}{2}} . \left\| y_{2m(k)-1} - y_{2n(k)+1} \right\|^{\frac{p}{2}}, \\ & \frac{1}{2}\left[\left\| y_{2m(k)-1} - y_{2m(k)} \right\|^p + \left\| y_{2n(k)} - y_{2n(k)+1} \right\|^p \right] \}) \ ]^{\frac{1}{p}} \end{aligned}$$

Since $\varphi \in F$, by Lemma 5, (3.6) and (3.7), we have, as $k \to \infty$,

$$\varepsilon \leq \left[\varphi\left(a\varepsilon^p + (1-a)\max\{0,0,0,\varepsilon^p,0\} \right) \right]^{\frac{1}{p}} < \varepsilon \tag{3.8}$$

which is a contradiction. Therefore, $\{y_{2n}\}$ is Cauchy sequence in $\mathcal{X}$. This ends the proof ∎

Theorem 7. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$ be mappings from a Banach space $\mathcal{X}$ into itself having the conditions (3.1) and (3.2) such that one of $\mathcal{A}(\mathcal{X})$ or $\mathcal{B}(\mathcal{X})$ is closed. If the pairs $\{\mathcal{A},\mathcal{S}\}$ and $\{\mathcal{B},\mathcal{T}\}$ are weakly compatible, then $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point in $\mathcal{X}$.*

Proof. By Lemma 6, $\{y_n\}$ is Cauchy sequence in $\mathcal{X}$. Since $\mathcal{X}$ is complete, $\{y_n\}$ converges to some element $z \in \mathcal{X}$, as do subsequences $\{\mathcal{A}x_{2n}\}$, $\{\mathcal{B}x_{2n+1}\}$, $\{\mathcal{T}x_{2n+1}\}$ and $\{\mathcal{S}x_{2n}\}$ of $\{y_n\}$ i.e.

$$\lim_n \mathcal{A}x_{2n} = \lim_n \mathcal{B}x_{2n+1} = \lim_n \mathcal{T}x_{2n+1} = \lim_n \mathcal{S}x_{2n} = z$$

Suppose that $\mathcal{B}(\mathcal{X})$ is closed, since $\mathcal{B}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$, there exists then a point $u \in \mathcal{X}$ such that $z = \mathcal{S}u$. Using (3.2), we obtain

$$\begin{aligned} \left\| \mathcal{A}u - \mathcal{B}x_{2n+1} \right\|^p &\leq \varphi(a \left\| \mathcal{S}u - \mathcal{T}x_{2n+1} \right\|^p + (1-a)\max\{\ \alpha \left\| \mathcal{S}u - \mathcal{A}u \right\|^p, \\ & \beta \left\| \mathcal{T}x_{2n+1} - \mathcal{B}x_{2n+1} \right\|^p, \left\| \mathcal{S}u - \mathcal{A}u \right\|^{\frac{p}{2}} . \left\| \mathcal{T}x_{2n+1} - \mathcal{A}u \right\|^{\frac{p}{2}}, \\ & \left\| \mathcal{T}x_{2n+1} - \mathcal{A}u \right\|^{\frac{p}{2}} \left\| \mathcal{S}u - \mathcal{B}x_{2n+1} \right\|^{\frac{p}{2}}, \\ & \frac{1}{2}\left[\left\| \mathcal{S}u - \mathcal{A}u \right\|^p + \left\| \mathcal{T}x_{2n+1} - \mathcal{B}x_{2n+1} \right\|^p \right] \}) \end{aligned}$$

