Quasi-subgeometry partitions of projective spaces

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Abstract

The notion of a subgeometry partition of a finite projective space $PG(2m-1, q^2)$ by $PG(m-1, q^2)$'s and $PG(2m-1, q)$'s or a partition of $PG(2m, q^2)$ by $PG(2m, q)$'s is generalized to quasi-subgeometry partitions of $PG(2m-1, q^d)$ by $PG(dm/e-1, q^e)$'s for a set of divisors e of d and, partitions of $PG(2m, q^{2d})$ by $PG(d(2m+1)/f-1, q^f)$'s for a set of divisors f of d. In all cases, there are associated vector space spreads that are unions of 'fans'.

More generally, in the arbitrary dimensional case, a complete theory of quasi-subgeometry partitions of $PG(V-1, D)$ corresponding to generalized spreads admitting D^* as a fixed-point-free collineation group is obtained. When D is a quadratic extension of a base field, 'subgeometry' partitions are obtain.

1 Introduction.

In 1976, A. Bruen and J.A. Thas [3] showed that it was possible to find partitions of the points of $PG(2s-1, q^2)$ by projective subgeometries isomorphic to $PG(s-1, q^2)$'s and $PG(2s-1, q)$'s. These might be called 'mixed subgeometry partitions'. Bruen and Thas showed that there is an associated translation plane of order q^{2s} and kernel containing $GF(q)$. There is an another construction using Segre varieties given in Hirschfeld and Thas [4] which generalizes and includes Baer subgeometry partitions

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of $PG(2s, q^2)$ by $PG(2s, q)$'s. In this latter case, there is an associated translation plane of order q^{2s+1} and kernel containing $GF(q)$.

Recently, in [6], there is an interpretation of the above construction from the viewpoint of the translation plane. That is, starting with a translation plane of a certain type, a 'retraction' method is possible which reverses the construction and produces either a mixed or a Baer subgeometry partition of a projective space. The main consideration is that a group isomorphic to $GF(q^2)^*$ acts as a collineation group of the translation plane.

In this article, we generalize these constructions several ways. First of all, finiteness is not required. Second, to apply the group action more generally, we define what we call a 'quasi-subgeometry' of a projective space to realize that certain spreads produce 'quasi-subgeometry partitions' of projective spaces. In the finite case, we could have a projective space isomorphic to $PG(2s-1, q^d)$ partitioned by quasi-subgeometries isomorphic to $PG(ds/e-1, q^e)$ for various divisors e of d. Third, it is realized that subgeometry or quasi-subgeometry partitions of a projective space need not actually produce a spread but do produce a 'generalized spread'. Furthermore, a generalized spread admitting a fixed-point-free field acting as a collineation group produces a quasi-subgeometry partition. Hence, our work generalizes to the consideration of generalized spreads and how these might produce partitions of projective spaces.

As mentioned, the fundamental device in our work is the consideration of the nature of collineation groups acting on the generalized spread $\mathcal{S}$ that arise from extension fields F of K. Such groups define within the vector space what we call 'F_L-fans'. These fans when considered projectively are the quasi-subgeometries in question.

Fans are only considered from the vector space point of view and we are interested in the associated quasi-subgeometry partitions. Hence, we form a generalization and integration of the André and Bruck-Bose approaches to the study of 'spreads'.

We then have a hybrid of the vector space and projective space variations which allows a complete generalization of the theory of subgeometry partitions of projective spaces and their associated translation planes to quasi-subgeometry partitions and associated generalized spreads.

Also included is a discussion a theory of subgeometry partitions of $PG(ds/2-1, q^2)$ by subgeometries isomorphic to $PG(s/2-1, q^2)$ and $PG(s-1, q)$ if s is even, or by $PG(s-1, q)$'s if s is odd. This is further generalized to a consideration of arbitrary quasi-subgeometry partitions of $PG(ds/e-1, q^e)$ by appropriate quasi-subgeometries.

In terms of examples in the finite case, we construct a variety of examples of finite quasi-subgeometry partitions from generalized André planes.

Our main theorems are as follows:

Theorem 1. *Let V be a vector space over a field K and let $\mathcal{S}$ be a generalized spread of V. Assume that there exists a field D containing K that contains the scalar mapping group K^* and the multiplicative group D^* acts as a fixed-point-free collineation group in $GL(V, K)$ on $\mathcal{S}$.*

Then there is a quasi-subgeometry partition of $PG(V-1, D)$ by quasi-subgeometries isomorphic to $PG(L_i-1, D_{L_i})$ where L_i is a component of $\mathcal{S}$ and $K \subseteq D_{L_i} \subseteq D$,

D_{L_i} a field extension of K and a subfield of D, for $i \in \Lambda$. Furthermore, $\mathcal{S}$ is a union of D_{L_i}-fans, $i \in \Lambda$.

Theorem 2. *Let $\mathcal{P}$ be a quasi-subgeometry partition of $PG(V-1, D)$ by quasi-subgeometries isomorphic to $PG(L-1, D_L)$ where L is a component of $\mathcal{S}$ and $K \subseteq D_L \subseteq D$, D_L a field extension of K and a subfield of D.*

Then there is a generalized spread $\mathcal{S}$ that is a union of D_L-fans and admits D^* as a fixed-point-free collineation group in $GL(V, K)$.

2 André versus Bruck-Bose.

There are two well-known approaches to the study of translation planes; one using vector spaces, due to André [1] and another using projective spaces due to Bruck-Bose [2]. We shall employ a combination of these two methods noting that it is often more expedient when working with translation planes to work in the associated vector space.

Definition 1. *A 'finite-dimensional vector space spread' of a $2n$-dimensional vector space V_{2n} over a skewfield K is a set $\mathcal{S}$ of n-dimensional vector subspaces with the following properties:*

(i) $\cup\{W; W \in \mathcal{S}\} = V_{2n}$ and

(ii) if W, Q are distinct elements of $\mathcal{S}$ then the direct sum $W \oplus Q = V_{2n}$.

Given a finite-dimensional vector spread spread $\mathcal{S}$, an affine translation plane $\pi_{\mathcal{S}}$ may be defined by defining 'points' to be the vectors of V_{2n} and 'lines' to be the additive cosets of the elements of $\mathcal{S}$.

Remark 1. *It is possible to have a vector space-approach without assuming that there is a finite-dimensional vector space.*

Let V be a vector space over a skewfield K that can be decomposed as the external direct sum $W_o \oplus W_o$, where W_o is some K-vector space. Let $\mathcal{S}$ be a set of mutually disjoint vector subspaces of V each of which is K-isomorphic to W_o. Then, $\mathcal{S}$ is a 'vector-space spread' of V provided

(i) $\cup\{W; W \in \mathcal{S}\} = V$ and

(ii) if W, Q are distinct elements of $\mathcal{S}$ then the (internal) direct sum $W \oplus Q = V$.

Definition 2. *Given a vector spread spread $\mathcal{S}$, an affine translation plane $\pi_{\mathcal{S}}$ may be defined by defining 'points' to be the vectors of V and 'lines' to be the additive cosets of the elements of $\mathcal{S}$.*

When a translation plane π is obtained using a vector space spread as above, we shall say that it is obtained by the 'André method'.

Hence, using the André method or vector space-approach, a 'finite (vector space) spread' is a set of mutually disjoint half-dimensional vector subspaces which cover the vector space as above. Actually, it is also the associated affine space corresponding to the vector space that is of importance in interconnecting the methods under discussion. For these interconnections, we adopt some non-standard notation which is useful in these contexts.

Notation 1. *Let V be a vector space over a field K. Then, we denote $AG(V,K)$ to be the affine space whose 'points' are the vectors of V and whose 'subspaces' are the vector translates of the vector subspaces of V.*

Now there are two projective spaces associated with V and/or $AG(V,K)$:

(i) Extend $AG(V,K)$ to a projective space by the method of adjunction of a 'hyperplane at infinity'. We shall call this projective space $PG(V,K)$.

(ii) Form the projective space obtained from V by taking the 'points' to be the 1-dimensional K-subspaces and the set of 'projective subspaces' to be the lattice of vector subspaces. We shall use the (somewhat non-standard) notation $PG(V-1,K)$ to denote this projective space.

Note that if V is k-dimensional over K, $AG(V,K)$ and $PG(V,K)$ are denoted by $AG(k,K)$ and $PG(k,K)$, respectively. Furthermore, $PG(V-1,K)$ is then denoted by $PG(k-1,K)$.

(iii) If $\mathcal{S}$ is a set of vector subspaces of V, we write $P(\mathcal{S})$ to denote the set $\{PG(W-1,K); W \in \mathcal{S}\}$ in $PG(V-1,K)$.

Definition 3. *Let K be a skewfield and let $\mathcal{P}$ be a projective space isomorphic to $PG(2n-1,K)$. A 'finite-dimensional projective spread' of $\mathcal{P}$ is a set $\mathcal{S}_{\mathcal{P}}$ of mutually disjoint $(n-1)$-dimensional projective subspaces such that $\cup\{W_{\mathcal{P}}; W_{\mathcal{P}} \in \mathcal{S}_{\mathcal{P}}\} = \mathcal{P}$.*

More generally, we may define a projective spread over a possibly infinite-dimensional projective space.

Definition 4. *Let K be a skewfield, V a vector space which is the external direct sum $W_o \oplus W_o$, where W_o is a K-vector space. Let $\mathcal{P}$ denote the lattice of projective subspaces of V.*

A 'projective spread' of $\mathcal{P}$ is a set $\mathcal{S}_{\mathcal{P}}$ of mutually disjoint projective subspaces of $PG(V-1,K)$ such that

(i) each element of $\mathcal{P}$ is isomorphic to $PG(W_o-1,K)$, any two distinct elements of $\mathcal{P}$ generate $PG(V-1,K)$ (in the sense that all points are on lines joining pairs of points from the two projective subspaces) and

(ii)
$$\cup\{W_{\mathcal{P}}; W_{\mathcal{P}} \in \mathcal{S}_{\mathcal{P}}\} = \mathcal{P}.$$

In either setting an element of a spread (vector space or projective) is called a 'component'.

Let Σ denote the projective space (isomorphic to) $PG(V-1,K)$. Now embed Σ into a projective space Σ^+ so that Σ is a (projective) hyperplane of Σ^+. This is accomplished as follows: Let V^+ denote the associated vector space over K such that $V^+ = V \oplus Q$, where Q is a 1-dimensional K-vector space and the sum is considered an 'external direct sum'.

Hence, $\pm$ is isomorphic to $PG(V-1,K)$ and Σ^+ is isomorphic to $PG(V^+-1,K)$. Furthermore, let $W_o^+ = W_o \oplus Q$ (again, an external direct sum).

We now define an affine translation plane $\pi_{\mathcal{S}_{\mathcal{P}}}$ as follows: the 'points' are the points of $\mathcal{P}^+ - \mathcal{P}$ and the 'lines' are the projective subspaces of $\mathcal{P}^+$ isomorphic to $PG(W_o^+-1,K)$ that intersect $\mathcal{P}$ in an element of $\mathcal{S}_{\mathcal{P}}$.

When an affine translation plane is obtained as above using a projective spread, we shall say that the plane is obtained using the 'Bruck-Bose method'.

So, using the Bruck-Bose method or projective space-approach, a finite projective spread is a set of mutually skew $n-1$ -dimensional projective spaces covering the points of $PG(2n-1,q)$.

The essentially difference between the André and Bruck-Bose approaches then depends on the methods used in constructing the affine translation plane corresponding to the spread whether it be a vector space or a projective spread.

In the following discussion, we shall give the interconnections between these two methods, generally considering an arbitrary vector space over a skewfield K of the form the external direct sum $W_o \oplus W_o$, for W_o a K-vector space. It might be noted that the Bruck-Bose approach has only been considered for finite dimensional projective spaces. However, with our slightly different definition of 'projective spreads', the method can also be accomplished for possibly infinite dimensional vector spaces.

Here we do consider such an approach more generally for arbitrary dimensional projective spaces. The question is, of course, whether vector space spreads and projective spreads are equivalent. We provide a proof for the following since it provides something of a mirror in which to reflect a more general construction that is the point of this article.

