Difference methods for quasilinear hyperbolic differential functional systems on the Haar pyramid

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Abstract

The paper deals with the local Cauchy problem for first order partial functional differential systems. A general class of difference methods is constructed. The convergence of explicit difference schemes is proved by means of consistency and stability arguments. It is assumed that the given functions satisfy nonlinear estimates of Perron type with respect to functional variables. Differential systems with deviated variables and differential integral problems can be obtained from a general case by specializing the given operators. The results are illustrated by numerical examples.

1 Introduction

For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. We denote by $M_{k\times n}$ the space of all real $k\times n$ matrices. For $x=(x_1,\dots,x_n)\in R^n$, $p=(p_1,\dots,p_k)\in R^k$ and $X\in M_{k\times n}$, $X=[x_{ij}]_{i=1,\dots,k,\,j=1,\dots,n}$, we put

$$||x||=\sum_{i=1}^{n}|x_i|,\quad ||p||=\max\{|p_i|:1\le i\le k\},$$

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$$||X|| = \max\{\sum_{j=1}^{n} |x_{ij}| : 1 \le i \le k\}.$$

Unless otherwice noted, we use in the paper the above norms and they are denoted by the same symbol $||\cdot||$. Let E be the Haar pyramid

$$E = \Big\{ (t,x) = (t, x_1, \ldots, x_n) \in R^{1+n} : \ t \in [0,a], \ \ x \in [-b + Mt, \ b - Mt] \Big\}$$

where $a > 0$, $b = (b_1, \ldots, b_n)$, $M = (M_1, \ldots, M_n) \in R^n_+$, $R_+ = [0, +\infty)$, and $b > Ma$. Write

$$E_0 = [-r_0, 0] \times [-b, \, b], \ \ E_t = (E_0 \cup E) \cap ([-r_0, t] \times R^n), \ \ 0 < t \le a,$$

where $r_0 \in R_+$, and

$$S_t = [-b, \, b] \ \text{ for } \ t \in [-r_0, 0], \quad S_t = [-b + Mt, \ b - Mt] \ \text{ for } \ t \in [0, a].$$

Set $\Omega = E \times C(E_0 \cup E, \, R^k)$ and assume that

$$\varrho : \Omega \to M_{k\times n}, \quad \varrho = [\varrho_{ij}]_{i=1,\ldots,k,\, j=1,\ldots,n},$$

$$f : \Omega \to R^k, \ f = (f_1, \ldots, f_k) \quad \varphi : E_0 \to R^k, \ \varphi = (\varphi_1, \ldots, \varphi_k)$$

are given functions. We consider the system of differential functional equations

$$\partial_t z_i(t,x) = \sum_{j=1}^{n} \varrho_{ij}(t,x,z)\, \partial_{x_j} z_i(t,x) + f_i(t,x,z), \ \ 1 \le i \le k, \tag{1}$$

with the initial condition

$$z(t,x) = \varphi(t,x) \quad \text{for} \quad (t,x) \in E_0. \tag{2}$$

Let us denote by $z|_{E_t}$, $0 \le t \le a$, the restriction of the function $z : E_0 \cup E \to R^k$ to the set E_t. The function $\varrho : \Omega \to M_{k\times n}$ is said to satisfy the Volterra condition if for each $(t,x) \in E$ and for z, $\bar{z} \in C(E_0 \cup E, R^k)$ such that

$$z|_{E_t} = \bar{z}|_{E_t} \ \text{ we have } \ \varrho(t,x,z) = \varrho(t,x,\bar{z}).$$

Note that the Volterra condition for ϱ means that the value of ϱ at the point (t,x,z) of the space Ω depends on (t,x) and on the restriction of z to the set E_t. In the same way we define the Volterra condition for f. In the paper we assume that ϱ and f satisfy the Volterra condition and we consider classical solutions of the above problem.

Numerical methods for nonlinear first order partial differential functional equations were considered by many authors and under various assumptions. Difference methods for initial boundary value problems were studied in [7], [11]. Initial problems on the Haar pyramid and a general class of difference schemes with suitable interpolating operators were considered in [6], [12], [16]. The convergence of difference methods for functional parabolic problems was studied in [10], [13] - [15]. The main problem in these investigations is to find a difference functional problem which is stable and satisfies consistency conditions with respect to the original problem.

The method of difference inequalities or simple theorems on reccurent inequalities are used in the investigations of the stability.

The numerical method of lines for partial differential functional equations was considered in [17], [18]. By using a discretization with respect to the spatial variable, the partial differential equation with a functional dependence is replaced by a sequence of ordinary functional differential equations with initial conditions. The proof of the convergence of the method of lines is based on differential inequalities techniques. For further bibliographic information concerning numerical methods for partial functional differential equations see the survey paper [2] and the monograph [8].

The results given in [6], [7], [11] for nonlinear functional differential problems are not applicable to quasilinear systems of the form (1). In the paper we prove that there is a class of difference methods for (1), (2) which are convergent. The stability of the methods is investigated by using a theorem on recurrent inequalities. We give a few numerical examples.

Differential systems with deviated variables and differential integral systems can be obtained by specializing the operators ϱ and f. Existence results for quasilinear hyperbolic problems are given in [1], [3], [5], [8]. For bibliography on applications of functional partial differential equations see the monograph [8] and the survey paper [9].

2 Discretization

We denote by $\mathbf{F}(A, B)$ the class of all functions defined on A and taking values in B, where A and B are arbitrary sets. Let $\mathbf{N}$ and $\mathbf{Z}$ be the sets of natural numbers and integers respectively. For $x, \bar{x} \in R^n$, $x = (x_1, \dots, x_n)$, $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$, we write $x * \bar{x} = (x_1\bar{x}_1, \dots, x_n\bar{x}_n)$. We define a mesh on the set $E_0 \cup E$ in the following way. Let $h = (h_0, h')$ where $h' = (h_1, \dots, h_n)$ stand for steps of the mesh. Denote by Δ the set of all $h = (h_0, h')$ such that there exist $\tilde{N}_0 \in \mathbf{Z}$ and $N = (N_1, \dots, N_n) \in \mathbf{Z}^n$ with the properties: $\tilde{N}_0 h_0 = r_0$ and $N * h' = b$. We assume that $\Delta \neq \emptyset$ and that there exists a sequence $\{h^{(j)}\}$, $h^{(j)} \in \Delta$, such that $\lim_{j\to+\infty} h^{(j)} = 0$. We define nodal points as follows:

$$t^{(i)} = ih_0, \quad x^{(m)} = m * h', \quad x^{(m)} = (x_1^{(m_1)}, \dots, x_n^{(m_n)})$$

where $(i, m) \in \mathbf{Z}^{1+n}$. Define $N_0 \in \mathbf{N}$ as follows: $N_0 h_0 \leq a < (N_0 + 1)h_0$.

Let

$$R_h^{1+n} = \{(t^{(i)}, x^{(m)}) : \ (i, m) \in \mathbf{Z}^{1+n}\}$$

and

$$E_h = E \cap R_h^{1+n}, \quad E_{0.h} = E_0 \cap R_h^{1+n},$$

$$E'_h = \{ \, (t^{(i)}, x^{(m)}) \in E_h : \ \ (t^{(i)} + h_0, x^{(m)}) \in E_h \, \}.$$

For a function $z : E_{0.h} \cup E_h \to R^k$ we write $z^{(i,m)} = z(t^{(i)}, x^{(m)})$. Now we formulate a difference problem corresponding to (1), (2). Let $e_j = (0, \dots, 0, 1, 0, \dots, 0) \in R^n$, $1 \leq j \leq n$, where 1 is the j-th coordinate and let $w : E_{0.h} \cup E_h \to R$. We consider difference operators $\delta_0, \delta = (\delta_1, \dots, \delta_n)$ defined in the following way:

$$\delta_0 w^{(i,m)} = \frac{1}{h_0}\left(w^{(i+1,m)} - Aw^{(i,m)} \right), \quad Aw^{(i,m)} = \frac{1}{2n}\sum_{j=1}^{n}\left(w^{(i,m+e_j)} + w^{(i,m-e_j)} \right), \tag{3}$$

$$\delta_j w^{(i,m)} = \frac{1}{2h_j}\left(w^{(i,m+e_j)} - w^{(i,m-e_j)} \right), \quad 1 \le j \le n. \tag{4}$$

For a function $z = (z_1, \dots, z_k) : E_{0.h} \cup E_h \to R^k$ we write

$$\delta_0 z^{(i,m)} = \left(\delta_0 z_1^{(i,m)}, \dots, \delta_0 z_k^{(i,m)} \right).$$

Put $\Omega_h = E'_h \times \mathbf{F}(E_{0.h} \cup E_h, R^k)$ and assume that

$$\varrho_h : \Omega_h \to M_{k\times n}, \quad \varrho_h = [\, \varrho_{h.ij} \,]_{i=1,\dots,k,\, j=1,\dots,n}\,,$$

$$f_h : \Omega_h \to R^k, \; f_h = (f_{h.1}, \dots, f_{h.k}), \quad \varphi_h : E_{0.h} \to R^k, \; \varphi_h = (\varphi_{h.1}, \dots, \varphi_{h.k})$$

are given functions. Let the operator $F_h = (F_{h.1}, \dots, F_{h.k})$ be defined by

$$F_{h.\nu}[\, z\,]^{(i,m)} = \sum_{j=1}^{m} \varrho_{h.\nu j}(\, t^{(i)}, x^{(m)}, z\,)\, \delta_j z_\nu^{(i,m)} + f_{h.\nu}(\, t^{(i)}, x^{(m)}, z\,), \quad 1 \le \nu \le k. \tag{5}$$

We will approximate classical solutions of problem (1), (2) by means of solutions of the difference problem

$$\delta_0 z^{(i,m)} = F_h[\, z\,]^{(i,m)}, \tag{6}$$

$$z^{(i,m)} = \varphi_h^{(i,m)} \;\text{ on }\; E_{0.h}. \tag{7}$$

We assume that the steps of the mesh satisfy the condition $h' \le Mh_0$. Now we formulate the Volterra condition for the operator F_h. Put

$$E_{i.h} = \{\, (t^{(j)}, x^{(m)}) \in E_{0.h} \cup E_h : \quad j \le i \,\}$$

where $0 \le i \le N_0$. The function ϱ_h is said to satisfy the Volterra condition if for each $(t^{(i)}, x^{(m)}) \in E'_h$ and for $z, \bar z \in \mathbf{F}(E_{0.h} \cup E_h, R^k)$ such that $z = \bar z$ on $E_{i.h}$ we have

$$\varrho_h(t^{(i)}, x^{(m)}, z) = \varrho_h(t^{(i)}, x^{(m)}, \bar z).$$

In the same way we define the Volterra condition for f_h.

