Absence of local and global solutions to weakly coupled systems of parabolic inequalities

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Abstract

The paper deals with the local and global nonexistence of weak solutions to a class of weakly coupled systems of parabolic inequalities with a fractional power of the Laplacian. Our results include nonexistence results as well as necessary conditions for the local and global solvability.

1 Introduction and main results

The broad goal of this paper is to discuss the nonexistence of weak solutions to systems of parabolic inequalities with a fractional power of the Laplacian $(-\Delta)^{\frac{\beta}{2}}, 0 < \beta \leq 2$, namely

$$\begin{cases} u_t \geq -(-\Delta)^{\frac{\alpha}{2}} u + h_1(x,t)|v|^p, \\ v_t \geq -(-\Delta)^{\frac{\beta}{2}} v + h_2(x,t)|u|^q, \end{cases} \tag{1.1}$$

posed in $S_T := \mathbb{R}^N \times (0,T), 0 < T \leq +\infty$, where $0 < \alpha, \beta \leq 2$ and $p > 1, q > 1$.

The nonexistence result of global solutions was first addressed by Escobedo and Herrero [4]. The authors showed that the problem

$$\begin{cases} u_t = \Delta u + v^p, \\ v_t = \Delta v + u^q, \end{cases} \tag{1.2}$$

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has no nonnegative global solution, except the trivial one if $2\max\{p+1, q+1\} \geq N(pq-1)$.
Moreover, if $2\max\{p+1, q+1\} < N(pq-1)$ the both non global and global solutions may exist.

In [5] Fila *et al.* studied (1.2) with $\delta\Delta u, 0 < \delta \leq 1$, instead of Δu. The authors proved, among other results, the following.
If $N > 2$ and $\max\{a, b\} < (N-2)/N$, where $a = (p+1)/(pq-1), b = (q+1)/(pq-1)$, then global non–trivial nonnegative solutions exist which belong to $L^\infty(\mathbb{R}^N \times \mathbb{R}_+)$ and satisfy

$$0 < u(x,t) \leq c|x|^{-2a}, 0 < v(x,t) \leq c|x|^{-2b},$$

for large $|x|$ and for all $t > 0$, where c depends only upon the initial data. For the case $\max\{a, b\} \geq \frac{N}{2}$ the only global trivial solution is acceptable. Later on the problem

$$\begin{cases} u_t &= \Delta u + |x|^{\sigma_1} v^p, \\ \\ v_t &= \Delta v + |x|^{\sigma_2} u^q, \end{cases} \tag{1.3}$$

was considered by Mochizuki and Huang [19] where $p, q \geq 1$ and $pq > 1$. The authors extend the above results to the case $0 \leq \sigma_1 < N(p-1)$ and $0 \leq \sigma_2 < N(q-1)$. In particular, under the restrictions

$$\max\{r, s\} < N, \quad q > 1 + \frac{2+\sigma_2}{N},$$

where

$$r = \frac{2(p+1) + \sigma_2 p + \sigma_1}{pq-1}, s = \frac{2(q+1) + \sigma_1 q + \sigma_2}{pq-1},$$

and

$$\limsup_{|x|\to+\infty} u_0|x|^a, \limsup_{|x|\to+\infty} v_0|x|^b < +\infty,$$

where

$$a > r, b > s,$$

it is shown that (1.3) has a global solution for $\| u_0|x|^a\|_\infty + \| v_0|x|^b\|_\infty$ small enough.

Returning to (1.1), it seems that the methods of [4],[5], [19] can not be used, since the proofs are based on the positivity and some estimates of solutions via the heat kernel.

Using the nonlinear capacity approach due to [3] and developed by [16], [17] Guedda and Kirane [11] studied

$$\begin{cases} u_t &= -(-\Delta)^{\frac{\alpha}{2}} u + h(x,t) v^p, \\ \\ v_t &= -(-\Delta)^{\frac{\beta}{2}} v + h(x,t) u^q, \end{cases} \tag{1.4}$$

where $p, q > 1$ and h behaves like $t^{\sigma_1}|x|^{\sigma_2}$ at infinity. The authors proved the existence of $N_c = N_c(\sigma_1, \sigma_2, p, q)$ such that for any $N \leq N_c$ Problem (1.4) has no global non trivial weak solution (u, v) satisfying the following

$$\int_{\mathbb{R}^N} u(x,0)dx \geq 0, \quad \int_{\mathbb{R}^N} v(x,0)dx \geq 0.$$

This result is extended by Kirane and Quafsaoui [13] to systems of the type

$$\begin{cases} u_t = \Delta(a(u,v,t,x)u) + A(x).\nabla v^r + hv^q, \\ v_t = \Delta(b(u,v,t,x)v) + B(x).\nabla u^s + gu^p. \end{cases}$$

It is the purpose of this paper to provide a sufficient condition for the local and global non solvability of (1.1) from a different angle. We investigate, for any fixed $p > 1$ and $q > 1$, in contrast to the Fujita-type result, the effect of the behavior of initial data and $h_i, i = 1, 2$, at infinity on the nonexistence of local and global weak solutions.