We begin with a fundamental lemma which shows that the projective space $PG(V^+-1,K)$ obtained from the vector space V^+ over K by defining the incidence geometry as the lattice structure of vector subspaces of V^+ is isomorphic to $PG(V,K)$, the projective space obtained by adjunction of a hyperplane at infinity to the affine space $AG(V,K)$ associated with V. In the following, we adopt the notation developed in the previous definitions.

Lemma 1. *(1) $PG(V^+-1,K)$ is isomorphic to $PG(V,K)$.*

(2) Bases may be chosen for V and V^+ so that

(a) vectors of V^+ may we represented in the form:

$$((x_i),x_\infty) \text{ for all } i\in\rho,\ \rho \text{ an index set, where } x_i,x_\infty\in K,$$

$$((x_i),0) \text{ for all } i\in\rho,\ \rho \text{ an index set, where } x_i\in K$$

represent vectors in V and

(b) regarding two non-zero 'tuples' above to be equal if and only if they are K-scalar multiples of each produces the 'homogeneous coordinates' of the associated projective spaces $PG(V-1,K)$ and $PG(V^+-1,K)$,

(c)

$$((x_i),1) \text{ for all } i\in\rho,\ \rho \text{ an index set, where } x_i\in K$$

represents homogeneous coordinates for a subset isomorphic to $AG(V,K)$.

(3) Furthermore, we may consider $PG(V,K)$ as the adjunction of $PG(V-1,K)$ as the hyperplane at infinity of $AG(V,K)$.

Proof. Let $\mathcal{B}$ be a vector basis for the vector space V over K. Let ρ be an index set for $\mathcal{B}$ so that

$$\mathcal{B}=\{e_i;i\in\rho\}.$$

Consider the vector space a left K-space. A vector $v=\sum x_ie_i$ where $x_i\in K$ and $e_i\in\mathcal{B}$ and $x_i=0$ for all but finitely many e_i's in $\mathcal{B}$.Then, 'points' of $AG(V,K)$ may be considered as tuples $((x_i);i\in\rho)$ such that $x_i=0$ for all but finitely many i's in

ρ. Now we consider a new 'tuple' of the form $((x_i), x_\infty)$ where x_i, $x_\infty \in K$, for $i \in \rho$ but agree to identify two such tuples if and only if they are scalar multiples of each; $((x_i), x_\infty) = ((x_i)', x'_\infty)$ if and only if there is a non-zero $\alpha \in K$ such that $x_i = \alpha x'_i$ and $x_\infty = \alpha x_\infty$. We further require that not all elements of either tuple are 0.

If $x_\infty = 1$, consider the mapping $((x_i)) \longmapsto ((x_i), 1)$. This is an injection and hence an embedding of the points of $AG(V, K)$ into the new tuple system. If $x_\infty = 0$ then the set of scalar multiples of tuples $((x_i), 0)$ represents a projective space isomorphic to $PG(V - 1, K)$. The set of tuples as represented is the projective space $PG(V, K)$ obtained by adjunction of the hyperplane at infinity isomorphic to $PG(V - 1, K)$ to $AG(V, K)$.

Now considering $V^+ = V \oplus Q$, choose a basis $\{e_\infty\}$ for Q. Then, $\mathcal{B} \cup \{e_\infty\}$ is a basis for V^+. We may then represent a vector of V^+ in the form $((x_i), x_\infty)$, where now

$$((x_i), x_\infty) = \sum x_i e_i + x_\infty e_\infty.$$

Now form the projective space $PG(V^+ - 1, K)$ by defining the projective subspaces using the lattice of vector subspaces of V^+. Since a 'point' of $PG(V^+ - 1, K)$ is a 1-dimensional K-subspace $\langle (x_i, x_\infty); i \in \rho \rangle$, this is equivalent to requiring two 'tuples' to be equal if and only if they are scalar multiples of each other. Hence, $PG(V^+ - 1, K)$ is isomorphic to $PG(V, K)$. ∎

2.1 André implies Bruck-Bose.

Theorem 3. *André implies Bruck-Bose: Hence, from a vector space V over a skewfield K and spread $\mathcal{S}$, we obtain a translation plane $\pi_{\mathcal{S}}$ and a corresponding projective partial spread $P(\mathcal{S})$ of $PG(V - 1, K)$.*

Proof. Let $\mathcal{L}_i$ for $i \in \rho$, ρ an index set, be components of a corresponding vector space spread $\mathcal{S}$ of V. Now take the lattice of K-subspaces $PG(V - 1, K)$. Clearly, $P(\mathcal{S})$ is a projective spread of $PG(V - 1, K)$. ∎

Note that although it is immediate that from a vector space spread $\mathcal{S}$ a projective space spread $P(\mathcal{S})$ is obtained, we have not determined whether the translation plane $\pi_{\mathcal{S}}$ obtained using the André method is isomorphic to the translation plane $\pi_{P(\mathcal{S})}$ obtained using the Bruck-Bose method. We shall now show this to be the case while considering the more complicated converse to the above theorem.

2.2 Bruck-Bose implies André.

Theorem 4. *Bruck-Bose implies André: Let V be a vector space over a skewfield K, which is the external direct sum $W_o \oplus W_o$, where W_o is a K-vector space. If Q is any 1-dimensional K-subspace, form the external direct sum $V \oplus Q = V^+$ and let the lattice of K-subspaces be denoted by $PG(V^+ - 1, K)$.*

(1) Then, $PG(V - 1, K)$ may be considered a hyperplane of $PG(V^+ - 1, K)$ ('co-dimension 1-subspace'), which we call the 'hyperplane at infinity'.

(2) Let $\mathcal{S}$ be a projective spread in $PG(V-1,K)$ and let $\mathcal{L}$ be an element of $\mathcal{S}$. Now $\mathcal{L}$ is a subspace of V that is K-isomorphic to W_o. Consider $\mathcal{L}^+ = \mathcal{L} \oplus Q$. Then, in $PG(V^+-1,K)$,

(a) $PG(\mathcal{L}^+-1,K)$ is isomorphic to $PG(W_o,K)$ and

(b) $PG(\mathcal{L}^+-1,K)$ intersects $PG(V-1,K)$ in $\mathcal{L}$.

(3) From $PG(V^+-1,K)$, remove the hyperplane at infinity $PG(V-1,K)$ to produce

(a) an affine space isomorphic to $AG(V,K)$ and

(b) a corresponding vector space spread $V(\mathcal{S})$ obtained by taking

$$V(\mathcal{S}) = \left\{PG(\mathcal{L}^+-1,K) - PG(V-1,K) \cap PG(\mathcal{L}^+-1,K); \mathcal{L} \in \mathcal{S}\right\}.$$

The proof to the above result will become clear once we compare the translation planes obtained from the two processes.

2.3 The Translation Planes are Isomorphic.

Notation 2. *If $\mathcal{S}$ is a projective spread, we shall use the notation introduced in the theorem of the previous subsection to obtain a vector space spread $V(\mathcal{S})$. If $\mathcal{Z}$ is a vector space spread, we shall use the notation $P(\mathcal{Z})$ to denote the corresponding projective spread obtained using lattices.*

Theorem 5. *(1) The translation plane $\pi_{\mathcal{S}}$ obtained from the projective spread $\mathcal{S}$ by using the Bruck-Bose method is isomorphic to the translation plane $\pi_{V(\mathcal{S})}$ obtained from the vector space spread $V(\mathcal{S})$ using the André method.*

(2) The translation plane $\pi_{\mathcal{Z}}$ obtained from a vector space spread $\mathcal{Z}$ using the André method is isomorphic to the translation plane $\pi_{P(\mathcal{Z})}$ using the Bruck-Bose method.

(3) $\pi_{\mathcal{S}} \simeq \pi_{V(\mathcal{S})} \simeq \pi_{P(V(\mathcal{S}))}$ and $\pi_{\mathcal{Z}} \simeq \pi_{P(\mathcal{Z})} \simeq \pi_{V(P(\mathcal{Z}))}$, using the notations of (1) and (2).

Proof. Again, we let $\mathcal{L}_i$ for $i \in \rho^{\mathcal{Z}}$, $\rho^{\mathcal{Z}}$ an index set, for 'components of a corresponding vector space spread $\mathcal{Z}$ of V. Now take the lattice of K-subspaces $PG(V-1,K)$. As noted previously, $P(\mathcal{Z})$ is a projective spread of $PG(V-1,K)$. The question basically comes down to asking whether $\pi_{V(P(\mathcal{Z}))}$ is isomorphic to $\pi_{\mathcal{Z}}$. This will be true if and only if there is an element of $\Gamma L(V,K)$ that maps $V(P(\mathcal{Z}))$ onto $\mathcal{Z}$.

That is, all of the previous connections between the two translation planes are valid if we can show that embedding $PG(V-1,K)$ into $PG(V^+-1,K)$ and then restricting to the associated affine space $AG(V,K)$ produces the same vector space spread.

Since we may extend to $PG(V^+-1,K)$ by taking the lattice of K-subspaces of $V^+ = V \oplus Q$, letting $\mathcal{L}_i^+ = \mathcal{L}_i \oplus Q$, the corresponding elements of the extension become $PG(\mathcal{L}_i^+-1,K)$. As argued in the previous lemma, choosing a basis $\mathcal{B}$ for V and a basis $\{e_\infty\}$ for Q, we obtain that the points of $PG(V^+-1,K)$ are the 'homogeneous coordinates' $(v,0)$ and $(w,1)$ where $w,v \in V$. Letting the set $\{(v,0); v \in V\}$ denote the hyperplane at infinity, removal of this set produces the set of points $\{(w,1); w \in V\}$ of an affine space. Note that two points $(w,1)$ and $(v,1)$ of the affine space are joined by a line consisting of the lattice of K-subspaces of the 2-dimensional K-subspace generated by $(w,1)$ and $(v,1)$ by removing the point with homogeneous coordinates $(t,0)$. This becomes the translate of a 1-dimensional K-subspace.

Hence, to obtain the corresponding 1-dimensional vector subspace, we may take $\langle (0,1),(v,1) \rangle$, for $v \neq 0$, as a 2-dimensional K-subspace and then remove the point $\langle (v,0) \rangle$ to obtain the set $(\alpha v, 1)$ for all $\alpha \in K$, which we identify with $[\alpha v]$.

In this way, the projective space $PG(\mathcal{L}_i^+ - 1, K)$ becomes the set of vectors $[u]$ for all $u \in \mathcal{L}_i$. Since any two such projective spaces $PG(\mathcal{L}_i - 1, K)$ generate $PG(V-1,K)$, then any point of $PG(V-1,K)$ is a point of a line joining a point of $PG(\mathcal{L}_i - 1, K)$ and a point of $PG(\mathcal{L}_j - 1, K)$ for $i \neq j$. Equivalently, all 1-dimensional K-subspaces of V are in the internal direct sum $\mathcal{L}_i \oplus \mathcal{L}_j$, which is equivalent to $V = \mathcal{L}_i \oplus \mathcal{L}_j$. This proves the result. ∎

The model that we want to emphasize to the reader when going from vector space to affine space to projective spaces is summarized in the following remark.

Remark 2. *Take a vector space spread $\mathcal{Z}$ and form the associated affine space. Extend the affine space to a projective space and extend the affine spaces corresponding to spread components to projective subspaces. Construct a projective spread on the hyperplane at infinity by intersection of the projective 'component' subspaces. Thus, abusing language somewhat we have:*

'Bruck-Bose is André at infinity.

3 Subgeometries.

If Σ is a projective space and Π is a subset of points of Σ such that if a line ℓ of Σ intersects Π in at least two points A and B then we define a 'line' of Π to be the set of points $\ell \cap \Pi$. If the points of Π and the 'lines' induced from lines of Σ form a projective space, we say that Π is a 'subgeometry' of Σ. If Σ is a projective space over a skewfield F, it is not completely clear when Π is a projective space over a skewfield T that T may be taken as a subskewfield of F. However, all of the subgeometies that have been considered thus far are of this type. Furthermore, the subgeometries studied in this article will be assumed to be of this type. Hence, to be clear, we formulate our definition of subgeometry as follows.