If ϱ_h and f_h satisfy the Volterra condition then the relation $h' \le Mh_0$ implies that there exists exactly one solution $u_h = (u_{h.1}, \dots, u_{h.k}) : E_{0.h} \cup E_h \to R^k$ of problem (6), (7). Indeed, suppose that there is a solution of the above problem on $E_{i.h}$, $0 \le i < N_0$, and $(t^{(i+1)}, x^{(m)}) \in E_h$. Then condition $h' \le Mh_0$ implies that

$$(t^{(i)}, x^{(m+e_j)}), \; (t^{(i)}, x^{(m-e_j)}) \in E_{0.h} \cup E_h \;\text{ for }\; 1 \le j \le n.$$

It follows from (3)-(7) that $u_h^{(i+1,m)}$ can be calculated and consequently u_h is defined on $E_{i+1.h}$. Then by induction the solution exists and it is unique on $E_{0.h} \cup E_h$.

The motivation for the definition of the set E'_h is the following. Approximate solutions of problem (1), (2) are functions defined on $E_{0.h} \cup E_h$. We write equation

(6) at each point $(t^{(i)}, x^{(m)})$ of the set E'_h and we calculate all the values of u_h on $E_{0.h} \cup E_h$.

Suppose that $v : E_0 \cup E \to R^k$ is a solution of the functional differential problem (1), (2). Let $v_h = v\,|_{E_{0.h} \cup E_h}$. For each $h \in \Delta$ there exists $\alpha(h)$ such that

$$|| \, u_h^{(i,m)} - v_h^{(i,m)} \, || \leq \alpha(h) \;\text{ on }\; E_h. \tag{8}$$

The above inequality gives the error estimate for the numerical method (6), (7). Suppose that there exists a function $\alpha_0 : \Delta \to R_+$, such that

$$||\varphi^{(i,m)} - \varphi_h^{(i,m)}|| \leq \alpha_0(h) \;\text{ on }\; E_{0.h} \;\text{ and }\; \lim_{h \to 0} \alpha_0(h) = 0.$$

We say that method (6), (7) is convergent if there is $\alpha : \Delta \to R_+$ such that condition (8) holds and $\lim_{h\to 0} \alpha(h) = 0$.

For a function $z : E_{0.h} \cup E_h \to R^k$ we write

$$\| \, z \, \|_{i.h} = \max \{ \, \|z^{(j,m)}\| : \;\; (t^{(j)}, x^{(m)}) \in E_{i.h} \, \}$$

where $0 \leq i \leq N_0$. Let $I_0 = [-r_0, 0]$, $I = [0, a]$ and

$$I_{0.h} = \{ \, t^{(i)} : \; -\tilde{N}_0 \leq i \leq 0 \, \}, \quad I_h = \{ \, t^{(i)} : \;\; 0 \leq i \leq N_0 \, \}, \quad I'_h = I_h \setminus \{ \, t^{(N_0)} \, \}.$$

For a function $\omega : I_{0.h} \cup I_h \to R$ we write $\omega^{(i)} = \omega(t^{(i)})$, $-\tilde{N}_0 \leq i \leq N_0$, and

$$\|\omega\|_{i.h} = \max \{ |\omega^{(j)}| : \;\; -\tilde{N}_0 \leq j \leq i \, \}.$$

In the sequel we will need the following operator

$$V_h : \mathbf{F}(E_{0.h} \cup E_h, R^k) \to \mathbf{F}(I_{0.h} \cup I_h, R_+).$$

If $z : E_{0.h} \cup E_h \to R_k$, then $V_h[z]$ is given by

$$V_h[z](t^{(i)}) = \max \{ \, \|z^{(i,m)}\| : \;\; (t^{(i)}, x^{(m)}) \in E_{0.h} \cup E_h \, \},$$

where $-\tilde{N}_0 \leq i \leq N_0$.

3 Functional difference equations

Now we formulate general conditions for the convergence of method (6), (7). Our result will be proved by means of consistency and stability arguments.

Assumption H $[\sigma_h]$**.** Suppose that the function $\sigma_h : I'_h \times \mathbf{F}(I_{.h} \cup I_h, R_+) \to R_+$ satisfies the conditions:

1) σ_h is nondecreasing with respect to the functional variable and fulfils the Volterra condition,

2) $\sigma_h(t, \theta_h) = 0$ for $t \in I'_h$ where $\theta_h^{(i)} = 0$ for $-\tilde{N}_0 \leq i \leq N_0$ and the difference problem

$$\eta^{(i+1)} = \eta^{(i)} + h_0 \, \sigma_h(\, t^{(i)}, \eta \,) \quad \text{for} \;\; 0 \leq i \leq N_0 - 1, \tag{9}$$

$$\eta^{(i)} = 0 \quad \text{for} \quad -\tilde{N}_0 \leq i \leq 0, \tag{10}$$

is stable in the following sense: if $\eta_h : I_{0.h} \cup I_h \to R_+$ is a solution of the problem

$$\eta^{(i+1)} = \eta^{(i)} + h_0\, c\, \sigma_h(\, t^{(i)}, \eta\,) + h_0 \gamma(h) \quad \text{for } \ 0 \le i \le N_0 - 1, \tag{11}$$

$$\eta^{(i)} = \alpha_0(h) \quad \text{for} \quad -\tilde{N}_0 \le i \le 0, \tag{12}$$

where $c \ge 1$ and

$$\alpha_0,\ \gamma : \Delta \to R_+, \quad \lim_{h \to 0} \alpha_0(h) = 0, \quad \lim_{h\to 0} \gamma(h) = 0,$$

then there exists a function $\beta : \Delta \to R_+$ such that

$$\eta_h^{(i)} \le \beta(h) \ \text{ for } \ 0 \le i \le N_0 \quad \text{and} \quad \lim_{h \to 0} \beta(h) = 0.$$

Assumption H $[\varrho_h, f_h]$**.** Suppose that the functions ϱ_h and f_h satisfy the Volterra condition and there is a function $\sigma_h : I'_h \times \mathbf{F}(I_{0.h} \cup I_h, R_+) \to R_+$ satisfying Assumption H $[\sigma_g]$ and such that

$$\|\varrho_h(t,x,z) - \varrho_h(t,x,\bar{z})\| \le \sigma_h(\, t, V_h[z - \bar{z}]\,),$$

$$\|f_h(t,x,z) - f_h(t,x,\bar{z})\| \le \sigma_h(\, t, V_h[z - \bar{z}]\,),$$

on Ω_h.

Remark 3.1. *The functions ϱ_h and f_h are generated by ϱ and f and corresponding interpolating operators. Adequate examples are given in Section 4.*

Now we formulate a theorem on the convergence of method (6), (7).

Theorem 3.2. *Suppose that Assumption H $[\varrho_h, f_h]$ is satisfied and*
1) $h \in \Delta$ and

$$\frac{1}{n} - \frac{h_0}{h_j}\, |\varrho_{h.\nu j}(t,x,z)\,| \ge 0 \ \text{ on } \ \Omega_h \ \text{ for } \ 1 \le \nu \le k, \ \ 1 \le j \le n, \tag{13}$$

2) $h' \le M h_0$ and $u_h : E_{0.h} \cup E_h \to R^k$ is the solution of the difference functional problem (6), (7) and there exists a function $\alpha_0 : \Delta \to R_+$ such that

$$\|\varphi^{(i,m)} - \varphi_h^{(i,m)}\| \le \alpha_0(h) \ \text{ on } \ E_{0.h} \ \text{ and } \ \lim_{h\to 0} \alpha_0(h) = 0, \tag{14}$$

3) the function $v \in C(E_0 \cup E, R^k)$ is a solution of problem (1), (2) and the function $v\,|_E$ is of class C^2,
4) there exists a function $\tilde{\beta} : \Delta \to R_+$ such that

$$\|\varrho_h(t,x,v_h) - \varrho(t,x,v)\| \le \tilde{\beta}(h),$$

$$\|f_h(t,x,v_h) - f(t,x,v)\| \le \tilde{\beta}(h), \ (t,x) \in E'_h,$$

and $\lim_{h\to 0} \tilde{\beta}(h) = 0$ where $v_h = (v_{h.1}, \dots, v_{h.k})$ is the restriction of v to the set $E_{0.h} \cup E_h$.

Under these assumptions there exists a function $\alpha : \Delta \to R_+$ such that

$$\|v_h^{(i,m)} - u_h^{(i,m)}\| \le \alpha(h) \ \text{ on } \ E_h \ \text{ and } \ \lim_{h \to 0} \alpha(h) = 0.$$

Proof. Let

$$\Gamma_h = (\Gamma_{h.1}, \ldots, \Gamma_{h.k}) : E'_h \to R^k, \quad \Lambda_h = (\Lambda_{h.1}, \ldots, \Lambda_{h.k}) : E'_h \to R^k$$

be the functions defined by

$$\Gamma^{(i,m)}_{h.\nu} = \delta_0 v^{(i,m)}_{h.\nu} - \partial_t v^{(i,m)}_{\nu} + \sum_{j=1}^{n} \varrho_{\nu j}(t^{(i)}, x^{(m)}, v) \left[\partial_{x_j} v^{(i,m)}_{\nu} - \delta_j v^{(i,m)}_{h.\nu} \right], \ 1 \le \nu \le k, \tag{15}$$

and

$$\Lambda^{(i,m)}_{h.\nu} = f_\nu(t^{(i)}, x^{(m)}, v) - f_{h.\nu}(t^{(i)}, x^{(m)}, u_h) \tag{16}$$

$$+ \sum_{j=1}^{n} \left[\varrho_{\nu j}(t^{(i)}, x^{(m)}, v) - \varrho_{h.\nu j}(t^{(i)}, x^{(m)}, u_h) \right] \delta_j v^{(i,m)}_{h.\nu}, \ 1 \le \nu \le k.$$