By letting $n \to \infty$ in the above inequality, it comes

$$\begin{aligned}\|\mathcal{A}u - z\|^p &\le \varphi((1-a)\max\{\,\alpha\,\|z - \mathcal{A}u\|^p, 0, \|z - \mathcal{A}u\|^p, 0, \\ &\quad \frac{1}{2}\|z - \mathcal{A}u\|^p\,\}) \\ &\le \varphi(\|z - \mathcal{A}u\|^p)\end{aligned}$$

which is a contradiction. Thus, $z = \mathcal{A}u = \mathcal{S}u$. But the pair of maps $\{\mathcal{A}, \mathcal{S}\}$ are weakly compatible, then $\mathcal{AS}u = \mathcal{SA}u$ i.e, $\mathcal{A}z = \mathcal{S}z$. We claim that z is a fixed point of $\mathcal{A}$, hence for $\mathcal{S}$. Suppose not. Then by (3.2), we get

$$\begin{aligned}\|\mathcal{A}z - \mathcal{B}x_{2n+1}\|^p &\le \varphi(a\,\|\mathcal{S}z - \mathcal{T}x_{2n+1}\|^p + (1-a)\max\{\,\alpha\,\|\mathcal{S}z - \mathcal{A}z\|^p, \\ &\quad \beta\,\|\mathcal{T}x_{2n+1} - \mathcal{B}x_{2n+1}\|^p, \|\mathcal{S}z - \mathcal{A}z\|^{\frac{p}{2}}.\|\mathcal{T}x_{2n+1} - \mathcal{A}z\|^{\frac{p}{2}}, \\ &\quad \|\mathcal{T}x_{2n+1} - \mathcal{A}z\|^{\frac{p}{2}}\,\|\mathcal{S}z - \mathcal{B}x_{2n+1}\|^{\frac{p}{2}}, \\ &\quad \frac{1}{2}[\|\mathcal{S}z - \mathcal{A}z\|^p + \|\mathcal{T}x_{2n+1} - \mathcal{B}x_{2n+1}\|^p]\,\})\end{aligned}$$

Therefore, since φ is u.s.c.,

$$\begin{aligned}\|\mathcal{A}z - z\|^p &\le \varphi(a\,\|\mathcal{A}z - z\|^p + (1-a)\max\{\,0, 0, \\ &\quad 0, \|\mathcal{A}z - z\|^p, 0\,\}) \\ &\le \varphi(\|z - \mathcal{A}z\|^p)\end{aligned}$$

Which is a contradiction. Thus, $z = \mathcal{A}z = \mathcal{S}z$.

Now, $\mathcal{A}(\mathcal{X}) \subset \mathcal{T}(\mathcal{X})$ implies that $\mathcal{A}z = \mathcal{T}v$, for some $v \in \mathcal{X}$. Consequently, $z = \mathcal{A}z = \mathcal{S}z = \mathcal{T}v$. Then, using (3.2) again, we have

$$\begin{aligned}\|z - \mathcal{B}v\|^p &= \|\mathcal{A}z - \mathcal{B}v\|^p \\ &\le \varphi(a\,\|\mathcal{S}z - \mathcal{T}v\|^p + (1-a)\max\{\,\alpha\,\|\mathcal{S}z - \mathcal{A}z\|^p, \\ &\quad \beta\,\|\mathcal{T}v - \mathcal{B}v\|^p, \|\mathcal{S}z - \mathcal{A}z\|^{\frac{p}{2}}.\|\mathcal{T}v - \mathcal{A}z\|^{\frac{p}{2}}, \\ &\quad \|\mathcal{T}v - \mathcal{A}z\|^{\frac{p}{2}}\,\|\mathcal{S}z - \mathcal{B}v\|^{\frac{p}{2}}, \\ &\quad \frac{1}{2}[\|\mathcal{S}z - \mathcal{A}z\|^p + \|\mathcal{T}v - \mathcal{B}v\|^p]\,\})\end{aligned}$$

It follows that

$$\begin{aligned}\|z - \mathcal{B}v\|^p &\le \varphi((1-a)\max\left\{0, \beta\,\|z - \mathcal{B}v\|^p, 0, 0, \frac{1}{2}\|z - \mathcal{B}v\|^p\right\} \\ &= \varphi(\|z - \mathcal{B}v\|^p)\end{aligned}$$