Definition 5. *Let Σ be a projective space isomorphic to $PG(V-1,F)$, where F is a skewfield. If Π is a subgeometry Σ that arises from a subskewfield T of F, we shall say that Π is an 'induced' subgeometry.*

Remark 3. *(1) Let V be an F-vector space such that the points of Σ are 1-dimensional F-subspaces and lines of Σ are 2-dimensional F-subspaces. If Π is an induced subgeometry isomorphic to $PG(Z-1,T)$ then the points of Π are assumed to be exactly those 1-dimensional F-subspaces $\langle z \rangle_F$ with a generator a non-zero vector z of Z considered as a subspace of V over T and T is a subskewfield of T.*

(2) In the following any reference to 'subgeometries' will intrinsically mean 'induced subgeometries'.

Considering subgeometries in the above manner, there is a natural generalization to what we shall call 'quasi-subgeometries'.

Definition 6. *A 'quasi-subgeometry' of a projective space is a subset of points that can be made into a projective space such that the lines of the projective set are subsets of lines of the projective space.*

If Σ is a projective space isomorphic to $PG(V-1,F)$, where F is a skewfield and Π is a subset of Σ isomorphic to $PG(Z-1,T)$, where T is a subskewfield of F, if the points of Π are induced in the same manner as in the above definition of subgeometry and 'lines' of Π as a $PG(Z-1,T)$ are subsets of lines of Σ, we shall say that Π is an 'induced quasi-subgeometry'.

In the following any reference to a 'quasi-subgeometry' shall intrinsically mean an induced quasi-subgeometry.

Remark 4. *In either a subgeometry or a quasi-subgeometry, we also assume that the 'lines' of the structure are induced from the set of 1-dimensional F-subspaces that lie in a 2-dimensional F-subspace and are generated by vectors of Z.*

Definition 7. *A partition of a projective space by quasi-subgeometries is called a 'quasi-subgeometry partition'. If all of the quasi-subgeometries are subgeometries, the partition is called a 'subgeometry partition'.*

What we are trying to accomplish in this article is to construct a variety of new partitions of projective geometries by either subgeometries or by quasi-subgeometries.

We have seen that if the subgeometries or quasi-subgeometries are all what might be called 'half-dimensional' then the ideas of 'Bruck-Bose is André at infinity' show that there is a corresponding translation plane. However, if the quasi-subgeometries involved in a partition could be essentially chosen from any set of projective geometries, it is not at all clear that there are connections between vector space spreads and quasi-subgeometry partitions or even if there are spreads producing translation planes. In order to see what sorts of vector space spreads or generalizations of these might conceivably correspond to quasi-subgeometry partitions, we define and study 'fans' first in vector space spreads and then more generally in 'generalized spreads'. We note that all of the quasi-subgeometries that we obtain from 'fans' are induced quasi-subgeometries. We first consider 'partial spreads' and 'generalized spreads'.

4 Partial Spreads.

Definition 8. *A 'partial vector space spread' of a vector space V is merely a set of vector subspaces any two distinct elements of which direct sum to the vector space and are each isomorphic to a fixed subspace.*

In a similar manner, one way define a 'partial projective spread'. The elements of a partial spread are called 'components'.

From a partial vector space spread $\mathcal{P}$, we may form a 'translation net' $T(\mathcal{P})$ by taking the 'points' of the net to be the vectors and the 'lines' of the net to be the vector translates of the partial spread components.

Remark 5. *The analysis of André versus Bruck-Bose works for partial spreads just as for spreads. Hence, a partial vector space spread produces a partial projective spread and conversely.*

Definition 9. *Let π be a translation plane defined using the André method. The group $\mathcal{T}$ of translations of the underlying vector space V over a skewfield K are collineations of π. This group is called the 'translation group' of π.*

Let $\mathcal{Z}$ denote the vector space spread defining π, so that $\pi = \pi_{\mathcal{Z}}$. Let $\Gamma L(V,K)$ denote the semi-linear group of V over K. If σ is a collineation of π then σ is an element of the semi-direct product of $\Gamma L(V,K)$ by $\mathcal{T}$. Let $\mathcal{F}$ denote the full collineation group of π. Then, $(\Gamma L(V,K)\cap\mathcal{F})\mathcal{T} = \mathcal{F}$.

We note that $\Gamma L(V,K)\cap\mathcal{F}$ may be regarded as the stabilizer subgroup of the zero vector $0 : \mathcal{F}_0$.

The group $\Gamma L(V,K)\cap\mathcal{F}$ is called the 'translation complement of π with respect to K'.

If $GL(V,K)$ is the general linear subgroup of $\Gamma L(V,K)$ then $GL(V,K)\cap\mathcal{F}$ is called the 'linear translation complement' of π with respect to K'.

If $K^+$is the maximal skewfield containing K such that the spread components are also K^+-subspaces, we shall say that K^+ is the 'kernel' of π.

In this case, if $\mathcal{F}$ is the full collineation group of π then also

$$(\Gamma L(V,K^+)\cap\mathcal{F})\mathcal{T} = \mathcal{F}.$$

We then use the terms 'translation complement' and 'linear translation complement' when referring to the 'translation and linear translation complements with respect to the kernel K^+.

Note that the translation complement is the subgroup of $\Gamma L(V,K^+)$ that permutes the spread components.

Remark 6. *We may extend the previous definitions to collineation groups of partial vector space spreads and speak of the translation complement of a translation net as the subgroup of $\Gamma L(V,K)$ that permutes the partial spread components. However, it may also be when considering the action on partial spreads that the associated vector space V may be allowed to be a vector space over a field L different from K and L need not be a superfield of K and a collineation might be in $\Gamma L(V,L)$ instead of $\Gamma L(V,K)$.*

4.1 Generalized Partial Spreads.

Definition 10. *(1) Let V be a vector space over a skewfield K. Let $\{T_i; i\in\Omega\} = \mathcal{T}$ be a set of mutually disjoint K-vector subspaces, then $\mathcal{T}$ shall be called a 'generalized partial spread'. If $V = \cup_{i\in\Omega}T_i$, then $\{T_i; i\in\Omega\}$ shall be called a 'generalized spread' of V.*

Note that this allows that the individual subspaces can be of different dimensions.

(2) Assume that V is isomorphic to the external direct sum $\sum_{i\in\lambda}W_{o,i}$, of dimension $|\lambda|\,w_o$, where all $W_{o,i}$ are K-vector spaces that are K-isomorphic to a fixed subspace W_o over K and all T_i are K-isomophic to W_o, and $dim_K W_o = w_o$, we shall say that $\{T_i; i\in\Omega\}$ is a '$|\lambda|$-generalized spread of size (w_o,K)'.

In all cases, the subspaces T_i are called the 'components' of the generalized spread.

For example, if V is a dw_o-dimensional over K for d finite and W_o is w_o-dimensional, we obtain a 'd-generalized spread of size $(w_o$,$K)$'.

(3) If $d = 2$ and all components of the generalized spread pairwise generate V then we have a spread.

If K is finite of order q, and w_o is also finite, we shall say that we have a 'd-generalized spread of size (w_o, q)'. We shall also use the terminology 'd-spread' to indicate this same structure.

We shall be interested in the case when K is a field.

Definition 11. *We shall say that an extension field D of K 'acts' on a generalized spread if and only if*

(i) the multiplicative group D^* is in $GL(V, K)$,

(ii) D^* contains the scalar group K^*,

(iii) permutes the spread components and

(iv) acts fixed-point-free (each non-identity element fixes only the zero vector).

Hence, a field D acting on a generalized spread maps components of a given dimension onto components of the same dimension.

Any orbit of components of a generalized spread under a field D shall be called a 'fan'. If the subfield D_L of D is induced from the stabilizer of a component L, we shall call the orbit $O(L)$ under D a 'D_L-fan'.

Definition 12. *Let $\mathcal{S}$ be a d-spread of size (s, K) of a vector space V over a skewfield K. If we define 'points' as vectors and 'lines' as translates of spread components, we obtain an incidence geometry of points and lines such that the line set is partitioned into a set of parallel classes. If $d = 2$, then distinct lines from different parallel classes must intersect, whereas if $d > 2$, this is not the case. Such an incidence geometry is called a 'translation space' and is an example of a 'Sperner space'.*

Remark 7. *We shall be interested in the collineation group of a translation space as well as the collineation group of a generalized spread.*

We define the 'linear' translation complement of a generalized spread over the skewfield to be the subgroup of $GL(V, K)$ that permutes the components of the spread.

5 Fans.

In this section, we consider that K is a field.

Let V be a K-vector space. Let D be a field extension of K that acts on a generalized partial spread $\mathcal{P}$ of V. For any component L of $\mathcal{P}$, then $D_L^* \cup \{0\} = D_L$ is a subfield of D since L is a K-subspace. Furthermore, L is a D_L-vector space if D^* is fixed-point-free (no non-identity element fixes a non-zero vector).

Notation 3. *We shall consider the vector spaces V and L over various fields F, T Consistent with our previous notation, we shall denote the corresponding projective spaces of the lattice of F-subspaces as $PG(V-1, E)$ and $PG(L-1, E)$ respectively, where E could be F or T. Note that when V is finite n-dimensional , care must be taken to change to the appropriate $PG(k-1, E)$, where k is the E-dimension of V.*

Definition 13. *If D is a 2-dimensional field extension of K, we say that the associated fan is a 'quadratic fan' provided the stabilizer of a component is not D, that the fan is not simply a single component.*

Lemma 2. *Every quadratic fan is a K-regulus in some associated projective space over K.*

Proof. We have that D acts on the D_L-fan and D is a field extension of K of degree 2. Since D^* contains the kernel homology group K^* by assumption, then $K^* \subseteq D_L^* \subseteq D^*$, implying $D_L^* = K$. Hence, the D-orbits on the quadratic fan are 1-dimensional D-subspaces which then are 2-dimensional K-subspaces partitioned by their intersections with the orbit components in the quadratic fan. Let L and M be distinct components of the orbit $O(L)$ under D^*. Since each of L and M are K-vector spaces, we have that $L \oplus M$ is a K-vector space. Moreover, there exists an element $\alpha \in D^*$ such that $M = L\alpha$. Similarly, any component N in the orbit $O(L)$ is $L\beta$ for some $\beta \in D^*$. We note that for $v \in L - \{0\}$, then $v\delta$ for all $\delta \in D^*$ union the zero vector is a 1-dimensional D-vector space and thus L and $L\alpha$ must each intersect $\{v\delta; \delta \in D\}$ in a 1-dimensional K-subspace since D^* contains the K^*-scalar group. Thus, there are vectors w and t from L and $L\alpha$ respectively that generate the 2-dimensional K-space $\{v\delta; \delta \in D\}$. Thus, for any $\alpha \in D^*$ such that $L \neq L\alpha$ then $L\beta \subseteq L \oplus L\alpha$, for all $\beta \in D^*$. If $L\beta \neq L\delta$ then it also follows that L and $L\alpha \subseteq L\beta \oplus L\delta \subseteq L \oplus L\alpha = V^L$. So, we have that for any two distinct components N and J of the orbit $O(L)$ of D^*, it then follows that $N \oplus J = V^L$. Hence, we have a partial spread, inducing a translation net relative to V_L. So, we have a net which is covered by Pappian subplanes. That is, we have a 'subplane covered net'. By the results of the author [5], the net is a regulus net in some projective space. This proves the result. ■

5.1 Folding the Fan.

Our definition of a 'fan' is an orbit of components under a group D^*. These components share the zero vector and are otherwise disjoint as sets. When we consider a 'folding' of a fan, we basically fold all of the components into a single one in the orbit and define a projective space arising from the fold as follows:

5.1.1 The Incidence Geometry.

Let $O(L) = \eta$ be an D_L-fan, for L a component and let the stabilizer of L in D^* define a field D_L. Then, by taking the lattice of D-subspaces, we obtain a projective space $PG(L-1, D_L)$. Since D acts on the vector space V, V considered over D produces $PG(V-1, D)$ by the lattice definition. The goal is to see that $PG(L-1, D_L)$ may be considered the 'folded fan' and as a quasi-subgeometry of $PG(V-1, D)$, in the sense that we have a subset $G(\eta)$ of points and a set of 'subsets' of lines of $PG(V-1, D)$ that forms a projective space $PG(L-1, D_L)$.

We shall show that this provides a quasi-subgeometry in $PG(V-1, D)$. To begin with, we assume that $D_L = K$ and produce a subgeometry. After this, we realize that the arguments are valid more generally and produce quasi-subgeometries.

So, for the moment, let $D_L = K$.