Let $z_h = v_h - u_h$, $z_h = (z_{h.1}, \ldots, z_{h.k})$ and

$$P_{i.m} = \left(t^{(i)}, x^{(m)}, u_h \right). \tag{17}$$

The function z_h satisfies the recursive equations

$$z^{(i+1,m)}_{h.\nu} = h_0 \left[\Gamma^{(i,m)}_{h.\nu} + \Lambda^{(i,m)}_{h.\nu} \right] + A z^{i,m)}_{h.\nu} + h_0 \sum_{j=1}^{n} \varrho_{h.\nu j}(\, P_{i.m}\,)\, \delta_j z^{(i,m)}_{h.\nu}, \ 1 \le \nu \le k,$$

where $(t^{(i)}, x^{(m)}) \in E'_h$. It follows from (3), (4) that

$$z^{(i+1,m)}_{h.\nu} = h_0 \left[\Gamma^{(i,m)}_{h,\nu} + \Lambda^{(i,m)}_{h.\nu} \right]$$

$$+ \frac{1}{2} \sum_{j=1}^{n} z^{(i,m+e_j)}_{h.\nu} \left[\frac{1}{n} + \frac{h_0}{h_j} \varrho_{h.\nu j}(\, P_{i.m}\,) \right] + \frac{1}{2} \sum_{j=1}^{n} z^{(i,m-e_j)}_{h.\nu} \left[\frac{1}{n} - \frac{h_0}{h_j} \varrho_{h.\nu j}(\, P_{i.m}\,) \right],$$

where $1 \le \nu \le k$. Let ω_h be a function defined by $\omega_h = V_h[z_h]$. It follows from (13) that

$$\| z^{(i+1,m)}_h \| \le \omega^{(i)}_h + h_0 \left[\| \Gamma^{(i,m)}_h \| + \| \Lambda^{(i,m)}_h \| \right] \ \text{ for } (t^{(i)}, x^{(m)}) \in E'_h. \tag{18}$$

It follows from assumption 3) that there exists $c_0 \in R_+$ such that

$$\| \partial_{x_j} v(t,x) \| \le c_0, \ \ (t,x) \in E, \ \ 1 \le j \le n. \tag{19}$$

Then $\| \delta_j v^{(i,m)}_h \| \le c_0$ on E'_h for $1 \le j \le n$. It follows from Assumption H $[\varrho_h, f_h]$ and conditon 4) that

$$\| \Lambda^{(i,m)}_h \| \le \| f(t^{(i)}, x^{(m)}, v) - f_h(t^{(i)}, x^{(m)}, v_h) \| + \| f_h(t^{(i)}, x^{(m)}, v_h) - f_h(t^{(i)}, x^{(m)}, u_h) \|$$

$$+ c_0 \| \varrho(t^{(i)}, x^{(m)}, v) - \varrho_h(t^{(i)}, x^{(m)}, v_h) \| + c_0 \| \varrho_h(t^{(i)}, x^{(m)}, v_h) - \varrho_h(t^{(i)}, x^{(m)}, u_h) \|$$

$$\le (1 + c_0)\, \sigma_h(\, t^{(i)}, \omega_h\,) + (1 + c_0) \tilde{\beta}(h).$$

We conclude from assumption 3) that there is $\tilde{\gamma} : \Delta \to R_+$ such that

$$\| \Gamma^{(i,m)}_h \| \le \tilde{\gamma}(h) \ \text{ on } \ E'_h \ \text{ and } \ \lim_{h \to 0} \tilde{\gamma}(h) = 0.$$

The above estimates and (18) imply

$$\omega_h^{(i+1)} \le \omega_h^{(i)} + h_0(1+c_0)\,\sigma_h(\,t^{(i)}, \omega_h\,) + h_0\gamma(h),\ 0 \le i \le N_0 - 1, \tag{20}$$

where $\gamma(h) = (1+c_0)\tilde{\beta}(h) + \tilde{\gamma}(h)$ and

$$\omega_h^{(i)} \le \alpha_0(h) \quad \text{for} \quad -\tilde{N}_0 \le i \le 0. \tag{21}$$

Let $\eta_h : I_{0.h} \cup I_h \to R_+$ be a solution of the Cauchy problem

$$\eta^{(i+1)} = \eta^{(i)} + h_0(1+c_0)\,\sigma_h(\,t^{(i)}, \eta\,) + h_0\gamma(h), \ \ 0 \le i \le N_0 - 1, \tag{22}$$

$$\eta^{(i)} = \alpha_0(h) \quad -\tilde{N}_0 \le i \le 0. \tag{23}$$

Using the Volterra condition and the monotonicity property of σ_h and (20)-(23) we get by induction that $\omega_h^{(i)} \le \eta_h^{(i)}$ for $0 \le i \le N_0$. Then we obtain the assertion of Theorem 3.2 from the stability of problem (9), (10).

Remark 3.3. *Note that condition (13) for method (6), (7) and inequality $h' \le Mh_0$ imply*

$$\frac{1}{n}M \ge (\,|\varrho_{h.\nu 1}(t,x,z)|\,, \ldots, |\varrho_{h.\nu n}(t,x,z)|\,) \quad on \ \ \Omega$$

where $1 \le \nu \le k$.

Now we consider the difference functional problem (6), (7), where $F_h = (F_{h.1}, \ldots, F_{h.k})$ is given by (5) and the operators δ_0, $\delta = (\delta_1, \ldots, \delta_n)$ are calculated in the following way:

$$\delta_0 z_\nu^{(i,m)} = \frac{1}{h_0}\left(\,z_\nu^{(i+1,m)} - z_\nu^{(i,m)}\,\right), \ \ 1 \le \nu \le k, \tag{24}$$

$$\delta_j z_\nu^{(i,m)} = \frac{1}{h_j}\left(\,z_\nu^{(i,m+e_j)} - z_\nu^{(i,m)}\,\right) \ \text{ if } \ \varrho_{h.\nu j}(t^{(i)}, x^{(m)}, z) \ge 0, \tag{25}$$

$$\delta_j z_\nu^{(i,m)} = \frac{1}{h_j}\left(\,z_\nu^{(i,m)} - z_\nu^{(i,m-e_j)}\,\right) \ \text{ if } \ \varrho_{h.\nu j}(t^{(i)}, x^{(m)}, z) < 0 \tag{26}$$

where $1 \le j \le n$, $1 \le \nu \le k$. It is easily seen that if ϱ_h and f_h satisfy the Volterra condition and $h' \le Mh_0$ then there exists exactly one solution $u_h = (u_{h.1}, \ldots, u_{h.k}) : E_{h.k} \cup E_h \to R^k$ of problem (6), (7) with δ_0 and δ given by (24) - (26).

Theorem 3.4. *Suppose that Assumption H [ϱ_h, f_h] is satisfied and*

1) $h \in \Delta$ and

$$1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j}\,|\varrho_{h.\nu j}(t,x,z)| \ge 0 \ \ on \ \ \Omega_h \ \ for \ \ 1 \le \nu \le k, \tag{27}$$

2) $h' \le Mh_0$ and $u_h : E_{0.h} \cup E_h \to R^k$ is the solution of the difference functional problem (6), (7) with the difference operators given by (24)-(26) and there exists a function $\alpha_0 : \Delta \to R_+$ such that condition (14) holds,

3) the function $v \in C(E_0 \cup E, R^k)$ is a solution of problem (1), (2) and the function $v\,|_E$ is of class C^1,

4) assumption 4) of Theorem 3.2 is satisfied.

Under these assumptions there exists a function $\alpha : \Delta \to R_+$ such that

$$||v_h^{(i,m)} - u_h^{(i,m)}|| \le \alpha(h) \ \ on \ \ E_h \ \ and \ \ \lim_{h \to 0} \alpha(h) = 0.$$

Proof. Let $\Gamma_h : E'_h \to R^k$ and $\Lambda_h : E'_h \to R^k$ be the functions defined by (15) and (16) with δ_0 and δ given by (24) - (26). Write $z_h = v_h - u_h$ and

$$J^{(i,m)}_{\nu.+} = \{j:\ 1 \le j \le n \ \text{ and } \ \varrho_{h.\nu j}(\,P_{i.m}\,) \ge 0\},$$

$$J^{(i,m)}_{\nu.-} = \{1,\dots,n\} \setminus J^{(i,m)}_{\nu.+}.$$

Then we have

$$z^{(i+1,m)}_{h.\nu} = h_0 \left[\, \Gamma^{(i,m)}_{h.\nu} + \Lambda^{(i,m)}_{h.\nu} \,\right] \tag{28}$$

$$+z^{(i,m)}_{h.\nu} \left[1 - h_0 \sum_{j\in J^{(i,m)}_{\nu.+}} \frac{1}{h_j} \varrho_{h.\nu j}(\,P_{i.m}\,) + h_0 \sum_{j\in J^{(i,m)}_{\nu.-}} \frac{1}{h_j} \varrho_{h.\nu j}(\,P_{i.m}\,) \right]$$

$$+h_0 \sum_{j\in I^{(i,m)}_{\nu.+}} \frac{1}{h_j} \varrho_{h.\nu j}(\,P_{i.m}\,)\, z^{(i,m+e_j)}_{h.\nu} - h_0 \sum_{j\in J^{(i,m)}_{\nu.-}} \frac{1}{h_j} \varrho_{h.\nu j}(\,P_{i.m}\,)\, z^{(i,m-e_j)}_{h\nu},$$

where $(t^{(i)}, x^{(m)}) \in E'_h$ and $P_{i.m}$ is given by (17). Let ω_h be a function defined by $\omega_h = V_h[z_h]$. It follows from (27), (28) that for $(t^{(i)} x^{(m)}) \in E'_h$ we have

$$\|\, z^{(i+1,m)}_h \,\| \le \omega^{(i)}_h + h_0 \left[\, \|\Gamma^{(i,m)}_h\| + \|\Lambda^{(i,m)}_h\| \,\right] \tag{29}$$

There exists $c_0 \in R_+$ such that condition (19) is satisfied. Then $||\delta_j v^{(i,m)}_h|| \le c_0$ on E_h for $1 \le j \le n$. It follows from Assumption H $[\varrho_h, f_h]$ and (16) that

$$\|\Lambda^{(i,m)}_h\| \le (1 + c_0)\, \sigma_h(t^{(i)}, \omega_h) + (1 + c_0)\tilde\beta(h).$$

We conclude that there is $\tilde\gamma : \Delta \to R_+$ such that

$$\|\, \Gamma^{(i,m)}_h \,\| \le \tilde\gamma(h) \ \text{ on } \ E'_h \quad \text{and} \quad \lim_{h\to 0} \tilde\gamma(h) = 0.$$

The above estimates and (29) imply

$$\omega^{(i+1)}_h \le \omega^{(i)}_h + h_0(1 + c_0)\sigma_h(t^{(i)}, \omega_h) + h_0\gamma(h), \ \ 0 \le i \le N_0 - 1,$$

where $\gamma(h) = (1 + c_0)\tilde\beta(h) + \tilde\gamma(h)$ and

$$\omega^{(i)}_h \le \alpha_0(h) \ \text{ for } \ -\tilde N \le i \le 0.$$

Let $\eta_h : I_{0.h} \cup I_h \to R_+$ be the solution of the Cauchy problem (22), (23). Using the Volterra condition and the monotonicity property of σ_h we get by induction that $\omega^{(i)}_h \le \eta^{(i)}_h$ for $0 \le i \le N_0$, and we obtain the assertion of Theorem 3.4 from the stability of problem (9), (10).