Which is a contradiction, so we have $z = \mathcal{B}v = \mathcal{T}v$. Thus, $z = \mathcal{B}v = \mathcal{T}v = \mathcal{A}z = \mathcal{S}z$. But pair of maps $\{\mathcal{B}, \mathcal{T}\}$ are weakly compatible, so $\mathcal{BT}v = \mathcal{TB}v$ i.e. $\mathcal{B}z = \mathcal{T}z$. Moreover,

$$\begin{aligned}\|z - \mathcal{B}z\|^p &= \|\mathcal{A}z - \mathcal{B}z\|^p \\ &\le \varphi(a\,\|\mathcal{S}z - \mathcal{T}z\|^p + (1-a)\max\{\,\alpha\,\|\mathcal{S}z - \mathcal{A}z\|^p, \\ &\quad \beta\,\|\mathcal{T}z - \mathcal{B}z\|^p, \|\mathcal{S}z - \mathcal{A}z\|^{\frac{p}{2}}.\|\mathcal{T}z - \mathcal{A}z\|^{\frac{p}{2}}, \\ &\quad \|\mathcal{T}z - \mathcal{A}z\|^{\frac{p}{2}}\,\|\mathcal{S}z - \mathcal{B}z\|^{\frac{p}{2}}, \frac{1}{2}[\|\mathcal{S}z - \mathcal{A}z\|^p + \|\mathcal{T}z - \mathcal{B}z\|^p]\,\}) \\ &= \varphi(a\,\|z - \mathcal{B}z\|^p + (1-a)\max\{0, 0, 0, \|z - \mathcal{B}z\|^p, 0\}) \\ &= \varphi(\|z - \mathcal{B}z\|^p)\end{aligned}$$

Consequently, we have $z = \mathcal{B}z$. Hence, $z = \mathcal{B}z = \mathcal{T}z = \mathcal{A}z = \mathcal{S}z$, and z is common fixed point of both $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$, and $\mathcal{T}$.

Finally, we prove that z is unique. For, let w be another common fixed point of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$, and $\mathcal{T}$ such that $z \neq w$. Then, by (3.2) we have,

$$\begin{aligned} \|z - w\|^p &= \|\mathcal{A}z - \mathcal{B}w\|^p \\ &\leq \varphi(a\,\|\mathcal{S}z - \mathcal{T}w\|^p + (1-a)\max\{\,\alpha\,\|\mathcal{S}z - \mathcal{A}z\|^p\,, \\ &\quad \beta\,\|\mathcal{T}w - \mathcal{B}w\|^p\,, \|\mathcal{S}z - \mathcal{A}z\|^{\frac{p}{2}}\,.\,\|\mathcal{T}w - \mathcal{A}z\|^{\frac{p}{2}}\,, \\ &\quad \|\mathcal{T}w - \mathcal{A}z\|^{\frac{p}{2}}\,\|\mathcal{S}z - \mathcal{B}w\|^{\frac{p}{2}}\,, \\ &\quad \frac{1}{2}\,[\|\mathcal{S}z - \mathcal{A}z\|^p + \|\mathcal{T}w - \mathcal{B}w\|^p]\,\}) \\ &= \varphi(a\,\|z - w\|^p + (1-a)\max\{0, 0, 0, \|z - w\|^p\,, 0\} \\ &= \varphi\,(\|z - w\|^p) \end{aligned}$$

Therefore, $z = w$. Similarly, one can obtain this conclusion by supposing $\mathcal{A}(\mathcal{X})$ is closed. ■

If we let $\mathcal{A} = \mathcal{B}$, and $\mathcal{S} = \mathcal{T}$ in Theorem 6, then we get the following:

Corollary 8. *Let $\mathcal{A}$ and $\mathcal{S}$ be mappings from a Banach space $\mathcal{X}$ into itself having the conditions*

$$\mathcal{A}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X}) \tag{3.9}$$

$$\begin{aligned} \|\mathcal{A}x - \mathcal{A}y\|^p \leq \varphi(a\,\|\mathcal{S}x - \mathcal{S}y\|^p &+ (1-a)\max\{\,\alpha\,\|\mathcal{S}x - \mathcal{A}x\|^p\,, \\ &\beta\,\|\mathcal{S}y - \mathcal{A}y\|^p\,, \|\mathcal{S}x - \mathcal{A}x\|^{\frac{p}{2}}\,.\,\|\mathcal{S}y - \mathcal{A}x\|^{\frac{p}{2}}\,, \\ &\|\mathcal{S}y - \mathcal{A}x\|^{\frac{p}{2}}\,\|\mathcal{S}x - \mathcal{A}y\|^{\frac{p}{2}}\,, \\ &\frac{1}{2}\,[\|\mathcal{S}x - \mathcal{A}x\|^p + \|\mathcal{S}y - \mathcal{A}y\|^p]\,\}). \end{aligned} \tag{3.10}$$

for all x, y in $\mathcal{X}$, where $0 < a \leq 1$, $0 < \alpha, \beta \leq 1$, $p \geq 1$, and $\varphi \in F$. If the pairs of maps $\{\mathcal{A}, \mathcal{S}\}$ are weakly compatible and $\mathcal{A}(\mathcal{X})$ is closed, then $\mathcal{A}$ and $\mathcal{S}$ have a unique common fixed point in $\mathcal{X}$.

If we put in Theorem 6 $\mathcal{A} = \mathcal{B}$, and $\mathcal{S} = \mathcal{T} = \mathcal{I}_X$ (: the identity mapping of $\mathcal{X}$) and we drop the closedness condition, then we get the corollary:

Corollary 9. *Let $\mathcal{A}$ be a mapping from a Banach space $\mathcal{X}$ into itself satisfying the condition*

$$\begin{aligned} \|\mathcal{A}x - \mathcal{A}y\|^p \leq \varphi(a\,\|x - y\|^p &+ (1-a)\max\{\,\alpha\,\|x - \mathcal{A}x\|^p\,, \\ &\beta\,\|y - \mathcal{A}y\|^p\,, \|x - \mathcal{A}x\|^{\frac{p}{2}}\,.\,\|y - \mathcal{A}x\|^{\frac{p}{2}}\,, \\ &\|y - \mathcal{A}x\|^{\frac{p}{2}}\,\|x - \mathcal{A}y\|^{\frac{p}{2}}\,, \frac{1}{2}\,[\|x - \mathcal{A}x\|^p + \|y - \mathcal{A}y\|^p]\,\}). \end{aligned}$$

for all x, y in $\mathcal{X}$, where $0 < a \leq 1$, $0 < \alpha, \beta \leq 1$, $p \geq 1$, and $\varphi \in F$. Then $\mathcal{A}$ has a unique fixed point in $\mathcal{X}$.

Remarks

1. Theorem 7 remains valid if we have $\mathcal{T}(\mathcal{X})$ or $\mathcal{S}(\mathcal{X})$ is closed (resp. $\mathcal{S}$ or $\mathcal{T}$ is surjective) in lieu of $\mathcal{A}(\mathcal{X})$ or $\mathcal{B}(\mathcal{X})$ is closed.

2. By letting $p = 1$ and $\alpha = \beta = 1$ in Corollaries 8 and 9, then we can obtain more corollaries .

3. In Corollary 9, if we let $\varphi(t) = kt$, $0 \leq k < 1$, $p = 1$, $a = 1$, we get the Banach's fixed point theorem. The analogue of Corollary 9 in [8] is Corollary 3.8 where it is noted by Example 3.1 there that, in his hypothesis, the continuity requirement on $\mathcal{A}$ can not be eliminated which is not the case in our consideration.

4. It is obvious that Theorem 7 is a generalization of the result of H. K Pathak et al [8], since no continuity hypothesis is assumed here. Further, in view of the example given above, our theorem apply to a wider class of mappings than the results on other type of compatible maps since they constitute a proper subclass of weakly compatible maps. In fact weak compatibility is the least condition for maps to have common fixed point. Thus, our theorem generalizes and extends also main results of [1]-[3], [7] and [9].

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