Hence, we define an incidence geometry $G(\eta)$ as follows: The 'points' are those D^*-orbits union the zero vector of vectors that lie in $O(L)$. Given two non-zero vectors u and v in different D^*-orbits that lie components on $O(L)$ note that the

2-dimensional D-subspace $\langle u,v\rangle_D$ is the same subspace when u and v are considered on the same component L^*. Within $\langle u,v\rangle_D$, a 'line' will be the set of 'points' as D^*-orbits arising from vectors in $\langle u,v\rangle_K$, where u and v are vectors on a component $L^*$of $O(L)$ that are in different D^*-orbits. The question is $\langle u,v\rangle_D \cap G(\eta) = \{\langle \alpha_o v + \beta_o v\rangle_D\,; \alpha_o, \beta_o \in K\}$?

We now utilize the model that a quadratic fan is a K-regulus.

Lemma 3. *Let Z be a vector space and R a regulus in $PG(Z-1,K)$ for K a field. Then, R is a partial spread and we let $N(R)$ denote the associated regulus net in the associated K-vector space Z.*

Then

(a) any pair of subplanes π_o and π_1 that share the zero vector can be embedded in a unique derivable subnet $\mathcal{D}_{\pi_o,\pi_1}$ of $N(R)$ that contains π_o and π_1.

(b) Furthermore, the only subplanes incident with the zero vector that non-trivially intersect $\pi_o \oplus \pi_1$ are in the derivable subnet $\mathcal{D}_{\pi_o,\pi_1}$.

Proof. Part (a) follows from Johnson [5] (15.25) p. 187. Each subplane is a 2-dimensional K-vector space. Now $\pi_o \oplus \pi_1$ is a 4-dimensional K-vector space V_o. Since the supernet defines a K-regulus, the subnet is also a regulus in an associated $PG(3,K)$ corresponding to the lattice of K-subspaces of V_o. Any subplane within V_o must be a line of the opposite regulus so is in the derivable net. Furthermore, if π_2 is a subplane incident with the zero vector and P_2 is a non-zero point of π_2, then π_2 is the unique subplane containing P_2 and 0 so that if P_2 intersects the derivable subnet then there is a subplane of the derivable subnet containing 0 and P_2, implying that π_2 is a subplane of the derivable subnet. But, P_2 is a point of the derivable subnet only if it also lies within $\pi_o \oplus \pi_1$. ∎

Lemma 4. *If D acts on the quadratic fan then*

$$\langle u,v\rangle_D \cap G(\eta) = \{\langle \alpha_o v + \beta_o v\rangle_D\,; \alpha_o, \beta_o \in K\}\,.$$

Proof. Since in the previous lemma, π_o and π_1 are 1-dimensional D, subspaces, let u and v be in π_o and π_1 respectively and assume that u and v are on the same component of the quadratic fan. $\pi_o \oplus \pi_1$ is a 2- dimensional D-space $\langle u,v\rangle_D$. Since the only subplanes in $\pi_o \oplus \pi_1$ are in the derivable subnet and these subplanes are all 1-dimensional D-subspaces, all subplanes within the derivable net can be represented by $\langle \alpha_o v + \beta_o v\rangle_D$. This proves the result. ∎

Hence, we also obtain:

Lemma 5. *Given two distinct points A and B of $G(\eta)$, there is a unique line AB incident with A and B. This line is a subline of the line in $PG(V-1,D)$ obtained from the 2-dimensional D-space $\langle A,B\rangle_D$.*

Furthermore, for any two distinct points C, D of AB, $AB = CD$.

Proof. Let L_1 and L_2 be two lines of $O(L)$ and let u_i, v_i be non-zero vectors in A and B respectively and u_i, v_i are in L_i for $i = 1,2$. We need to show that the 'line' arising from D^*-orbits of vectors in $\langle u_1,v_1\rangle_{D_L}$ is the same as the 'line' arising from D^*-orbits of $\langle u_2,v_2\rangle_{D_L}$. We note that there is an element g of D^* that maps

L_1 onto L_2. Now since $g(u_1)$ and u_2 are in the same D^*-orbit and are also in L_2, it follows that they are D_L-scalar multiples of each other. Hence, there exists an element $\alpha_1 \in D_L^*$ such that $\alpha_1 g(u_1) = u_2$. Similarly, there exists an element $\alpha_2 \in D_L^*$ such that $\alpha_2 g(v_1) = v_2$. Now the 'points' originating from $\langle u_1, v_1\rangle_{D_L}$ are of the form $(\delta u_1 + \rho v_1)D$ and the 'points' from $\langle u_2, v_2\rangle_{D_L}$ are of the form $(\lambda u_2 + \gamma v_2)D$ for $\delta, \rho, \lambda, \gamma \in D_L$. Now $(\delta u_1 + \rho v_1)D = g((\delta u_1 + \rho v_1))D$
$= (\delta g(u_1) + \rho g(v_1))D = (\delta \alpha_1^{-1} u_2 + \rho \alpha_2^{-1} v_2)D$. Thus, the two 'lines' have the same 'points' and hence we have a unique such line. Since the 'lines' are defined from the set of points on a line of $PG(V-1, D)$, the lines of $G(\eta)$ are 'sublines' of $PG(V-1, D)$.

Take any 2-dimensional D_L-space Ψ on L. Then, there is a unique set of D^*-orbits defined by the images of vectors within Ψ. By the above remarks, this set is a 'line'. Take any two such orbits R and T. Then R and T intersect L in distinct 1-dimensional D_L-subspaces of L in Ψ and hence generate Ψ. Since the corresponding set of orbits of elements of Ψ then become our 'line', then since every 1-dimensional D_L-subspace of L defines a unique orbit 'point', it follows that any 2-dimenisonal D_L-subspace of L produces and/or corresponds to a 'line' of $G(\eta)$ and any 'line' of $G(\eta)$ corresponds to a 2-dimensional D_L-subspace of L. ∎

We wish to show that $G(\eta)$ is a projective space. Since the 'point set' corresponds to the set of 1-dimensional D_L-vector subspaces on L, and the line set corresponds to the set of 2-dimensional D_L-subspaces on L, we may define the associated subgeometry based on the lattice of D_L-subspaces to realize this as a projective space isomorphic to $PG(L-1, D_L)$.

Theorem 6. *Let η be an K-fan (a quadratic fan). Define an incidence geometry $G(\eta)$ as follows:*

(i) 'points' of $G(\eta)$ are the point-orbits of D acting on the partial spread $\mathcal{P}$ and

(ii) 'lines' are defined by pairs of distinct 'points' A and B as follows: If $A = uD$ and $B = vD$, for u, v on same component of $O(L)$, then the line AB is the set of points arising as D-orbits of vectors in $\langle u, v\rangle_{D_L}$.

Then $G(\eta)$ is a projective space isomorphic to $PG(L-1, D_L)$ that may be considered a subgeometry of $PG(V-1, D)$.

Our main result on quadratic fans is that a vector space generalized spread that is the union of fans produces a subgeometry partition of a projective space.

Theorem 7. *Let $\mathcal{Z}$ be a generalized spread whose components are K-subspaces with underlying vector space V over a field K. Let D be a quadratic field extension of K. Assume that D acts on $\mathcal{Z}$.*

Let the orbits of D^* be denoted by $O(L)$ where L is a component of the spread of π.

For $O(L)$, let D_L denote the subfield of D leaving L invariant. If the generalized spread $\mathcal{Z}$ for π is

$$\cup\{O(L); L \in \mathcal{Z}\},$$

let L_i be a representative for the orbit $O(L_i) = \eta_i$, for $i \in \Lambda$, an index set for the set of orbits.

Then

(a) there is a subgeometry partition $Sub(\mathcal{Z})$ of $PG(V-1,D)$ defined as follows:

$$Sub(\mathcal{Z}) = \{G(\eta_i) \simeq PG(L_i - 1, D_{L_i}); i \in \Lambda\}.$$

That is, a generalized spread $\mathcal{Z}$ that is a union of D_{L_i}-fans produces a subgeometry partition of $PG(V-1,D)$ by $PG(L_i-1,D_{Li})$'s for all $i \in \Lambda$, where D_{L_i} is either K or D.

(b) Each orbit of length > 1, $O(L_i)$ under D^* is a K-regulus in some projective space $PG(V^{L_i}-1,K)$. Note that we are not assuming that the dimensions of L_i and L_j are equal over K.

We now take up the more general problem.

5.2 Quasi-Subgeometries.

In the general situation, a D_L-fan produces a projective geometry that sits within $PG(V-1,D)$ by defining lines to arise from the D-1-subspaces originating from a vector on L; that is by $\{\langle\alpha_o v + \beta_o v\rangle_D; \alpha_o,\beta_o \in K\} \subseteq \langle u,v\rangle_D \cap G(\eta)$. However, although this would produce a projective geometry isomorphic to $PG(W_o-1,D_L)$ that sits in $PG(V-1,D)$, it may not actually be the complete set of intersection points and hence would not properly be considered a 'subgeometry'. We have define this embedded projective geometry as a 'quasi-subgeometry', so then from any spread that is a union of D_{L_i}-fans, we would obtain a partition of $PG(V-1,D)$ by quasi-subgeometries isomorphic to $PG(W_o-1,D_{L_i})$ for $i \in \Phi$.

The analogous theorems for quasi-subgeometry partitions are as follows and note that a re-reading of the previous results reveals that the proofs are essentially the same as for the subgeometry partitions, except that a 'subline' given may not be the intersection of the pointset by a given line, although it will be a subset of a line.

Theorem 8. *Let η be a D_L-fan. Define an incidence geometry $G(\eta)$ as follows:*

(i) 'points' of $G(\eta)$ are the point-orbits of D acting on the generalized partial spread $\mathcal{P}$ and

(ii) 'lines' are defined by pairs of distinct 'points' A and B as follows: If $A = uD$ and $B = vD$, for u, v on same component of $O(L)$, then the line AB is the set of points arising as D-orbits of vectors in $\langle u,v\rangle_{D_L}$ of the form $\langle\alpha_o v + \beta_o v\rangle_D; \alpha_o,\beta_o \in D_L$.

Then $G(\eta)$ is a projective space isomorphic to $PG(L-1,D_L)$ that may be considered a quasi-subgeometry of $PG(V-1,D)$.

Theorem 9. *$\mathcal{Z}$ be a generalized spread $\mathcal{S}$ with underlying vector space V with components K-spaces where K is a field. Let D be a field acting on $\mathcal{Z}$.*

Let the orbits of D^* be denoted by $O(L)$ where L is a component of the generalized spread.

For $O(L)$, let D_L denote the subfield of D leaving L invariant. If the generalized spread $\mathcal{Z}$ for π is

$$\cup\{O(L); L \in \mathcal{Z}\},$$

let L_i be a representative for the orbit $O(L_i) = \eta_i$, for $i \in \Lambda$, an index set for the set of orbits.

Then there is a quasi-subgeometry partition $Sub(\mathcal{Z})$ *of* $PG(V-1,D)$ *defined as follows:*

$$Sub(\mathcal{Z}) = \{G(\eta_i) \simeq PG(L_i - 1, D_{L_i}); i \in \Lambda\}$$

That is, a generalized spread $\mathcal{Z}$ *that is a union of* D_{L_i}*-fans produces a quasi-subgeometry partition of* $PG(V-1,D)$ *by* $PG(L_i - 1, D_{Li})$*'s for all* $i \in \Lambda$.

5.3 Unfolding the Fan.

We think of a particular quasi-subgeometry as a 'projective' folded fan and argue that this can be unfolded into an associated vector space fan.

Let V be a K-vector space, for K a field and D be an extension field of K. Let L be a K-vector subspace such that V is a D-space and L is a D_L-space. Hence, assume that we have a (induced) quasi-subgeometry Σ_o isomorphic to $PG(L-1, D_L)$, where Σ_o is in Σ_1, and Σ_1 is the lattice of subspaces $PG(V-1,D)$. We want to realize this quasi-subgeometry as a D_L-fan in the vector space V over D. Since we assume that D_L is a subfield of D, then V is also an D_L-vector space. We also assume that D_L contains the field K.