Remark 3.5. *Suppose that $M_j > 0$ for $1 \le j \le n$. Note that condition (27) for method (6), (7) with the difference operators given by (24)-(26) and the assumption $h' \le Mh_0$ imply*

$$1 \ge \sum_{j=1}^{n} \frac{1}{M_j} \mid \varrho_{h.\nu j}(t, x, z) \mid \ \text{ on } \ \Omega$$

where $1 \le \nu \le k$. If $M_1 = M_2 = \ldots = M_n = \tilde M$ then $\tilde M \ge \| \varrho_h(t, x, z)$ on Ω.

Remark 3.6. *The stability of difference problems generated by hyperbolic systems of conservations laws is strictly connected with the Courant - Friedrichs - Levy conditions, see [4], Chapter III. Inequalities (13), (27) can be considered as the Courant-Friedrichs-Levy conditions for functional differential systems.*

4 Difference schemes for quasilinear systems

We give examples of functions ϱ_h and f_h corresponding to ϱ and f. We also give error estimates for the difference methods. We adopt additional assumptions for the mesh $E_{0.h} \cup E_h$. We assume that the steps of the mesh satisfy the condition: $h' = Mh_0$. Then we can write the definitions of the sets $E_{0.h}$ and E_h in the following way:

$$E_{0.h} = \{(t^{(i)}, x^{(m)}) : \ -\tilde{N}_0 \le i \le 0, \ \ -N \le m \le N\},$$

$$E_h = \{(t^{(i)}, x^{(m)}) : \ \ 0 \le i \le N_0, \ \ |m_i| \le N_i - i \ \text{ for } i = 1, \dots, n\}.$$

Put $B = [-b, b] \subset R^n$ and $B_{h'} = \{ x^{(m)} : -N \le m \le N \}$. We define the operator $T_{h'} : \mathbf{F}([-r_0, a] \times B_{h'}, R) \to \mathbf{F}([-r_0, a] \times B, R)$ as follows. Write

$$S_+ = \{ s = (s_1, \dots, s_n) : \ s_i \in \{0, 1\} \text{ for } 1 \le i \le n \}.$$

Let $w \in \mathbf{F}([-r_0, a] \times B_{h'}, R)$ and $t \in [-r_0, a], \ x \in B$. There exists $m \in \mathbf{Z}^n$ such that $x^{(m)}, \ x^{(m+1)} \in B_{h'}$ where $m + 1 = (m_1 + 1, \dots, m_n + 1)$ and $x^{(m)} \le x \le x^{(m+1)}$. We define

$$T_{h'}[w](t, x) = \sum_{s \in S_+} w(t, x^{(m+s)}) \left(\frac{x - x^{(m)}}{h'} \right)^s \left(1 - \frac{x - x^{(m)}}{h'} \right)^{1-s}$$

where

$$\left(\frac{x - x^{(m)}}{h'} \right)^s = \prod_{i=1}^n \left(\frac{x_i - x_i^{(m_i)}}{h_i} \right)^{s_i}$$

$$\left(1 - \frac{x - x^{(m)}}{h'} \right)^{1-s} = \prod_{i=1}^n \left(1 - \frac{x_i - x_i^{(m_i)}}{h_i} \right)^{1-s_i}$$

and we take $0^0 = 1$ in the above formulas. Then the function $T_{h'}[w](t, \cdot)$ is continuous on B.

We define the operator $T_h : \mathbf{F}(E_{0.h} \cup E_h, R) \to \mathbf{F}(E_0 \cup E, R)$ in the following way. Suppose that $w : E_{0.h} \cup E_h \to R$. For $(t, x) \in E_0 \cup E$ and $-r_0 \le t \le a$ three cases will be distinguished.

I. Suppose that $(t, x) \in E_0$. Then there is $(i, m) \in \mathbf{Z}^{1+n}$ such that

$$(t^{(i)}, x^{(m)}), \ (t^{(i+1)}, x^{(m+1)}) \in E_{0.h} \ \text{ and } \ t^{(i)} \le t \le t^{(i+1)}, \ x^{(m)} \le x \le x^{(m+1)}.$$

We define

$$T_h[w](t, x) = \left(1 - \frac{t - t^{(i)}}{h_0} \right) T_{h'}[w](t^{(i)}, x) + \frac{t - t^{(i)}}{h_0} T_{h'}[w](t^{(i+1)}, x). \tag{30}$$

II. Suppose that $(t, x) \in E$ and there is $(i, m) \in \mathbf{Z}^{1+n}$ such that

$$[\, t^{(i)}, t^{(i+1)} \,] \times [\, x^{(m)}, x^{(m+1)} \,] \subset E$$

and $t^{(i)} \le t < t^{(i+1)}, \ x^{(m)} \le x \le x^{(m+1)}$. Then we define $T_h[w](t, x)$ by formula (30).

III. Suppose that $(t, x) \in E$ and

(a) $t^{(i)} \le t < t^{(i+1)}$ for some $i, \ 0 \le i \le N_0 - 1, x^{(m)} \le x \le x^{(m+1)}$ for some $m \in \mathbf{Z}^N$,

(b) $(t^{(i)}, x^{(m)}),\ (t^{(i)}, x^{(m+1)}) \in E$ and

$$(t^{(i)}, x^{(m)}) \in \partial_0 E \ \text{ or } \ (t^{(i)}, x^{(m+1)}) \in \partial_0 E$$

where $\partial_0 E = \partial E \cap (\,[0,a] \times R^n\,)$ and ∂E is the boundary of E. Define the sets of integers $I_+[i,m]$, $I_-[i,m]$, $I_0[i,m]$ (possibly empty) as follows :

$$I_+[i,m] = \{\, j : \ 1 \le j \le n \ \text{ and } \ x_j^{(m_j+1)} = b_j - M_j t^{(i)}, \ \}$$

$$I_-[i,m] = \{\, j : \ 1 \le j \le n \ \text{ and } \ x_j^{(m_j)} = -b_j + M_j t^{(i)}\,\},$$
$$I_0[i,m] = \{1, \dots, n\} \setminus (\, I_+[i,m] \cup I_-[i,m]\,)\,.$$

We define $Ux = (\bar{x}_1, \dots, \bar{x}_n)$ and $Wx = (\tilde{x}_1, \dots, \tilde{x}_n)$ as follows:

$$\bar{x}_j = x_j^{(m_j)} + \frac{h_0}{t^{(i)} + h_0 - t}\left(x_j - x_j^{(m_j)}\right) \ \text{ and } \ \tilde{x}_j = x_j^{(m_j)} \ \text{ for } \ j \in I_+[i,m],$$

$$\bar{x}_j = x_j^{(m_j+1)} + \frac{h_0}{t^{(i)} + h_0 - t}\left(x_j - x_j^{(m_j+1)}\right) \ \text{ and } \ \tilde{x}_j = x_j^{(m_j+1)} \ \text{ for } j \in I_-[i,m],$$
$$\bar{x}_j = \tilde{x}_j = x_j \ \text{ for } \ j \in I_0[i,m].$$

Then we write

$$T_h[w](t,x) = \left(1 - \frac{t - t^{(i)}}{h_0}\right) T_{h'}[w](t^{(i)}, Ux) + \frac{t - t^{(i)}}{h_0} T_{h'}[w](t^{(i+1)}, Wx).$$

If $(t,x) \in E_0 \cup E$ and $N_0 h_0 < t \le a$ then we put $T_h[w](t,x) = T_h[w](Nh_0, x)$. Then we have defined $T_h[w] : E_0 \cup E \to R$ and $T_h[w]$ is a continuous function on $E_0 \cup E$.

The above interpolating operator was introduced and widely studied in [6].

It $z = (z_1, \dots, z_k) : E_{0.h} \cup E_h \to R^k$ then we put $T_h[z] = (T_h[z_1], \dots, T_h[z_k])$. We will denote by $\|\cdot\|_t$ the maximum norm in the space $C(E_t, R^k)$, $0 \le t \le a$.

Lemma 4.1. *Suppose that $v : E_0 \cup E \to R^k$ is of class C^2 and denote by v_h the restriction of v to the set $E_{0.h} \cup E_h$. Let $\tilde{C}$ be such a constant that*

$$\|\partial_{tt} v(t,x)\| \le \tilde{C}, \ \ \|\partial_{tx_j} v(t,x)\| \le \tilde{C}, \ \ \|\partial_{x_i x_j} v(t,x)\| \le \tilde{C}, \ \ 1 \le i,j \le n,$$

on $E_0 \cup E$, and

$$C_0 = \frac{1}{2}\tilde{C}\left[1 + 2\sum_{j=1}^{n} M_i + \sum_{i,j=1}^{n} M_i M_j\right]$$

Then $\|T_h[v_h] - v\|_t \le C_0 h_0^2$ for $t \in [0, N_0 h_0]$.

The proof o the above lemma is silmilar to the proof of Theorem 3.1 in [6]. We omit details.

Now we consider functional differential problem (1), (2) and the difference functional system

$$\delta_0 z_\nu^{(i+1,m)} = \sum_{j=1}^{n} \varrho_{\nu j}(\, t^{(i)}, x^{(m)}, T_h[z]\,)\, \delta_j z_\nu^{(i,m)} + f_\nu(\, t^{(i)}, x^{(m)}, T_h[z]\,), \quad 1 \le \nu \le k, \ (31)$$

with the initial condition

$$z^{(i,m)} = \varphi_h^{(i,m)} \quad \text{on} \quad E_{0.h}, \tag{32}$$

where $\varphi_h : E_{0.h} \to R^k$ is a given function and the operators δ_0, $\delta = (\delta_1, \dots, \delta_n)$ are defined by (3), (4).