Lemma 6. *Under the above assumptions and notation, there is a vector space* V^+ *over* D *defining a projective space* Σ_2 *as* $PG(V^+ - 1, D)$ *so that*

$$\Sigma_o \subseteq \Sigma_1 \subseteq \Sigma_2.$$

Note that Σ_o *is a quasi-subgeometry of* Σ_1 *and* Σ_1 *is a projective subspace of* Σ_2.

Proof. Let Q be any 1-dimensional D-vector space, and consider the external direct sum $V \oplus Q = V^+$ as a D-vector space. Let Σ_2 denote the lattice of D-subspaces of V^+. In this way, we have embedded Σ_1 in the projective geometry Σ_2, which is $PG(V^+ - 1, D)$. And, as above, Σ_o may be considered isomorphic to the lattice of subspaces of the vector space W_o over D_L:

$$\begin{array}{rcl} PG(L-1, D_L) & \simeq & \Sigma_o,\ PG(V-1,D) \simeq \Sigma_1, PG(V^+ - 1, D) \simeq \Sigma_2, \text{ where} \\ \Sigma_o & \subseteq & \Sigma_1 \subseteq \Sigma_2. \end{array}$$

∎

We now make explicit a coordinate description of the above embedding. Note that this is the reflection of our previous discussion on 'Bruck-Bose is André at infinity'. We restate our Lemma 1 in terms of the field D.

Lemma 7. *(1)* $PG(V^+ - 1, D)$ *is isomorphic to* $PG(V, D)$.

(2) Bases may be chosen for V *and* V^+ *so that*

(a) vectors of V^+ *may we represented in the form:*

$$((x_i), x_\infty) \textit{ for all } i \in \rho,\ \rho \textit{ an index set, where } x_i, x_\infty \in D,$$

$$((x_i), 0) \textit{ for all } i \in \rho,\ \rho \textit{ an index set, where } x_i \in D$$

represent vectors in V and

(b) regarding two non-zero 'tuples' above to be equal if and only if they are K-scalar multiples of each produces the 'homogeneous coordinates' of the associated projective spaces $PG(V-1,D)$ and $PG(V^{+}-1,D)$,

(c)

$$((x_i),1) \text{ for all } i \in \rho,\ \rho \text{ an index set, where } x_i \in D$$

represents homogeneous coordinates for a subset isomorphic to $AG(V,D)$.

(3) Furthermore, we may consider $PG(V,D)$ as the adjunction of $PG(V-1,D)$ as the hyperplane at infinity of $AG(V,D)$.

What we try now to do is to integrate the idea that Σ_o is isomorphic to a projective space $PG(L-1,D_L)$ while at the same time being embedded into Σ_1, the projective space $PG(V-1,D)$ as a quasi-subgeometry. This is somewhat problematic since we need to think simultaneously of a point of Σ_o being a 1-dimensional D_L-vector subspace as well as a 1-dimensional D-vector subspace and we need to realize that Σ_o is only isomorphic to $PG(L-1,D_L)$ as a quasi-subgeometry of $PG(V-1,D)$.

Lemma 8. *(1) (a) Using the representation of the previous lemma, a point of Σ_o may be represented by either a 1-dimensional D-subspace generated by $((x_i),0)$ or defines a 'point' of Σ_1 having homogeneous coordinate $((x_i),1)$.*

(b) We adopt the notation $\langle(v,0)\rangle$ for some v of L, for the first situation and note that D-scalar multiplication may be defined as

$$\alpha(v,0)=(\alpha v,0).$$

(c) Note that $(\alpha v,1)$, for any $\alpha \in D$ and $v \in L$, is a 'point' of Σ_1 and Σ_2.

(d) $\alpha v=\beta w$ for v,w in $L-\{0\}$ and $\alpha,\beta \in D$ if and only if $\alpha^{-1}\beta w=v$ implying that $\alpha^{-1}\beta \in D_L$.

Remark 8. *We recall that the lattice $PG(L-1,D_L)$ corresponds to Σ_o . Hence, when we consider the points of Σ_o defined using the notation $(\alpha v,x_{\infty})$, for $v \in L$, we are using the above setup. Since Σ_o is a subset of points of $PG(V-1,D)$, there is a preimage set Σ_o^{+} of 1-dimensional D-subspaces; a subset of vectors of V. In our notation, we have a point not in $PG(V-1,D)$ of the form $(\alpha v,1)$ for all $v \in L$ and for all $\alpha \in D$.*

We define some sets of points which will become projective quasi-subgeometries of $PG(V^{+}-1,D)$:

Let

$$\Sigma_{\alpha}=\{(\alpha v,1),\langle(v,0)\rangle_D\,; v \in L\} \text{ for fixed nonzero } \alpha \in D\}.$$

(1) (a) Σ_{α} is a projective quasi-subgeometry of $PG(V^{+}-1,D)$ that is isomorphic to $PG(L^{+}-1,D_L) \simeq PG(L,D_L)$.

(b) Under the above structure of 'lines', Σ_{α} induces on the point set Σ_o an isomorphic (as projective spaces) quasi-subgeometry.

(c) $\cap\Sigma_{\alpha}=\Sigma_o=\{\langle(v,0)\rangle_D\,; v \in L\}$.

(2) Let

$$\widehat{V}=\{(w,1); w \in V\}.$$

Then by defining $\beta(w,1)=(\beta w,1)$, $\widehat{V}$ is an D-vector space isomorphic to V over D.

Also, Σ_α^- may be considered a D_L-vector subspace over D_L, that admits the scalar multiplication over $D_L \subseteq D$.

(3) Let

$$\Sigma_o^+ = \{(\alpha v,1), \alpha \in D \text{ and } v \in L\} \subseteq \widehat{V}.$$

This set of vectors produces $\Sigma_o = PG(L-1,D_L)$ when considering Σ_o^+ as the lattice of D-subspaces of the subset Σ_o^+. Let A and B be distinct 1-dimensional D-subspaces within Σ_o^+. Hence, the 'line' of Σ_o AB containing A and B must be realized within the 2-dimensional D-subspace, $\langle A,B\rangle_D$ and must abstractly correspond to a 2-dimensional D_L-subspace of W_o. If $a,b \in L$, for $a \in A$ and $b \in B$ then the 'points' of the 'line' in the quasi-subgeometry are of the form $\langle \alpha_o u + \beta_o v\rangle_D$ for all α_o, β_o in D_L, for not both α_o and β_o zero.

Let

$$\Sigma_\alpha^- = \{(\alpha v,1);\ \text{for fixed non-zero } \alpha \in D \text{ and for all } v \in L\}.$$

Then Σ_α^- may be made into a D_L-vector space by defining $\beta(\alpha v,1)=(\beta\alpha v,1)$ where $\beta \in D_L$ and $v \in L$. Under our notation, L represents (or becomes) a subset of V as Σ_1^-.

Similarly, V is a subspace of V^+ and L is a subset of V, so a subset of V^+.

Proof. (1) We consider the 2-dimensional vector space of V^+ generated by vectors $(\alpha v,1)$ and $(\alpha u,1)$ over D:

$$\{\beta((\alpha v),1)+\gamma((\alpha v)1,);\beta,\gamma \in D\}.$$

The corresponding 'line' in $PG(V^+-1,D)$ is

$$\{(\beta/(\beta+\gamma)(\alpha v),1)+(\gamma/(\beta+\gamma)(\alpha u),1,),(\alpha(u-v),0);\ \beta,\gamma \in D, \beta+\gamma \neq 0\}.$$

If we restrict β and γ to D_L, we have a 'line' corresponding to D_L. Moreover, any two 'points' of the 'line' clearly determine the same subset as a 'line'. Hence, we see that we may regard Σ_α as a projective quasi-subgeometry of $PG(V^+-1,D)$, and Σ_α is obviously isomorphic to $PG(L^+-1,D_L)$.

Deleting the set $\left\{\langle (v,0)\rangle_{D_L}; v \in L\right\}$ from Σ_α as a subset of points of $PG(V^+-1,D)$ isomorphic to $PG(L-1,D_L)$ produces an affine space isomorphic to $AG(L,D_L)$ and Σ_α^- is this set.

Note that a line of Σ_α produces a line of Σ_α^- by the deletion of a point $\langle u-v,0\rangle$, so we see that (restrict β and γ to D_L) then

$$\{(\beta/(\beta+\gamma)(\alpha v),1)+(\gamma/(\beta+\gamma)(\alpha u),1,);\beta,\gamma \in D_L, \beta+\gamma \neq 0\}$$

is a subset of Σ_α^-; Σ_α^- then is an D_L-subspace by the action indicated in the statement of part (2). Finally, we note that the action given in Σ_α is that essentially induced from the action of the associated field. This 'subline' set will induce on Σ_o a subline set of a pair of distinct points (assuming that the dimension of Σ_α^- is larger than 1). ∎

5.4 The Associated Affine Spaces $\Sigma_{\alpha}^{-} \simeq AG(L, D_L)$.

Consider the group D^*/D_L^* and let $\mathcal{C}_L = \{\alpha_j; j \in \Omega\}$ be a coset representative set for D_L^*. Now fix $\alpha \in D - \{0\}$ and consider

$$\Sigma_{\alpha}^{-} = \{(\alpha v, 1); v \in L\}.$$

The previous lemma shows that in our defined action of D on $\widehat{V}$, we may regard that D_L^* leaves each Σ_{α}^{-} invariant.

As an affine space, we suppress the '1' and write $(\alpha v, 1)$ as $[v]_{\alpha}$ to place these elements back in V (or an isomorphic copy of V).

If β_o is in D_L, then more formally, we have $\beta_o(\alpha v, 1) = (\beta_o \alpha v, 1) = (\alpha \beta_o v, 1)$. Hence, $\beta_o[v]_{\alpha} = [\beta_o v]_{\alpha}$ for all $\beta_o \in D_L$ and for all $v \in L$.

To emphasize the previous, we repeat part of the statement of the previous remark.

Lemma 9. Σ_{α}^{-} *is a subset of* $AG(V, D)$ *which may be considered isomorphic to* $AG(L, D_L)$*. Furthermore, the associated vector space over* D_L *is isomorphic to* L*. In addition,* Σ_{α} *is a subset of* Σ_2 *isomorphic to* $PG(L, D_L)$*.*

(1) Let α_j *and* α_k *be distinct elements in* $\mathcal{C}_L$*, then*

$$\Sigma_{\alpha_j}^{-} \cap \Sigma_{\alpha_k}^{-} = (0, 1).$$

(2) $\left\{\Sigma_{\alpha_j}^{-}; j \in \Omega\right\}$ *is a set of mutually isomorphic and pairwise disjoint* D_L*-vector spaces.*

Proof. In this context, Σ_{α}^{-} is clearly D_L-isomorphic to L, by the mapping that maps

$$[v]_{\alpha} \longmapsto v.$$

Thus, we have Σ_{α}^{-} is an affine geometry isomorphic to $AG(L, D_L)$. Hence, Σ_{α} is a subspace of Σ_2 isomorphic to $PG(L, D_L)$.

Now assume that $(\alpha v, 1) = (\beta u, 1) \in \Sigma_{\alpha}^{-} \cap \Sigma_{\beta}^{-}$ if and only if $\alpha^{-1}\beta u = v$, and if both u and v are non-zero then this implies that $\alpha^{-1}\beta \in D_L$. Thus, two distinct elements of a coset representative set define affine spaces that are mutually disjoint as vector spaces. This completes the proof of the the lemma. ■

5.5 D Acts on the set of Affine Spaces Σ_{α}^{-} as D^*/D_L^*.

We have previously defined an action of D^* so that:

Lemma 10. *(1)* D^* *acts transitively on* $\left\{\Sigma_{\alpha_j}^{-}; j \in \Omega\right\}$ *and induces faithfully the group* D^*/D_L^* *on this set.*

(2) D^* *fixes* $(0, 1)$ *and fixes* Σ_o *pointwise.*

(3) D^* *acts as a collineation group of the affine space* $AG(V, D)$ *that fixes the zero vector of the associated vector space* V *and acts as a natural scalar group.*

Proof. We note that $\Sigma_\alpha = \Sigma_{\alpha k}$ for any $k \in D_L - \{0\}$.