We will estimate functions of several variables by means of functions of one variable. Therefore we will need the following operator $V : C(E_0 \cup E, R^k) \to C([-r_0, a], R_+)$. If $z \in C(E_0 \cup E, R^k)$ then

$$V[z](t) = \max \{ \|z(t,x)\| : \ x \in S_t \}, \ \ -r_0 \le t \le a.$$

Assumption H $[\sigma, f]$**.** Suppose that the functions $\varrho : \Omega \to M_{k\times n}$ and $f : \Omega \to R^k$ are continuous, they satisfy the Volterra condition and

1) there exists a continuous function $\sigma : R_+ \times C([-r_0, a], R_+) \to R_+$ such that

(i) σ is nondecreasing with respect to both variables,

(ii) σ satisfies the Volterra condition and $\sigma(t, \theta) = 0$ for $t \in R_+$ where $\theta(t) = 0$ for $t \in [-r_0, a]$,

(iii) for each $c \ge 1$ the maximal solution of the problem

$$\omega'(t) = c\,\sigma(t, \omega), \quad \omega(t) = 0 \ \text{ for } \ t \in [-r_0, 0]$$

is $\bar\omega(t) = 0$ for $t \in R_+$,

2) for (t, x, z), $(t, x, \bar z) \in \Omega$ we have the estimates

$$\| \varrho(t, x, z) - \varrho(t, x, \bar z) \| \le \sigma(t, V[z - \bar z]),$$

and

$$\| f(t, x, z) - f(t, x, \bar z) \| \le \sigma(t, V[z - \bar z]).$$

Theorem 4.2. *Suppose that Assumption H [ϱ, f] is satisfied and*

1) $h \in \Delta$ and

$$\frac{1}{n} - \frac{h_0}{h_j} \, | \, \varrho_{\nu j}(t, x, z) \, | \ge 0 \ \text{ on } \ \Omega$$

for $1 \le \nu \le k$,$1 \le j \le n$, and there is a function $\alpha_0 : \Delta \to R_+$ such that condition (14) holds,

2) $h' = Mh_0$ and the function $u_h : E_{0.h} \cup E_h \to R^k$ is a solution of problem (31), (32) with δ_0 and δ given by (3), (4),

3) $v : E_0 \cup E \to R^k$ is a solution of (1), (2) and v is of class C^2 and $v_h = v \,|_{E_{0.h} \cup E_h}$.

Then there is $\varepsilon > 0$ and a function $\alpha : \Delta \to R_+$ such that for $\|h\| < \varepsilon_0$ we have

$$\|u_h^{(i,m)} - v_h^{(i,m)}\| \le \alpha(h) \ \text{ on } \ E_h \ \text{ and } \ lim_{h\to 0}\alpha(h) = 0.$$

Proof. Let

$$L_{h_0} : \mathbf{F}(I_{0.h} \cup I_h) \to C([-r_0, a], R)$$

be the operator given by

$$(L_{h_0}\eta)(t) = \eta^{(i+1)} \frac{t - t^{(i)}}{h_0} + \eta^{(i)} \left(1 - \frac{t - t^{(i)}}{h_0} \right) \ \ for \ \ t^{(i)} \le t \le t^{(i+1)}$$

and

$$(L_{h_0}\eta)(t) = (L_{h_0}\eta)(N_0 h_0) \ \text{ for } \ n_0 h_0 < t \leq a,$$

where $\eta \in \mathbf{F}(I_{0.h} \cup I_h, R)$. We prove that the functions

$$\varrho_h(t,x,z) = \varrho(t,x,T_h[z]), \ \ f_h(t,x,z) = f(t,x,T_h[z]) \ \text{ where } \ (t,x,z) \in \Omega_h$$

and

$$\sigma_h(t,\eta) = \sigma(t, L_{h_0}\eta), \quad (t,\eta) \in I'_h \times \mathbf{F}(I_{0.h} \cup I_h, R_+),$$

satisfy all the assumptions of Theorem 3.2.

We first prove that problem (9), (10) is stable in the sense of Assumption H $[\sigma_h]$. Let $\eta_h : I_{0.h} \cup I_h \to R_+$ be the solution of (11), (12), where

$$\alpha_0, \ \gamma : \Delta \to R_+ \ \text{ and } \ \lim_{h\to 0} \alpha_0(h) = 0, \ \lim_{h\to 0} \gamma(h) = 0.$$

Denote by $\omega_h : [-r_0, a] \to R_+$ the maximal solution of the problem

$$\omega'(t) = c\,\sigma(t,\omega) + \gamma(h), \ \ \omega(t) = \alpha_0(h) \text{ for } t \in I_0.$$

There exists $\varepsilon > 0$ such that the solution ω_h is defined on $[-r_0, a]$ for $\|h\| < \varepsilon_0$ and

$$\lim_{h\to 0} \omega_h(t) = 0 \ \text{ uniformly on } \ [-r_0, a].$$

The function ω_h is convex on $[0, a]$, therefore we have

$$\omega_h^{(i+1)} \geq \omega_h^{(i)} + h_0\, c\,\sigma(t,\omega_h) + h_0 \gamma(h), \ \ 0 \leq i \leq N_0 - 1.$$

Since η_h satisfies (11), (12), then we have $\eta_h^{(i)} \leq \omega_h^{(i)}$ for $0 \leq i \leq N_0$, which proves the stability of problem (9), (10). For $(t,x,z) \in \Omega_h$, $\bar{z} \in \mathbf{F}(E_{0.h} \cup E_h, R^k)$ we have

$$\|\varrho_h(t,x,z) - \varrho_h(t,x,\bar{z})\| = \|\varrho(t,x,T_h[z]) - \varrho(t,x,T_h[\bar{z}])\|$$

$$\leq \sigma(\,t, V[\,T_h[z-\bar{z}]\,])\,) = \sigma_h(\,t, V_h[z-\bar{z}]\,)$$

and

$$\|f_h(t,x,z) - f_h(t,x,\bar{z})\| \leq \sigma_h(\,t, V_h[z-\bar{z}]\,).$$

It follows from Lemma 4.1 that there is $\tilde{\beta} : \Delta \to R_+$ such that

$$\|\varrho(t,x,T_h[v_h]) - \varrho(t,x,v)\| \leq \tilde{\beta}(h),$$

$$\|f(t,x,T_h[v_h]) - f(t,x,v)\| \leq \tilde{\beta}(h) \ \text{ on } \ E'_h$$

and $\lim_{h\to 0} \tilde{\beta}(h) = 0$. Then the assertion of Theorem 4.2 follows from Theorem 3.2.

Now we give an error estimate for method (31), (32) with the difference operators δ_0 and $\delta = (\delta_1, \ldots, \delta_n)$ defined by (3), (4). According to Theorem 3.2 we have the estimate

$$\|u_h - v_h\|_{i.h} \leq \tilde{\omega}_h^{(i)}, \ \ 0 \leq i \leq N_0,$$

where $\tilde{\omega}_h : I_{0.h} \cup I_h \to R_+$ satisfies the difference functional inequality

$$\tilde{\omega}_h^{(i+1)} \geq \tilde{\omega}_k^{(i)} + h_0(1 + c_0)\,\sigma_h(\,t^{(i)}, \tilde{\omega}_h\,) + h_0 \gamma(h), \ \ 0 \leq i \leq N_0 - 1,$$

where $\gamma(h) = (1 + c_0)\tilde{\beta}(h) + \tilde{\gamma}(h)$ and

$$\tilde{\omega}_h^{(i)} \geq \alpha_0(h) \;\text{ for }\; -\tilde{N}_0 \leq i \leq 0.$$

The constant c_0 and the functions α_0, $\tilde{\beta}$, $\tilde{\gamma} : \Delta \to R_+$ are defined by the relations

$$\| \varrho(t, x, T_h[v_h]) - \varrho(t, x, v) \| \leq \tilde{\beta}(h), \quad \| f(t, x, T_h[v_h]) - f(t, x, v) \| \leq \tilde{\beta}(h)$$

on E'_h and

$$\|\varphi_h(t, x) - \varphi(t, x)\| \leq \alpha_0(h) \;\text{ on }\; E_{0.h},$$
$$\|\partial_{x_j} v(t, x)\| \leq c_0 \;\text{ on }\; E, \;\; 1 \leq j \leq n,$$

and

$$\| \Gamma_h^{(i,m)} \| \leq \tilde{\gamma}(h) \;\text{ on }\; E'_h,$$

where $\Gamma_h = (\Gamma_{h.1}, \ldots, \Gamma_{h.k})$, and

$$\Gamma_{h.\nu}^{(i,m)} = \delta_0 v_{h.\nu}^{(i,m)} - \partial_t v_\nu^{(i,m)} + \sum_{j=1}^{n} \varrho_{\nu j}(t^{(i)}, x^{(m)}, v) \left[\partial_{x_j} v_\nu^{(i,m)} - \delta_j v_{h.\nu}^{(i,m)} \right], \quad 1 \leq \nu \leq k.$$

Assumption H$_L$ $[\varrho, f]$. Suppose that the functions $\varrho : \Omega \to M_{k\times n}$ and $f : \Omega \to R^k$ are continuous and there is $L \in R_+$ such that

$$\| \varrho(t, x, z) - \varrho(t, x, \bar{z}) \| \leq L \, \|z - \bar{z}\|_t,$$

$$\|f(t, x, z) - f(t, x, \bar{z})\| \leq L \, \|z - \bar{z}\|_t$$

on Ω.

Remark 4.3. *It follows from Assumption H_L $[\varrho, f\,]$ that ϱ and f satisfy the Volterra condition on Ω.*

Theorem 4.4. *Suppose that*

1) Assumption $H_L[\varrho, f]$ is satisfied and condition 1) - 3) of Theorem 4.2 hold,

2) the function $v\,|_E$ is of class C^3 and c_0, $\tilde{C}$, $\bar{C}$, $d \in R_+$ are such constants that

$$\|\partial_{x_j} v(t, x)\| \leq c_0 \;\text{ on }\; E, \;\; 1 \leq j \leq n, \tag{33}$$

$$\|\partial_{tt} v(t, x)\|, \; \|\partial_{t x_j} v(t, x)\|, \; \|\partial_{x_i x_j} v(t, x)\| \leq \tilde{C} \;\text{ on }\; E_0 \cup E, \tag{34}$$

where $1 \leq i, j \leq n$ and

$$\|\partial_{x_j x_j x_j} v(t, x)\| \leq \bar{C} \;\text{ on }\; E, \;\; 1 \leq j \leq n, \tag{35}$$

$$|\, \varrho_{\nu j}(t, x, v)\,| \leq d \;\text{ on }\; E, \;\; 1 \leq \nu \leq k, \;\; 1 \leq j \leq n. \tag{36}$$