Recall that the group action of D^* is by

$$\tau_\beta : \Sigma_\alpha \longmapsto \Sigma_{\alpha\beta} : ((\alpha v), 1) \longmapsto ((\alpha\beta v), 1) \text{ and } (w, 0) \longmapsto \beta(w, 0).$$

Hence, we note that the group D^* acts on the associated projective spaces Σ_α and also acts on the affine geometries Σ_α^-, each of which is isomorphic to $AG(L, D_L)$. If $\beta \in D_L$ then $(\alpha v, 1)$ and $(\alpha\beta v, 1)$ are both in Σ_α^- so that D_L^* is the elementwise stabilizer of this set. Since for any γ, $\tau_{\alpha^{-1}\gamma}$ maps Σ_α^- onto Σ_γ^-, it follows that we have the transitive action as maintained. Furthermore, the group clearly fixes $(0, 1)$ and fixes all points of Σ_o.

Now consider $\Sigma_2 - \Sigma_1$, points of the form $((x_i), 0)$. If we define D^* to act on these points by scalar multiplication, we see that D^* acts as a collineation group of the affine space $AG(V, D)$ which fixes the 'zero vector' and acts as a scalar mapping on the underlying vector space. This completes the proof of the lemma. ■

Now realize this vector space as over K. Note that the $\Sigma_\alpha^-{}'s$ are naturally D_L-subspaces under the scalar action of D restricted to D_L, since each is fixed by D_L.

Definition 14. *Suppose that $V = W_o \oplus W_o$. A quasi-subgeometry of $PG(V - 1, D)$ isomorphic to $PG(W_o - 1, D_L)$ shall be said to have the 'congruence generating property' if and only if as vector spaces,*

$$\Sigma_{\alpha_j}^- \oplus \Sigma_{\alpha_k}^- = V$$

for any distinct pair of elements $\alpha_j, \alpha_k \in \mathcal{C}_L$.

Note that there is a unique way to consider the direct sum that is dependent only on the pointsets involved.

Proposition 1. *If W_o is finite dimensional over K then $PG(L - 1, D_L)$ has the 'congruence generating property'.*

Proof. Assume that W_o is finite dimensional of dimension k over K so that the dimension of V over K is $2k$. Now if Π_1 and Π_2 are two mutually disjoint D_L-subspaces that are D_L-isomorphic to W_o then they are also K-isomorphic to W_o. Hence, any two of these subspaces will generate V. ■

Theorem 10. *If $PG(V - 1, D)$ has a quasi-subgeometry isomorphic to $PG(W_o - 1, D_L)$ that has the congruence generating property then the quasi-subgeometry produces an D_L-fan in the vector space V such that each pair of components in the associated generalized partial spread generate V as a direct sum.*

Proof. Under the assumptions, we have a set of mutually disjoint vector subspaces of a vector space V over K that are D_L-isomorphic to W_o and any two generate V. By our previous lemmas, the group D^* acts in the manner required, implying that we have constructed a D_L-fan acting on a partial spread. ■

Our main result of this section is as follows:

Theorem 11. *Let V be a vector space over a field K and assume there are field extensions D_{L_i} and D such that $K \subseteq D_{L_i} \subseteq D$, for $i \in \Lambda$, an index set, and L_i K-vector subspaces of V.*

Let $\mathcal{P}$ be a quasi-subgeometry partition of $PG(V-1, D)$ by quasi-subgeometries $\mathcal{G}_i$ isomorphic to $PG(L_i - 1, D_{L_i})$, so that

$$\mathcal{P} = \cup_{i \in \Lambda} \mathcal{G}_i.$$

(1) Then, by 'unfolding', there is a corresponding generalized spread $\mathcal{S}_{\mathcal{P}}$ of the vector space V such that D acts on V and $\mathcal{S}_{\mathcal{P}}$ consists of D_{L_i}-fans for $i \in \Lambda$.

(2) Conversely, the associated generalized spread admitting D produces, by 'folding', a quasi-subgeometry partition of $PG(V-1, D)$ consisting of $PG(L_i - 1, D_{L_i})$'s on the analgous pointsets of the partition.

(3) The constructed generalized spread is a spread if and only if the quasi-subgeometries generate K-subspaces that pairwise have the congruence generating property.

This is true for example if $L_i' \oplus L_j' = V$, for all distinct $i, j \in \Lambda$, for all subspaces L_i' of $O(L_i)$, the D^*-orbits.

(4) If D is a 2-dimensional field extension of K then the quasi-subgeometries are subgeometries and there is a bijective correspondence between generalized spreads admitting D and subgeometry partitions of $PG(V-1, D)$ by $PG(L_i - 1, K)$'s and $PG(L_j - 1, D)$'s. The generalized spreads are unions of K-reguli and subspaces fixed by D^*.

Proof. Let $\Sigma_o^1 \simeq PG(L_1 - 1, D_{L_1})$ and $\Sigma_o^2 \simeq PG(L_2 - 1, D_{L_2})$ be disjoint quasi-subgeometries of $PG(V-1, D)$.

There are subsets Σ_o^{i+} of V of 1-dimensional D-subspaces that give rise of Σ_o^i, for $i = 1, 2$. These two subsets must be disjoint on 1-dimensional D-subspaces and hence disjoint on non-zero vectors. In order to realize this within our notation, we assume that L and M are disjoint K-vector spaces. We then may consider the representations of Σ_o^{i+} as follows:

$$\Sigma_o^{i+} = \left\{ \alpha v^i ; \alpha \in D, \, v^i \in L_i \right\}, \text{ for } 1, 2.$$

Hence, it is then clear that we have two fans that are disjoint, one a D_{L_1}-fan and one a D_{L_2}-fan.

Now if there is a partition of $PG(V-1, D)$ by quasi-subgeometries isomorphic to $PG(L_i - 1, D_{L_i})$ for $i \in \Omega$, then we obtain a set of mutually disjoint D_{L_i}-fans for $i \in \Omega$. Since every 1-dimensional D-subspace lies within one of the subgeometries, then every vector of V must lie within one of the fans. Hence, there is a covering of V by D_{L_i}-fans for $i \in \Omega$ so we have a generalized spread that is the union of D_{L_i}-fans and admits D^* as a collineation group within $GL(V, K)$. This proves (1), (2). Part (3) is merely the requirement that we actually obtain a spread. To prove (4), we note in the quadratic extension case, all quasi-subgeometries become geometries, the K-fans become K-reguli and the D-fans are simply components of the generalized spread fixed by D^*. ∎

6 Finite Fans in Spreads.

In the previous sections, we have developed the connections with fans and quasi-subgeometries. Since all of our results are valid for arbitrary vector spaces, we may apply these results, in particular, for finite spreads; finite fans and finite quasi-subgeometries. In the general results, we considered a field extension D and discussed D_L-fans. In the finite case, we also do the same thing, and we consider the study of finite vector spaces of dimension $2ds$ over K isomorphic to $GF(q)$ that admit a fixed-point-free field group $D^* = F_d^*$ of order $q^d - 1$ (i.e. $F_d = F_d^* \cup \{0\}$ is a field) that contains the field group K^* or those that could admit $D^* = F_{2d}^*$ isomorphic to $GF(q^{2d})^*$. Note that for spreads of finite dimensional vector spaces V, the dimension over a subkernel field is necessarily even, say $2ds$. In this setting, the order of the associated translation plane is q^{ds}. We are interested in essentially two types of fields 'acting' as collineation groups of the translation plane. First, we consider whether it is possible that the field D could fix a component of the spread. Since a component has $q^{ds} - 1$ nonzero vectors and $D \simeq GF(q^w)$ is fixed point free, then w must divide ds in this context. So, in such a setting we take $w = d$ without loss of generality. On the other hand, assume that it would be required that D never fix a component. Since V is a $2ds$ dimensional $GF(q)$-vector space, then V would be a $2ds/w$-dimensional $GF(q^w)$-vector space. Hence, w divides $2ds$ and if it does not divide ds, then without loss of generality, we consider $w = 2d$. Hence, the fields in question could be isomorphic to $GF(q^d)$ or $GF(q^{2d})$, where $2d$ divides $2ds$ but not ds, so we assume that s is odd in the latter case to avoid reduction to the previous situation.

Hence, the q^e-fans that are under consideration have degree either $(q^d-1)/(q^e-1)$ where e divides d or degree $(q^{2d} - 1)/(q^e - 1)$ where e divides $2d$.

Definition 15. *A 'q^e-fan' in a $2ds$-dimensional vector space over K isomorphic to $GF(q)$ is a set of $(q^w - 1)/(q^e - 1)$ mutually disjoint K-subspaces of dimension ds that are in an orbit under a field group F_w^* of order $(q^d - 1)$, where F_w contains K and such that F_w^* is fixed-point-free, and where $w = d$ or $2d$ and in the latter case s is odd.*

Hence, applying our main results to the finite case, we obtain the following corollaries.

Corollary 1. *Assume a partial spread $\mathcal{Z}$ of order q^{ds} in a vector space of dimension $2ds$ over K isomorphic to $GF(q)$ admits a fixed-point-free field group F_w^* of order (q^w-1) containing K^*, for $w = d$ or $2d$ and s is odd if $w = 2d$. Then any component orbit Γ of length $(q^w - 1)/(q^e - 1)$ (a 'q^e-fan'), for e a divisor of d, produces a quasi-subgeometry isomorphic to a $PG(ds/e - 1, q^e)$ in the corresponding projective space $PG(2ds/w - 1, q^w)$, considered as the lattice of F_w-subspaces of V.*

Corollary 2. *Let π be a translation plane of order q^{ds} and kernel containing K isomorphic to $GF(q)$ that admits a fixed-point-free field group F_w^* of order $(q^w - 1)$ containing K^*, where $w = d$ or $2d$ and s is odd if $w = 2d$.*

(1) Then, for any component orbit Γ, there is a divisor e_Γ of w such the orbit length of Γ is $(q^w - 1)/(q^{e_L} - 1)$ so that Γ is a q^{e_Γ}-fan.

Hence, there is a quasi-subgeometry partition of $PG(2ds/w-1,q^w)$ by quasi-subgeometries isomorphic to $PG(ds/e_\Gamma-1,q^{e_\Gamma})$, for various divisors e_Γ of w.

(2) Conversely, for $w=d$ or $2d$, every quasi-subgeometry partition of $PG(2ds/w-1,q^w)$ by quasi-subgeometries isomorphic to $PG(ds/f-1,q^f)$, for various divisors f of w, produces a translation plane of order q^{ds} and kernel containing K isomorphic to $GF(q)$, that admits a fixed-point-free field collineation group of order q^w-1, F_w^* containing K^*. The projective quasi-subgeometries of type $PG(ds/f-1,q^f)$ correspond to q^f-fans.

Our general main result for the finite spread case is now summarized as follows.

Theorem 12. *Let π be a translation plane of order q^{ds} and kernel containing K isomorphic to $GF(q)$. Assume that there is a fixed-point-free collineation group GK^* such that GK is a field isomorphic to $GF(q^w)$, where $w=d$ or $2d$ and s is odd in the latter case. Let the set of divisors of w be $N=\{e_i;i=1,....,E\}$, including w and 1.*

(1) Each component L has a unique maximal subfield $GF(q^f)$ within $GF(q^w)$, for $f=e_k$ for some k, such that L is a $GF(q^f)$-subspace. In this case, the orbit length of L under GK^* is $(q^w-1)/(q^f-1)$ and the orbit is a q^f-fan.

Let k_i denote the number of GK^*-orbits of components of length $(q^w-1)/(q^{e_i}-1)$; of q^{e_i}-fans, and let N^- denote the subset of N containing the divisors e_i used in the construction.

Then $\sum_{i=1}^{N^-}k_i(q^w-1)/(q^{e_i}-1)=q^{ds}+1$.

(2) Consider the associated affine geometry $AG(2ds/w,q^w)$, embed in $PG(2ds/w,q^w)$ and let Δ denote the hyperplane at infinity isomorphic to $PG(2ds/w-1,q^w)$. Consider an orbit of components under $GF(q^w)^*$ of length $(q^w-1)/(q^{e_i}-1)$; the various q^{e_i}-fans. Each q^{e_i}-fan will produce a quasi-subgeometry isomorphic to $PG(ds/e_i-1,q^{e_i})$ in Δ. Hence, we obtain k_i such $PG(ds/e_i-1,q^{e_i})'s$.