Then

$$\|u_h - v_h\|_{i.h} \leq \tilde{\eta}_h^{(i)} \;\text{ for }\; 0 \leq i \leq N_0, \tag{37}$$

where $\tilde{\eta}_h^{(0)} = \alpha_0(h)$, and

$$\tilde{\eta}_h^{(i)} = \alpha_0(h)(1 + \tilde{L}h_0)^i + h_0 \gamma^\star(h_0) \sum_{j=0}^{i-1} \left(1 + \tilde{L}h_0\right)^j, \; 1 \leq i \leq N_0,$$

and

$$\gamma^\star(h_0) = (1+c_0)B^\star h_0^2 + Ah_0 + Bh_0^2,$$

$$A = \frac{1}{2}\tilde{C}\left[1 + \frac{1}{n}\sum_{j=1}^{n} M_j^2\right], \quad B = \frac{1}{6}d\bar{C}\sum_{j=1}^{n} M_j^2,$$

and

$$\tilde{L} = L(1+c_0), \quad B^\star = \frac{1}{2}\tilde{C}L\left[1 + 2\sum_{j=1}^{n} M_j + \sum_{i,j=1}^{n} M_i M_j\right]. \tag{38}$$

Proof. It follows that

$$\|\partial_t v^{(i,m)} - \delta_0 v_h^{(i,m)}\| \le \frac{1}{2}\tilde{C}h_0\left[1 + \sum_{j=1}^{n} M_j^2\right]$$

and

$$\|\partial_{x_j} v^{(i,m)} - \delta_j v_h^{(i,m)}\| \le \frac{1}{6}\bar{C}M_j^2 h_0^2, \ \ 1 \le j \le n,$$

where $(t^{(i)}, x^{(m)}) \in E'_h$. Then we have

$$\|\Gamma_h^{(i,m)}\| \le Ah_0 + Bh_0^2 \ \text{ on } \ E'_h.$$

According to Lemma 4.1 and Assumption $\mathrm{H}_L\ [\varrho, f]$, the terms

$$\|\varrho(t,x,T_h[v_h]) - \varrho(t,x,v)\|, \ \ \|f(t,x,T_h[v_h]) - f(t,x,v)\|, \ \ (t,x) \in E,$$

are bounded from above by $B^\star h_0^2$. By Theorem 3.2 we have the estimate (37) with $\tilde{\eta}_h : I_h \to R_+$ satisfying the equation

$$\eta^{(i+1)} = \left(1 + \tilde{L}h_0\right)\eta^{(i)} + h_0\gamma^\star(h_0), \ \ 0 \le i \le N_0 - 1,$$

and the intial condition $\eta^{(0)} = \alpha_0(h)$. This completes the proof.

Now we consider the difference functional system (31) with the initial condition (32), where δ_0 is defined by (24) and

$$\delta_j z_\nu^{(i,m)} = \frac{1}{h_j}\left(z_\nu^{(i,m+e_j)} - z_\nu^{(i,m)}\right) \ \text{ if } \ \varrho(t^{(i)}, x^{(m)}, T_h[z]) \ge 0, \tag{39}$$

$$\delta_j z_\nu^{(i,m)} = \frac{1}{h_j}\left(z_\nu^{(i,m)} - z_\nu^{(i,m-e_j)}\right) \ \text{ if } \ \varrho(t^{(i)}, x^{(m)}, T_h[z]) < 0 \tag{40}$$

where $1 \le j \le n$, $1 \le \nu \le k$.

We will consider solutions of (1), (2) which are of class C^1 on $E_0 \cup E$. Therefore we will need the following Lemma.

Lemma 4.5. *([8], Chapter 3) Suppose that the function $v : E_0 \cup E \to R^k$ is of class C^1. Let v_h be the restriction of v the the set $E_{0.h} \cup E_h$. Then there is $C^\star \in R_+$ such that*

$$\|\, T_h[v_h] - v\|_t \le C^\star\, \|h\|, \ \ 0 \le t \le a.$$

Theorem 4.6. *Suppose that Assumption H $[\varrho, f]$ is satisfied and*

1) $h \in \Delta$ and

$$1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j} \mid \varrho_{\nu j}(t,x,z) \mid \geq 0 \ \text{ on } \ \Omega,$$

2) $h' = Mh_0$ and the function $u_h : E_{0.h} \cup E_h \to R^k$ is a solution of problem (31), (32) with δ_0 and δ given by (24) and (39), (40) respectively,

3) there is a function $\alpha_0 : \Delta \to R_+$ such that condition (14) holds,

4) $v : E_0 \cup E \to R^k$ is a solution of (1),(2), v is of class C^1 and $v_h = v \mid_{E_{0.h} \cup E_h}$.

Then there is $\varepsilon_0 > 0$ and a function $\alpha : \Delta \to R_+$ such that for $\|h\| < \varepsilon_0$ we have

$$\|u_h^{(i,m)} - v_h^{(i,m)}\| \leq \alpha(h) \ \text{ on } \ E_h \quad \text{and} \quad \lim_{h \to 0} \alpha(h) = 0.$$

The proof of the above Theorem is similar to the proof of Theorem 4.2. Details are omited.

Now we formulate a result on the error estimate.

Theorem 4.7. *Suppose that Assumption $H_L[\varrho, f]$ is satisfied and*

1) conditions 1) - 3) of Theorem 4.6 hold,

2) $v : E_0 \cup E \to R^k$ is a solution of (1), (2) and v is of class C^2 on $E_0 \cup E$,

3) c_0, $\tilde{C}$, $d \in R_+$ are constants satisfying (33), (34), (36) respectively.

Then

$$\|u_h - v_h\|_{i.h} \leq \bar{\eta}_h^{(i)}, \quad 0 \leq i \leq N_0,$$

where $\bar{\eta}_h^{(0)} = \alpha_0(h)$ and

$$\bar{\eta}_h^{(i)} = \alpha_0(h) \left(1 + \tilde{L} h_0\right)^i + \bar{\gamma}(h_0) \sum_{j=0}^{i-1} \left(1 + \tilde{L} h_0\right)^j,$$

$$\bar{\gamma}(h_0) = (1 + c_0) B^{\star} h_0^2 + \bar{A} h_0, \quad \bar{A} = \frac{1}{2} \tilde{C} \left[1 + d \sum_{j=1}^{n} M_j\right],$$

and the constants $\tilde{L}$, $B^{\star}$ are given by (38).

The proof of the above theorem is similar to the proof of Theorem 4.4. We omit details.

5 Numerical examples

Example 5.1 For $n = 2$ we put

$$E = \{ \ (t,x,y) : \ t \in [0,1], \ \ -2 + t \leq x \leq 2 - t, \ \ -2 + t \leq y \leq 2 - t \ \}.$$

Let us denote by z an unknown function of the variables (t,x,y) and consider the differential integral equation

$$\partial_t z(t,x,y) = y \ \left\{1 + \left[\int_{-x}^{x} z(t,s,y)\,ds - 2txy\right]^2\right\}^{-1} \partial_x z(t,x,y) \tag{41}$$

$$+x\ \left\{1+\left[\int_{-y}^{y} z(t,x,r)\,dr-2txy\right]^2\right\}^{-1}\ \partial_y z(t,x,y)$$

$$+\int_{D(t,x,y)} z(t,s,r)\,drds-\frac{1}{2}(2-t)^2 z(t,x,y)+(x+y)(1-t)$$

with the initial condition

$$z(0,x,y)=0\ \text{ for }\ (x,y)\in[-2,\,2]\times[-2,\,2], \tag{42}$$

where

$$\int_{D(t,x,y)} z(t,s,r)\,drds=\int_{-1+0.5(x+t)}^{1+0.5(x-t)}\ \int_{-1+0.5(y+t)}^{1+0.5(y-t)} z(t,s,r)\,dr\,ds.$$

Note that if $(t,x,y)\in E$, then

$$\{\,t\,\}\times[-1+0.5(x+t),\ 1+0.5(x-t)]\times[-1+0.5(y+t),\ 1+0.5(y-t)]\in E.$$

The exact solution of this problem is known. It is $v(t,x,y)=t(x+y)$, $(t,x,y)\in E$.

We apply Theorem 4.6 to (41), (42). Let $h=(h_0,h_1,h_2)$ stand for the steps of the mesh on E. Let $T_h:\mathbf{F}(E_h,R)\to\mathbf{F}(E,R)$ be the interpolating operator, defined in Section 4 with $n=2$. It follows that for a point $(t^{(i)},x^{(j)},y^{(k)})\in E_h$ and for a function $z:E_h\to R$ we have

$$T_h[z](t^{(i)},x,y)=z^{(i,j,k)}\left(1-\frac{x-x^{(j)}}{h_1}\right)\left(1-\frac{y-y^{(k)}}{h_2}\right)$$

$$+z^{(i,j,k+1)}\left(1-\frac{x-x^{(j)}}{h_1}\right)\frac{y-y^{(k)}}{h_2}+z^{(i,j+1,k)}\frac{x-x^{(j)}}{h_1}\left(1-\frac{y-y^{(k)}}{h_2}\right)$$

$$+z^{(i,j+1,k+1)}\frac{x-x^{(j)}}{h_1}\,\frac{y-y^{(k)}}{h_2},\quad x^{(j)}\le x\le x^{(j+1)},\quad y^{(k)}\le y\le y^{(k+1)}$$

and consequently

$$\int_{x^{(j)}}^{x}\int_{y^{(k)}}^{y} T_h[z](t^{(i)},s,r)\,dr\,ds \tag{43}$$

$$=z^{(i,j,k)}\,\frac{x-x^{(j)}}{2}\left(2-\frac{x-x^{(j)}}{h_1}\right)\frac{y-y^{(k)}}{2}\left(2-\frac{y-y^{(k)}}{h_2}\right)$$

$$+z^{(i,j,k+1)}\,\frac{x-x^{(j)}}{2}\left(2-\frac{x-x^{(j)}}{h_1}\right)\frac{\left(y-y^{(k)}\right)^2}{2h_2}$$

$$+z^{(i,j+1.k)}\,\frac{\left(x-x^{(j)}\right)^2}{2h_1}\,\frac{y-y^{(k)}}{2}\left(2-\frac{y-y^{(k)}}{h_2}\right)$$

$$+z^{(i,j+1,k+1)}\,\frac{\left(x-x^{(j)}\right)^2}{2h_1}\,\frac{\left(y-y^{(k)}\right)^2}{2h_2}.$$

According to the above formula, we have

$$\int_{x^{(j)}}^{x^{(j+1)}}\int_{y^{(k)}}^{y^{(k+1)}} T_h[z](t^{(i)},s,r)\,dr\,ds \tag{44}$$

$$= \frac{h_1 h_2}{4} \left(z^{(i,j,k)} + z^{(i,j,k+1)} + z^{(i,j+1,k)} + z^{(i,j+1,k+1)} \right)$$

and

$$\int_{x^{(j)}}^{x^{(j+1)}} \int_{y^{(k)}}^{y} T_h[z](t^{(i)}, s, r)\, dr\, ds = \frac{h_1}{2} \frac{\left(y - y^{(k)} \right)^2}{2h_2} \left(z^{(i,j,k+1)} + z^{(i,j+1,k+1)} \right) \tag{45}$$

$$+ \frac{h_1}{2} \frac{y - y^{(k)}}{2} \left(2 - \frac{y - y^{(k)}}{h_2} \right) \left(z^{(i,j,k)} + z^{(i,j+1,k)} \right)$$

and

$$\int_{x^{(j)}}^{x} \int_{y^{(k)}}^{y^{(k+1)}} T_h[z](t^{(i)}, s, r)\, dr\, ds = \frac{\left(x - x^{(j)} \right)^2}{2h_1} \left(z^{(i,j+1,k)} + z^{(i,j+1,k+1)} \right) \tag{46}$$

$$+ \frac{h_2}{2} \frac{x - x^{(j)}}{2} \left(2 - \frac{x - x^{(j)}}{h_1} \right) \left(z^{(i,j,k)} + z^{(i,j,k+1)} \right).$$

Having disposed of this preliminary step, we formulate the difference problem corresponding to (41), (42). Consider the difference equation

$$\delta_0 z^{(i,j,k)} = y^{(k)} \left\{ 1 + \left[\int_{-x^{(j)}}^{x^{(j)}} T_h[z](t^{(i)}, s, y^{(k)})\, ds - 2t^{(i)} x^{(j)} y^{(k)} \right] \right\}^{-1} \delta_1 z^{(i,j.k)} \tag{47}$$

$$+ x^{(j)} \left\{ 1 + \left[\int_{-y^{(k)}}^{y^{(k)}} T_h[z](t^{(i)} x^{(j)}, r)\, dr - 2t^{(i)}, x^{(j)} y^{(k)} \right]^2 \right\}^{-1}$$

$$+ \int_{D(t^{(i)}, x^{(j)}, y^{(k)})} T_h[z](t^{(i)}, s, r)\, dr\, ds - \frac{1}{2}(2 - t^{(i)})^2\, z^{(i,j,k)} + (x^{(j)} + y^{(k)})\,(1 - t^{(i)})$$

with the initial condition

$$z(0, x^{(j)}, y^{(k)}) = 0 \ \text{ for } \ (x^{(j)}, y^{(k)}) \in [-2, 2] \times [-2, 2], \tag{48}$$

where

$$\delta_0 z^{(i,j,k)} = \frac{1}{h_0} \left[z^{(i+1,j,k)} - z^{(i,j,k)} \right],$$

$$\delta_1 z^{(i,j,k)} = \frac{1}{h_1} \left[z^{(i,j+1,k)} - z^{(i,j,k)} \right] \ \text{ if } \ y^{(k)} \geq 0,$$

$$\delta_1 z^{(i,j,k)} = \frac{1}{h_2} \left[z^{(i,j,k)} - z^{(i,j-1,k)} \right] \ \text{ if } \ y^{(k)} < 0$$

and

$$\delta_2 z^{(i,j,k)} = \frac{1}{h_2} \left[z^{(i,j,k+1)} - z^{(i,j,k)} \right] \ \text{ if } \ x^{(j)} \geq 0,$$

$$\delta_2 z^{(i,j,k)} = \frac{1}{h_2} \left[z^{(i,j,k)} - z^{(i,j,k-1)} \right] \ \text{ if } \ x^{(j)} < 0.$$

Note that results of the papers [6], [11], [16] cannot be applied to the above problem.

We can now formulate formulas for calculating integrals in equation (47). Our considerations start with the observation that

$$\int_{-x^{(j)}}^{x^{(j)}} T_h[z](t^{(i)}, s, y^{(k)})\, ds = \frac{h_1}{2}\left[z^{(i,-j,k)} + z^{(i,j,k)} \right] + h_1 \sum_{\nu=-j+1}^{j-1} z^{(i,\nu,k)} \qquad (49)$$

and

$$\int_{-y^{(k)}}^{y^{(k)}} T_h[z](t^{(i)}, x^{(j)}, r)\, dr = \frac{h_2}{2}\left[z^{(i,j,-k)} + z^{(i,j,k)} \right] + h_2 \sum_{\xi=-k+1}^{k-1} z^{(i,j,\xi)}. \qquad (50)$$

The last integral in equation (47) has the following property: if $(t^{(i)}, x^{(j)}, y^{(k)})$ is a grid point, then

$$\left(t^{(i)}, -1 + 0.5(x^{(j)} + t^{(i)}), -1 + 0.5(y^{(k)} + t^{(i)}) \right),$$

$$\left(t^{(i)}, 1 + 0.5(x^{(j)} - t^{(i)}), 1 + 0.5(y^{(k)} - t^{(i)}) \right)$$

in general, are not grid points. Therefore we need the following construction. Write

$$\theta_{ij} = 1 - 0.5(x^{(j)} + t^{(i)}), \quad \tilde{\theta}_{ij} = 1 + 0.5(x^{(j)} - t^{(i)}),$$

and

$$\eta_{ik} = 1 - 0.5(y^{(k)} + t^{(i)}), \quad \tilde{\eta}_{ik} = 1 + 0.5(y^{(k)} - t^{(i)}).$$

Then

$$x^{(j)} + \theta_{ij} = 1 + 0.5(x^{(j)} - t^{(i)}), \quad x^{(j)} - \tilde{\theta}_{ij} = -1 + 0.5(x^{(j)} + t^{(i)}),$$

and

$$y^{(k)} + \eta_{ik} = 1 + 0.5(y^{(k)} - t^{(i)}), \quad y^{(k)} - \tilde{\eta}_{ik} = -1 + 0.5(y^{(k)} + t^{(i)}).$$

There exist κ_{ij}, $\tilde{\kappa}_{ij}$, μ_{ik}, $\tilde{\mu}_{ik} \in \mathbf{N}$ and $\varepsilon_x^{(i,j)}, \tilde{\varepsilon}_x^{(i,j)} \in [0, h_1)$, $\varepsilon_y^{(i,k)}$, $\tilde{\varepsilon}_y^{(i,k)} \in [0, h_2)$ such that

$$\theta_{ij} = (\kappa_{ij} + 1)h_1 + \varepsilon_x^{(i,j)}, \;\; \tilde{\theta}_{ij} = \tilde{\kappa}_{ij} h_1 + \tilde{\varepsilon}_x^{(i,j)}$$

and

$$\eta_{ik} = (\mu_{ik} + 1)h_2 + \varepsilon_y^{(i,k)}, \;\; \tilde{\eta}_{ik} = \tilde{\mu}_{ik} h_2 + \tilde{\varepsilon}_y^{(i,k)}.$$

Write

$$A[i,j,k] = h_1 h_2 \sum_{\nu=-\tilde{\kappa}_{ij}+1}^{\kappa_{ij}-1} \; \sum_{\xi=-\tilde{\mu}_{ik}+1}^{\mu_{ik}-1} z^{(i,j+\nu,k+\xi)}$$

$$+\frac{h_1 h_2}{2} \sum_{\xi=-\tilde{\mu}_{ik}+1}^{\mu_{ik}-1} \left[z^{(i,j-\tilde{\kappa}_{ij},\xi)} + z^{(i,j+\kappa_{ij},\xi)} \right] + \frac{h_1 h_2}{2} \sum_{\nu=-\tilde{\kappa}_{ij}+1}^{\kappa_{ij}-1} \left[z^{(i,\nu,k-\tilde{\mu}_{ik})} + z^{(i,\nu,k+\mu_{ik})} \right]$$

$$+\frac{h_1 h_2}{4} \left[z^{(i,j-\tilde{\kappa}_{ij},k+\mu_{ik})} + z^{(i,j+\kappa_{ij},k+\mu_{ik})} + z^{(i,j-\tilde{\kappa}_{ij},k-\tilde{\mu}_{ik})} + z^{(i,j+\kappa_{ij},k-\tilde{\mu}_{i,k})} \right].$$

For simplicity of formulation of next formulas we write

$$x^{[j.i]} = 1 + 0.5(x^{(j)} - t^{(i)}), \;\; \tilde{x}^{[j.i]} = -1 + 0.5(x^{(j)} + t^{(i)}),$$

$$y^{[k.i]} = 1 + 0.5(y^{(k)} - t^{(i)}), \quad \tilde{y}^{[k.i]} = -1 + 0.5(y^{(k)} + t^{(i)}).$$

The integrals

$$B[i,\nu,k] = \int_{x^{(\nu)}}^{x^{(\nu+1)}} \left[\int_{y^{(k+\mu_{ik}+1)}}^{y^{[k.i]}} T_h[z](t^{(i)},s,r)\,dr + \int_{\tilde{y}^{[k.i]}}^{y^{(k-\tilde{\mu}_{ik})}} T_h[z](t^{(i)},s,r)\,dr \right] ds$$

and

$$C[i,j,\xi] = \int_{y^{(\xi)}}^{y^{(\xi+1)}} \left[\int_{\tilde{x}^{[j.i]}}^{x^{(j-\tilde{\kappa}_{ij})}} T_h[z](t^{(i)},s,r)\,ds + \int_{x^{(j+\kappa_{ij}+1)}}^{x^{[j.i]}} T_h[z](t^{(i)},s,r)\,ds \right] dr$$

where $-\tilde{\kappa}_{ij} \le \nu \le \kappa_{ij}$, $-\tilde{\mu}_{ik} \le \xi \le \mu_{ik}$, can be calculated using (45) and (46). Write

$$E[i,j,k] = \int_{x^{(j+\kappa_{ij}+1)}}^{x^{[j.i]}} \int_{y^{(k+\mu_{ik}+1)}}^{y^{[k.i]}} T_h[z](t^{(i)},s,r)\,dr\,ds +$$
$$\int_{x^{(j+\kappa_{ij}+1)}}^{x^{[j.i]}} \int_{\tilde{y}^{[k.i]}}^{y^{(k-\tilde{\mu}_{ik})}} T_h[z](t^{(i)},s,r)\,dr\,ds + \int_{\tilde{x}^{[j.i]}}^{x^{(j-\tilde{\kappa}_{ij})}} \int_{\tilde{y}^{[k.i]}}^{y^{(k-\tilde{\mu}_{ik})}} T_h[z](t^{(i)},s,r)\,dr\,ds$$
$$+ \int_{\tilde{x}^{[j.i]}}^{x^{(j-\tilde{\kappa}_{ij})}} \int_{y^{(k+\mu_{ik}+1)}}^{y^{[k.i]}} T_h[z](t^{(i)},s,r)\,dr\,ds.$$

We calculate $E[i,j,k]$ using (43).