(3) If $\sum_{i=1}^{N^-}k_i(q^w-1)/(q^{e_i}-1)=q^{ds}+1$, then there is a corresponding quasi-subgeometry partition of $PG(2ds/w-1,q^w)$ by k_i $PG(ds/e_i-1,q^{e_i})'s$ for $i=1,2,...,N$. We call this a partition of type $(k_1,....,k_{N^-})$.

(4) If $PG(2ds/w-1,q^w)$ admits a quasi-subgeometry partition by k_i $PG(ds/e_i-1,q^{e_i})'s$ for $i=1,2...,N^-$, for $e_i\in N^-\subseteq N$ and $N=\{e_i;i=1,2,...,E\}$ is the set of all divisors of d then necessarily

$$\sum_{i=1}^{N^-}k_i(q^w-1)/(q^{e_i}-1)=q^{ds}+1).$$

Furthermore, there is an associated translation plane of order q^{ds} and kernel containing $GF(q)$ admitting a collineation group GK^* such that the union with the zero mapping is a field isomorphic to $GF(q^w)$ that is fixed-point-free and there is a set of k_i mutually disjoint q^{e_i}-fans whose union is the spread for the translation plane.

We now show that there are a great variety of examples of quasi-subgeometry partitions based on net replacement. All of the associated translation planes that we shall consider are generalized André planes. It is pointed out that in the cases in question, in the following, the group acting is considered always of the type $GF(q^d)^*$.

7 Multiple André Replacement.

For additional background on André planes and generalized André planes, we refer the reader to Lüneburg [7].

Let Σ be a Desarguesian plane of order q^{ds} where q is a prime power. Let F_{ds} denote the field isomorphic to $GF(q^{ds})$ coordinatizing Σ.

Definition 16. *A 'generalized André plane' is a translation plane with spread*

$$x = 0, y = 0, y = x^{q^{\lambda(m)}}m; m \in F_{ds},$$

where λ is a function from F_{ds}^* to N, the set of natural numbers.

Definition 17. *Let the 'q-André net' A_α be defined as follows:*

$$A_\alpha = \{y = xm; m^{(q^{ds}-1)/(q-1)} = \alpha\}.$$

We define 'André replacement' as follows:

Choose any divisor e of d and consider the 'André replacement net' $A_\alpha^{q^{ef}}$ defined as follows:

$$A_\alpha^{q^{ef}} = \{y = x^{q^{ef}}m; m^{(q^{ds}-1)/(q-1)} = \alpha,\ (ef, d) = e\}.$$

Then if Σ is the associated Desarguesian with spread

$$A_\alpha \cup M$$

then there is a constructed translation plane with spread

$$A_\alpha^{q^{ef}} \cup M.$$

This translation plane is called an 'André plane'.

Definition 18. *The collineation subgroup of the associated Dessarguesian plane Σ that is in the linear translation complement and acts like the $GF(q^{ds})$-scalar group is called the '$GF(q^{ds})$-kernel group'.*

Then the $GF(q^{ds})$-kernel group acts on the André net and if we replace by André replacement as above, this group acts on the constructed André plane.

Choose any divisor e of d and consider the André replacement net $A_\alpha^{q^{ef}}$. Take the subfield F_d of F_{ds} isomorphic to $GF(q^d)$. This group F_d^* acts on $A_\alpha^{q^{ef}}$ with orbits of length $(q^d - 1)/(q^e - 1)$ and hence there are exactly $k_e = \frac{(q^{ds}-1)}{(q-1)}\frac{(q^e-1)}{(q^d-1)}$ q^e-fans. Note that this process can be done for any divisor e_β for any André net A_β. Hence, we obtain a variety of q^{e_β}-fans.

Theorem 13. *Let Σ be a Desarguesian affine plane of order q^{ds}. For each of the $q-1$, André nets A_α, choose a divisor e_α of d, (these divisors can possibly be equal and/or possibly equal to 1 or d). For each q-André net A_α, there is a corresponding set of k_α q^{e_α}-fans. Form the corresponding André plane $\Sigma_{(e_\alpha f_\alpha \forall \alpha \in GF(q))}$ obtained with spread:*

$$y = x^{q^{e_\alpha f_\alpha}}m \ for\ m^{(q^{ds}-1)/(q-1)} = \alpha, x = 0, y = 0; m \in GF(q^{ds}),\ (e_\alpha f_\alpha, d) = e_\alpha.$$

Then the spread $\Sigma_{(e_\alpha f_\alpha \forall \alpha \in GF(q))}$ is a union of $\sum_{\alpha=1}^{q-1}(k_{e_\alpha} = \frac{(q^{ds}-1)}{(q-1)}\frac{(q^{e_\alpha}-1)}{(q^d-1)})$ q^{e_α}-fans, together with two q^d-fans $x = 0$ and $y = 0$.

There is a corresponding quasi-subgeometry partition of $PG(2s-1, q^d)$ of k_{e_α} quasi-subgeometries isomorphic to $PG(ds/e_\alpha - 1, q^{e_\alpha})$, and two $PG(s-1, q^d)$ (corresponding to $x = 0$ and $y = 0$).

7.1 Multiple q^e-André Replacement.

Actually, a refinement of the above will produce a more general variety of partition, however, the associated translation planes are not necessarily André planes but are certainly generalized André planes. For example, we may partition any q-André net of cardinality $(q^{ds}-1)/(q-1)$ into $(q^e-1)/(q-1)$ q^e-André nets of cardinality $(q^{ds}-1)/(q^e-1)$. For the q^e-André nets, the basic replacement components must be the form $y = x^{q^{ef}}m$, however, we may choose the $f's$ independent of each other for the $(q^e-1)/(q-1)$ q^e-André nets. We then may choose another divisor e_1 of d/e to produce a set of $\frac{(q^{ds}-1)}{(q^e-1)}\frac{(q^{ee_1}-1)}{(q^d-1)}$ q^{ee_1}-fans from each of the $(q^e-1)/(q-1)$ q^e-André nets. Furthermore, the partitioning into relative sized André nets can be continued. For example for any one of the q^e-André nets, choose a divisor e_1 of d/e and partition this q^e-net into a set of q^{ee_1}-André nets. The point is that we may choose each of these fans with possibly different component set configurations. Furthermore, we can continue this partitioning and choice of component sets for the fans.

In this way, we obtain an explosion of possible subgeometry partitions as the number of possibilities is extremely large.

For example, we may require a q^e-fan for every divisor e of d. In other to obtain this, it suffices to require $q > N$, the number of divisors of d.

Theorem 14. *If $q-1 \geq$ the number of divisors (including d and 1) of d then there exists a generalized André translation plane obtained from a Desarguesian plane of order q^{ds} by multiple André replacement that produces a quasi-subgeometry partition of $PG(2s-1, q^d)$ such that for every divisor e of d, there exists quasi-subgeometries of the partition isomorphic to $PG(ds/e-1, q^e)$.*

Perhaps we should illustrate the above theorem with a few examples. Note that one of the above constructions give quasi-subgeometry partitions $PG(2s-1, q^d)$ of $\frac{(q^{ds}-1)}{(q-1)}\frac{(q^{e_\alpha}-1)}{(q^d-1)}$ subgeometies isomorphic to $PG(ds/e_\alpha - 1, q^{e_\alpha})$ and two $PG(s-1, q^d)$'s.

Perhaps the most wild situation is when $s = 1$.

Example 1. *Let $s = 1$ and $d = 6$. Then the superspace is isomorphic to $PG(1, q^6)$ and there are $\sum_{\alpha=1}^{q-1}(k_{e_\alpha} = \frac{(q^{ds}-1)}{(q-1)}\frac{(q^{e_\alpha}-1)}{(q^d-1)})$ $PG(ds/e_\alpha - 1, q^{e_\alpha})'s$ together with the two $PG(s-1, q^d)'s$, where $e_\alpha = 1, 2, 3$ or 6. Note that for each α in $GF(q)^*$, e_α may be chosen as indicated. Hence, to obtain all types we would require that $q-1$ is at least 4. But, in any case, there is a very large number of possibilities.*

7.2 Recognition of André type Partitions.

The question remains, if we are given a quasi-subgeometry partition of $PG(2s-1, q^d)$ by quasi-subgeometries isomorphic to $PG(ds/e-1, q^e)$ for a set of divisors e of d, when is the associated translation plane André or generalized André? Of course, in the translation plane, we can determine if the plane is André by consideration of its collineation group. However, this does not mean that the partition itself is André. Here we are intending this to mean that a quasi-subgeometry partition is obtained via q^e-fans of André type using André replacement in a Desarguesian affine plane. Note in all cases the translation plane associated are generalized André planes contructed by multiple André replacement.

Definition 19. *Any quasi-subgeometry partition obtained by using multiple André replacement shall be called 'Andre'.*

So, we shall list some open questions:

(1) When is a quasi-subgeometry (respectively, subgeometry) partition André?

(2) If a quasi-subgeometry (respectively, subgeometry) partition provides a generalized André translation plane, is the partition itself André?

(3) Do non-André quasi-subgeometry (respectively, subgeometry) partitions exist for $d > 2$? Note that when $d = 2$, we obtain simply a mixed partition and the q-fans become (are) K-reguli. Hence, non-André partitions certainly exist if $d = 2$.

We have not given examples of quasi-subgeometry partitions ((respectively, subgeometry) corresponding to q^e-fan spreads, when $GF(q^{2d})$-acts. However, when $d = 1$, there are a variety of examples. In this case the q-fans are K-reguli and produce subgeometries.

(4) Do quasi-subgeometry partitions of $PG(s-1, q^{2d})$ exist by quasi-subgeometries isomorphic to $PG(ds/e-1, q^e)$ for d and s odd?.

Now we turn to our more general results involving finite generalized spreads.

8 Generalized Spreads and Replaceable Translation Sperner Spaces.

When there is a spread, the vector space is $2ds$ dimensional over K isomorphic to $GF(q)$ and the components have q^{ds} vectors each. We consider a more general situation where the vector space is tds-dimensional and we have a generalized spread whose components have q^{ds} vectors each. Thus, we consider 't-spreads of size (ds, q)'. Here, we also have a 'translation space' whose points are the vectors of V and whose lines are the translates of the components of the t-spread.

To see a simple example of such t-spreads, take any vector space V of dimension t over $GF(q^{ds})$ and let $\mathcal{S}$ be the t-spread of V of $(q^{tds}-1)/(q^{ds}-1)$ 1-dimensional $GF(q^{ds})$-subspaces. Forming the 'translation Sperner space' by taking translates of these $(q^{tds}-1)/(q^{ds}-1)$ spread components, we obtain a Sperner space with q^{tds} total points and q^{ds}points per line.

Now assume that $ds = 2$, regard V as $2t$-dimensional over $GF(q)$ and take any proper subspace W of V of dimension 2 over $GF(q)$ which is not contained in a component of the t-spread. Hence, whenever, W intersects a spread component, it must intersect in a 1-dimensional $GF(q)$-subspace. Consider the subspread induced on W of $(q^2-1)/(q-1)$ 1-dimensional $GF(q)$-subspaces. This defines a subplane of the translation space that is Desarguesian of order q by taking the translates within W as lines of the subplane. We note that this subplane π_o cannot be considered a subspace of the associated affine space over $GF(q^2)$. Take the set of images of π_o under the $GF(q^2)^*$-scalar group to obtain a set of $(q+1)$ subplanes π_i, for

$i = 0, 1, ..., q$, that share the same set W^+ of $q+1$ 1-dimensional $GF(q^2)$-spaces. Replace W^+ with the set $\{\pi_i; i = 0, 1, ..., q\}$.

Now consider the original t-spread components of 1-dimensional $GF(q^2)$-subspaces. If we consider these as 2-dimensional $GF(q)$-spaces then the set of $(q+1)$-1-dimensional $GF(q)$-subspaces within each is a 2-spread of each original compoenent. Hence, if we take $\mathcal{S} - W^+$ as a partial $2t$-spread of V as a $2t$-dimensional $GF(q)$-subspace together with $\{\pi_i\}$, we obtain a 'new' $2t$-spread of size $(1, q)$ upon which $GF(q^2)$ acts. Then $GF(q^2)^*$ has exactly one orbit of length $q+1$, a q-fan and the remaining orbits are of length 1. Our previous analysis shows that we have a q-regulus in $PG(\pi_o \oplus \pi_1 - 1, q)$.