Note that we have actually proved that

$$\int_{D(t^{(i)},x^{(j)},y^{(k)})} T_h[z](t^{(i)},s,r)\,dr\,ds = A[i,j,k] + E[i,j,k] \tag{51}$$
$$+ \sum_{\nu=-\tilde{\kappa}_{ij}}^{\kappa_{ij}} B[i,\nu,k] + \sum_{\xi=-\tilde{\mu}_{ik}}^{\mu_{ik}} C[i,j,\xi].$$

We approximate the solution of the problem (41), (42) by means of solutions of the difference problem consisting of (47) - (51). Let $u_h : E_h \to R$ be the solution of this difference problem. Write

$$\varepsilon_h^{(i)} = \max\{\, |u_h^{(i,j,k)} - v_h^{(i,j,k)}| : \ (t^{(i)},x^{(j)},y^{(k)}) \in E_h \,\},\ 0 \le i \le N_0,$$

where v_h is the restriction of v to the set E_h. We take $h_0 = h_1 = h_2 = 10^{-3}$. The values of $\varepsilon_h^{(i)}$ are listed in the table.

Table of errors

$t^{(i)}$:	0.5	0.6	0.7	0.8	0.9	1.0
$\varepsilon_h^{(i)}$:	$5.0650\,10^{-3}$	$5.2852\,10^{-3}$	$5.4654\,10^{-3}$	$5.6056\,10^{-3}$	$5.7057\,10^{-3}$	$5.7657\,10^{-3}$

The results shown in the table are consistent with our mathematical analysis.

Example 5.2 For $n = 2$ we put

$$E = \{\,(t,x,y) : \ t \in [0,\ 0.5], \ \ -1+t \le x \le 1-t,\ -1+t \le y \le 1-t\,\}.$$

Let us denote by (u,v) an unknown function of the variables (t,x,y) and consider the system of differential equations with a deviated argument

$$\partial_t v(t,x,y) = \left\{1 - \frac{1}{1+[u(\alpha) - tv(\gamma) + f_{11}(t,x,y)]^2}\right\} \partial_x u(t,x,y)$$

$$- \left\{1 - \frac{1}{1+[u(\delta) - tv(\beta) + f_{12}(t,x,y)]^2}\right\} \partial_y u(t,x,y)$$

$$+u(t,x,y) - v(t,x,y) - (t-1)(xy - x - y) + xy,$$

$$\partial_t v(t,x,y) = \frac{x}{1+[u(\beta) + tv(\alpha) + f_{21}(t,x,y)]^2} \partial_x v(t,x,y)$$

$$+ \frac{y}{1+[u(\delta) + tv(\gamma) + f_{22}(t,x,y)]^2} \partial_y v(t,x,y)$$

$$+u(t,x,y) - v(t,x,y) - xy(1+t),$$

with the initial condition

$$u(0,x,y) = x+y, \quad v(0,x,y) = xy, \quad (x,y) \in [-1,\,1] \times [-1,\,1],$$

where

$$f_{11}(t,x,y) = (t-1)\left(tx + 0.5(x+y)\right), \quad f_{12}(t,x,y) = (t-1)\left(0.5(x+y) - 1 - t^2\right),$$

$$f_{21}(t,x,y) = -0.5(x+y)(1+t^2) - 0.5txy + (t-1)\left(1 + 0.5tx\right),$$

$$f_{22}(t.x,y) = -0.5(x+y)(t^2+1) + (1-t)(1+0.5tx) - 0.5txy,$$

and

$$\alpha = (\,t,\, 0.5(x-1+t),\, 0.5(y+1-t)\,), \quad \beta = (\,t,\, 0.5(x+1-t),\, 0.5(y+1-t)\,),$$

$$\gamma = (t,\, 0.5(x+1-t),\, 0.5(y-1+t)\,), \quad \delta = (\,t,\, 0.5(x-t+t),\, 0.5(y-1+t)\,).$$

Note that $\alpha,\, \beta,\, \gamma,\, \delta \in E$ for $(t,x,y) \in E$. The exact solution of this problem is known. It is

$$\tilde{u}(t,x,y) = txy + x + y, \quad \tilde{v}(t,x,y) = t(x+y) + xy, \quad (t,x,y) \in E.$$

Suppose that $h = (h_0, h_1, h_2)$ stand for the steps of the mesh on E and T_h : $\mathbf{F}(E_h, R) \to \mathbf{F}(E, R)$ is the interpolating operator defined in Section 4 for $n = 2$. For a function $z : E_h \to R$ we put

$$\delta_0 z^{(i,jk)} = \frac{1}{h_0}\left[z^{(i+1,j,k)} - Az^{(i,j,k))}\right],$$

where

$$Az^{(i.j.k)} = \frac{1}{4}\left[z^{(i,j+1,k)} + z^{(i,j-1,k)} + z^{(i,j,k+1)} + z^{(i,j,k-1)}\right]$$

and

$$\delta_1 z^{(i,j,k)} = \frac{1}{2h_1}\left[\, z^{(i,j+1,k)} - z^{(i,j-1,k)}\,\right], \quad \delta_2 z^{(i,j,k)} = \frac{1}{2h_2}\left[\, z^{(i,j,k+1)} - z^{(i,j,k-1)}\,\right].$$

Consider the system of difference equations

$$\delta_0 v^{(i,j,k)} = \left\{ 1 - \frac{1}{1 + \Big[\, T_h[u](\alpha_{ijk}) - t^{(i)}\, T_h[v](\gamma_{ijk}) + f_{11}^{(i,j,k)} \,\Big]^2} \right\} \delta_1 u^{(i,j,k)}$$

$$- \left\{ 1 - \frac{1}{1 + \Big[\, T_h[u](\delta_{ijk}) - t^{(i)}\, T_h[v](\beta_{ijk}) + f_{12}^{(i,j,k)} \,\Big]^2} \right\} \delta_2 u^{(i,j,k)}$$

$$+ u^{(i,j,k)} - v^{(i,j,k)} - (t^{(i)} - 1)(x^{(j)} y^{(k)} - x^{(j)} - y^{(k)}) + x^{(j)} y^{(k)},$$

$$\delta_0 u^{(i,j,k)} = \frac{x^{(j)}}{1 + \Big[\, T_h[u](\beta_{ijk}) + t^{(i)}\, T_h[v](\alpha_{ijk}) + f_{21}^{(i,j,k)} \,\Big]^2} \delta_1 v^{(i,j,k)}$$

$$+ \frac{y^{(j)}}{1 + \Big[\, T_h[u](\delta_{ijk}) + t^{(i)}\, T_h[v](\gamma_{ijk}) + f_{22}^{(i,j,k)} \,\Big]^2} \delta_2 v^{(i,j,k)}$$

$$+ u^{(i,j,k)} - v^{(i,j,k)} - x^{(j)} y^{(k)} (1 + t^{(i)})$$

with the initial condition

$$u(0, x^{(j)}, y^{(k)}) = x^{(j)} + y^{(k)}, \quad v(0, x^{(j)}, y^{(k)}) = x^{(j)} y^{(k)}, \;\; (x^{(j)}, y^{(k)}) \in [-1,\, 1] \times [-1,\, 1],$$

where

$$f_{\nu\xi}^{(i,j,k)} = f_{\nu\xi}(t^{(i)}, x^{(j)}, y^{(k)}), \;\; \nu,\, \xi = 1, 2,$$

and

$$\begin{aligned}
\alpha_{ijk} &= \Big(t^{(i)},\, 0.5(x^{(j)} - 1 + t^{(i)}),\, 0.5(y^{(k)} + 1 - t^{(i)}) \Big), \\
\beta_{ijk} &= \Big(t^{(i)},\, 0.5(x^{(j)} + 1 - t^{(i)}),\, 0.5(y^{(k)} + 1 - t^{(i)}) \Big), \\
\gamma_{ijk} &= \Big(t^{(i)},\, 0.5(x^{(j)} + 1 - t^{(i)}),\, 0.5(y^{(k)} - 1 + t^{(i)}) \Big), \\
\delta_{ijk} &= \Big(t^{(i)},\, 0.5(x^{(j)} - 1 + t^{(i)}),\, 0.5(y^{(k)} - 1 + t^{(i)}) \Big).
\end{aligned}$$

Note that if $(t^{(i)}, x^{(j)}, y^{(k)})$ is a grid point then $\alpha_{ijk},\ \beta ijk,\ \gamma_{ijk},\ \delta_{ijk}$ in general, are not grid points. Denote by $(u_h, v_h) : E_h \to R^2$ the solution of this difference problem. Write

$$\begin{aligned}
\varepsilon_{1.h}^{(i)} &= \max \{\, |u_h^{(i,j,k)} - \tilde{u}_h^{(i,j,k)}| :\; (t^{(i)}, x^{(j)}, y^{(k)}) \in E_h \,\}, \\
\varepsilon_{2.h}^{(i)} &= \max \{\, |v_h^{(i,j,k)} - \tilde{v}_h^{(i,j,k)}| :\; (t^{(i)}, x^{(j)}, y^{(k)}) \in E_h \,\},
\end{aligned}$$

where $0 \le i \le N_0$ and $(\tilde{u}_h, \tilde{v}_h)$ is the restriction ov $(\tilde{u}, \tilde{v})$ to the set E_h. Put

$$\varepsilon_h^{(i)} = \max \{ \varepsilon_{1.h}^{(i)},\; \varepsilon_{2.h}^{(i)} \}, \;\; 0 \le i \le N_0.$$

We take $h_0 = h_1 = h_2 = 10^{-4}$. The values of $\varepsilon_h^{(i)}$ are listed in the table.

Table of errors

$t^{(i)}$:	0.1	0.2	0.3	0.4	0.5
$\varepsilon_h^{(i)}$:	$4.5657\,10^{-4}$	$4.5732\,10^{-4}$	$4.5801\,10^{-4}$	$4.6031\,10^{-4}$	$4.6163\,10^{-4}$

The results shown in the table are consistent with our mathematical analysis.

Remark 5.1. *The methods described in Section 4 have the potential for applications in the numerical solving of differential integral equations or equations with a deviated argument. Difference methods considered in the paper have the following property: a large number of previous values* $z^{(i,m)}$ *must be preserved, because they are needed to compute an approximate solution corresponding to* $t = t^{(i+1)}$.

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