Thus, we see that we may fold the fan and produce a quasi-subgeometry partition of $PG(t-1, q^2)$ by quasi-subgeometries isomorphic to $PG(0, q^2)$ and $PG(1, q)$'s. Such partitions are perhaps are not all that interesting, but this gives a glimpse of what could occur. For example, there could be t-spreads of size (s, q) that admit a group isomorphic to $GF(q^2)$ without s necessarily equal to 2. Suppose that s is even and let $s/2 = s^*$, then, in this setting, there would be a partition of $PG(tds^* - 1, q^2)$ by either $PG(s^* - 1, q^2)'s$ where the component is fixed by $GF(q^2)$ or $PG(2s^* - 1, q)$'s; that is, either q^2-fans or q-fans.

In the previous setting, $t = 2$ produces the subgeometry partitions of $PG(2s^* - 1, q^2)$ that correspond to spreads. Here, it still might be possible for a t-spread of size (ds, q) to involve a union of 'reguli'. Furthermore, we have seen that these $GF(q^2)$-orbits of ds-spaces are still covered by subplanes; that is, these are still subplane covered nets that become (are) reguli. We know that if A and B are two 1-dimensional $GF(q^2)$ subspaces that lie within $O(L)$ then

$$\langle A, B\rangle_{GF(q^2)} \cap O(L) = \langle \alpha_o A + \beta_o B\rangle_{GF(q^2)} \ \forall \alpha_o, \beta_o \in GF(q).$$

Hence, each such orbit induces a 'subgeometry' in $PG(tds^* - 1, q^2)$ isomorphic to $PG(2s^* - 1, q)$.

If ds is odd, however, then it still might be possible for $GF(q^2)^*$ to act on the t-spread, but there could be no fixed components. This could occur if $(q^{tds}-1)/(q^{ds}-1)$ is divisible by $(q+1)$.

Theorem 15. *(1) Let $\mathcal{S}$ be a t-spread of V of size (ds, q).*

(a) If s is even, let $s/2 = s^*$. Assume that $GF(q^2)$ 'acts' on $\mathcal{S}$ then there is a subgeometry partition of $PG(tds^* - 1, q^2)$ consisting of $PG(s^* - 1, q^2)$'s and $PG(2s^* - 1, q)$'s. If $(q+1)$ divides $(q^{tds} - 1)/(q^{ds} - 1)$, it is possible that there are no $PG(s^* - 1, q^s)$'s.

(b) If ds is odd, but t is even, let $t^* = t/2$ and assume that $GF(q^2)$ acts on $\mathcal{S}$ then there is a subgeometry partition of $PG(t^*ds-1, q^2)$ by subgeometries isomorphic to $PG(s - 1, q)$'s.

(2) Conversely,

(a) any subgeometry partition of $PG(tds^* - 1, q^2)$ by $PG(s^* - 1, q^2)$'s and $PG(2s^* - 1, q)$'s produces a t-spread of size (ds, q) that is a union of q^2-fans and q-fans and

(b) any subgeometry partition of $PG(t^*ds - 1, q^2)$ by $PG(s - 1, q)$'s produce a t-spread of size (ds, q) that is a union of q-fans.

In either case (a) or (b), the t-spread corresponds to a translation Sperner space admitting $GF(q^2)^*$ as a collineation group.

(3) The t-spread is a union of components fixed by $GF(q^2)^*$ and a set of $GF(q)$-reguli each in a projective space isomorphic to $PG(2s-1,q)=PG(4s^*-1,q)$. ***Note*** *that these projective spaces need not be the same for different $GF(q)$-reguli.*

Problem 1. *If we have a t-spread of size (ds,q) where ds is odd, and $GF(q^2)$ acts on the t-spread then is it possible that there is a collineation group that acts transitively (2-transitively on the d-spread components)? If $t=2$ then a 2-spread is a spread and there would be a corresponding flag-transitive translation plane.*

Remark 9. *If we have a t-spread $\mathcal{Z}$ of size (ds,q), then we ask when $GF(q^w)$ could act on $\mathcal{Z}$. Certainly the associated vector space V of dimension tds over K isomorphic to $GF(q)$ is then a $GF(q^w)$-vector space so that w must divide tds. When $t=2$, we isolated on fields $GF(q^d)$ or $GF(q^{2d})$, where $2d$ does not divide ds. In the t-spread case, we could analogously consider $GF(q^d)$ and $GF(t^*d)$ where t^*d does not divide ds, and t^* divides t.*

Theorem 16. *(1) Let $\mathcal{S}$ be a t-spread of V of size (ds,q), and assume that $w=d$ or t^*d where t^*d does not divide ds, and t^* divides t.*

Assume that $GF(q^w)$ 'acts' on $\mathcal{S}$ then there is a quasi-subgeometry partition of $PG(tds/w-1,q^w)$ consisting of $PG(ds/e_i-1,q^{e_i})$'s where e_i divides w for $i\in\Lambda$.

(2) Conversely, any quasi-subgeometry partition of $PG(tds/w-1,q^w)$ by $PG(ds/e_i-1,q^{e_i})$'s for $i\in\Lambda$, and e_i a divisor of w, produces a t-spread of size (ds,q) that is a union of q^{e_i}-fans. The t-spread corresponds to a translation Sperner space admitting $GF(q^w)^*$ as a collineation group.

Clearly, we have barely scratched the surface of the many open questions and problems that the previous results generate. We shall be content here to list essentially one involving 'subgeometry' partitions.

Problem 2. *For any t-spread of type (s,q) :*

(a) If s is even, determine an infinite class of subgeometry partitions of $PG(ts/2-1,q^2)$ by $PG(s/2-1,q^2)$'s and $PG(s-1,q)$'s.

(b) If t is even, determine an infinite class of subgeometry partitions of $PG(ts/2-1,q^2)$ by $PG(s-1,q)$'s.

Example 2. *For example, for subgeometry partitions of $PG(5,q^2)$, we consider an associated vector space V of dimension $12=ts$ over $GF(q)$, producing an t-spread where $t=1,2,3,4,6$ or 12. We assume that such an t-spread admits $GF(q^2)$.*

(i) If $t=1$ then $s=12$ then we have one component admitting $GF(q^2)$, so this is a trivial partition and we obtain exactly one $PG(5,q^2)$.

(ii) If $t=2$ and $s=6$, the projective space could be partitioned by $PG(5,q)'s$ and $PG(2,q^2)'s$. In this case, a $PG(5,q)$ arises from a vector space of q^6 vectors which defines a q-fan and a $PG(2,q^2)$ produces a vector space of q^4 vectors which is fixed by $GF(q^2)$. The q-fans define reguli in $PG(11,q)=PG(2s-1,q)$.

(iii) If $t=3$ and $s=4$ then the partition subgeometries are $PG(1,q^2)$ and $PG(3,q)$'s. A $PG(1,q^2)$ produces a vector space of q^4 vectors fixed by $GF(q^2)^*$ and a $PG(3,q)$ arises from a vector space of q^4-vectors which defines a q-fan. The q-fans define reguli in various $PG(7,q)$'s=$PG(2s-1,q)$'s.

(iv) If $t = 4$ and $s = 3$ the we have a partition by $PG(2, q)$'s arising from q-fan's that define reguli in various $PG(5, q)$'s $=PG(2s-1, q)$'s.

(v) If $t = 6$ and $s = 2$, $PG(5, q^2)$ could be partitioned by $PG(0, q^2)$'s and $PG(1, q)$'s arising from q-fans that define reguli in various $PG(3, q)$'s$=$ $PG(2s-1, q)$'s and a $PG(0, q^2)$ produces a vector space of q^2 vectors fixed by $GF(q^2)^*$.

(vi) If $t = 12$ and $s = 1$, $PG(5, q^2)$ could be partitions by $PG(0, q)$'s arising from q-fans that define (trivially) reguli in various $PG(1, q)$'s.

9 Congruence Generating Partitions.

In our main result regarding the unfolding of a projective fan, at one point, to obtain a spread, we have assumed that the quasi-subgeometries are 'congruence generating'. Suppose that there are not. Then, we still obtain a partition of the vector space $V = W_o \oplus W_o$ by subvector spaces K-isomorphic to W_o. This would not necessarily be a spread since the dimension of W_o over K is not necesssarily finite. However, if we have the congruence generating property we obtain a spread. On the other hand, we would obtain a generalized spread. If we more generally study coverings of vectors spaces $V = W_o \oplus W_o$ by mutually disjoint vector subspaces isomorphic to W_o, we still could discuss 'fans' –say, 'pseudo-fans' to distinguish between the finite and infinite dimensional situations, and these would produce quasi-subgeometry partitions. We shall call such a partition of a vector space a 'pseudo-spread'. Conversely, without any further assumptions, quasi-subgeometry partitions of projective spaces produce pseudo fans. We formally list this observation.

Theorem 17. *Pseudo-spreads that are unions of pseudo-fans are equivalent to quasi-subgeometry partitions by projective spaces $PG(W_o - 1, T)$ arising from subfields T of the field F of the ambient projective space isomorphic to $PG(W_o \oplus W_o - 1, F)$.*

Of course, there is not a distinction between finite-dimensional pseudo-spreads and spreads. Hence, such differences necessarily lie with infinite-dimensional projective spaces.

10 Final Remarks.

We have seen that generalized spreads with fields D acting on them are equivalent to quasi-subgeometry partitions of projective spaces. There are subgeometry partitions corresponding to the action of quadratic extension fields of a base field. Furthermore, there are spreads that produce and correspond to both subgeometry and quasi-subgeometry partitions. We have provided examples of finite generalized André planes that produce new quasi-subgeometry and subgeometry partitions. Flag-transitive finite translation planes can produce subgeometry partitions but could conceivably also produce different subgeometry partitions. We have raised a variety of questions and problems involving the construction of both subgeometry and quasi-subgeometry partitions from t-spreads.

In the infinite case, since we have a much more varied class of fields to work with, conceivably there will be more and varied examples of infinite fans and infinite quasi-subgeometry partitions. In particular, we may construct the analogues of

André and generalized André planes using multiple André replacement from Pappian planes Σ defined over certain algebraic extension fields, analogous to finite extension fields of K isomorphic to $GF(q)$. Again, there are many areas of investigation related to quasi-subgeometry partitons that have not been previously considered. Clearly, there is a tremendous number of quasi-subgeometry partitions, so there are a variety of open questions and problems dealing with infinite partitions. Although the development of spreads partitioned by fans is of general interest in the infinite case, for space reasons we do not continue this theory here. Similarly, we do not here pursue the analysis and construction of t-spreads admitting fields D.

Instead and finally, we return to our main unswered questions involving spreads and pseudo-spreads associated with quasi-subgeometry partions and generalized spreads but we now do not restrict ourselves to finite fields.

Let K and D, $K \subseteq D$ be fields and V a K-vector space K-isomorphic to W_o.

Problem 3. ***Are there partitions of*** $PG(V-1, D)$ ***by quasi-subgeometries isomorphic to*** $PG(L-1, D_L)$***, for all intermediate subfields*** D_L ***between*** K ***and a specified field*** D, ***where*** L ***is a*** K***-subspace of*** V***, for a set of such subspaces.***

We have solved the next problem in the finite case, but we state this more generally for arbitrary fields.

Problem 4. ***For*** $V = W_o \oplus W_o$***, and*** W_o ***a*** K***-space, and when it makes sense to discuss André partitions, are there André partitions of*** $PG(V-1, D)$ ***by quasi-subgeometries isomorphic to*** $PG(W_o - 1, D_L)$***, for all intermediate subfields*** D_L ***between*** K ***and a specified field*** D?

Problem 5. ***For*** $V = W_o \oplus W_o$ ***and when it makes sense to discuss André partitions, are there NON-André partitions of*** $PG(V-1, D)$ ***by quasi-subgeometries isomorphic to*** $PG(W_o - 1, D_L)$***, for all intermediate subfields*** D_L ***between*** K ***and a specified field*** D?

Problem 6. ***Do proper pseudo-spreads exist? Is there a proper pseudo-spread that is the union of pseudo-fans?***

Problem 7. ***For*** $V = \sum_{i=1}^{n} W_i$ ***and*** W_i ***isomorphic to*** W_o***, a subspace over*** K***, determine subgeometry partitions of*** $PG(V-1, D)$ ***where*** D ***is a quadratic field extension of*** K***.***

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