This work is motivated by the paper [2] in which Baras and Kersner showed that the problem

$$u_t = \Delta u + h(x)u^p, \qquad u(x,0) = u_0(x) \geq 0, \tag{1.5}$$

has no local weak nonnegative solution if the initial data satisfies

$$\lim_{|x|\to+\infty} u_0^{p-1}h(x) = +\infty,$$

and any possible local weak nonnegative solution blows up at a finite time if

$$\lim_{|x|\to+\infty} u_0^{p-1}h(x)|x|^2 = +\infty.$$

Here, we attempt to extend this result to (1.1). The methods used are some modifications and adaptations of ideas from [2],[16].

Next we add to (1.1) the initial condition

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \mathbb{R}^N, \tag{1.6}$$

where it is assumed that u_0, v_0 are locally integrable functions and there exists $R_0 > 0$ such that

$$u_0(x) \geq 0, v_0(x) \geq 0,$$

and

$$h_i(x,t) \geq g_i(x) \geq 0, \quad i = 1, 2,$$

for all $t \geq 0$ and for all $|x| > R_0$.

The definition of local weak solutions to (1.1),(1.6) used here is the following.

Definition 1.1. *The couple $(u,v) \in L^1_{loc}(S_T) \times L^1_{loc}(S_T)$ is a local weak solution to to (1.1),(1.6) defined in $S_T, 0 < T < +\infty$ if $u \in L^q_{loc}(S_T, h_2dxdt), v \in L^p_{loc}(S_T, h_1dxdt)$ such that*

$$\int_{\mathbb{R}^N} u(x,0)\zeta(x,0)dx + \int_{S_T} h_1|v|^p\zeta dxdt \leq \int_{S_T} u(-\Delta)^{\frac{\alpha}{2}}\zeta dxdt - \int_{S_T} u\zeta_t dxdt, \tag{1.7}$$

and

$$\int_{\mathbb{R}^N} v(x,0)\zeta(x,0)dx + \int_{S_T} h_2|u|^q\zeta dxdt \leq \int_{S_T} v(-\Delta)^{\frac{\beta}{2}}\zeta dxdt - \int_{S_T} v\zeta_t dxdt, \tag{1.8}$$

hold for any nonnegative $\zeta \in C_0^\infty(S_T)$ such that $\zeta = 0$ at $t = T$.

The integral

$$\int_{\mathbb{R}^N} w(x,0)\zeta(x,0)dx$$

is understood in the weak sense, i.e.,

$$\int_{\mathbb{R}^N} w(x,t)\zeta(x,t)dx \to \int_{\mathbb{R}^N} w(x,0)\zeta(x,0), \text{ as } t \to 0^+, \quad \forall\zeta \in C_0(\mathbb{R}^N \times [0,T)).$$

Definition 1.2. *The couple* (u,v) *is a global weak solution to* (1.1),(1.6) *if it is a local solution to* (1.1),(1.6) *defined in* S_T *for any* $T > 0$.

Now we are ready to summarize our main results. Set

$$G = \inf\left\{g_1, g_2, g_2^{\frac{p-1}{q-1}}\right\}.$$

Theorem 1.1. *Let* $p, q > 1$. *Assume that*

$$\lim_{|x|\to+\infty} u_0 G^{\frac{1}{p-1}} \lim_{|x|\to+\infty} v_0 G^{\frac{1}{p-1}} = +\infty.$$

Then Problem (1.1),(1.6) *has no weak local solution for any* $T > 0$.

The absence of global weak solutions is formulated by the following.

Theorem 1.2. *Let* $p > 1, q > 1$. *Assume*

$$\lim_{|x|\to+\infty} \left(u_0 G^{p'-1}|x|^{\frac{1}{2}\inf\{\beta,\alpha\}\frac{p+q+2}{pq-1}}\right) \lim_{|x|\to+\infty} \left(v_0 G^{p'-1}|x|^{\frac{1}{2}\inf\{\beta,\alpha\}\frac{p+q+2}{pq-1}}\right) = +\infty, \quad (1.9)$$

then no global weak solution to (1.1),(1.6) *can exist.*

Remark 1.1. The above results still hold if Problem (1.1),(1.6) is posed in an exterior domain $\Omega \subset \mathbb{R}^N$ with a smooth boundary $\partial\Omega$. In this case we add the Dirichlet boundary condition $u_{|\partial\Omega} = v_{|\partial\Omega} = 0$.

The main step for the proofs of Theorems 1.1 and 1.2 is to establish necessary conditions for the local and global solvability. Those conditions follow from an estimate of

$$\int_{|x|>R} u_0(x)dx \int_{|x|>R} v_0(x)dx, R > R_0.$$

This is done in the next section. Here we assume that u_0 and v_0 are such that a local solution to (1.1),(1.6) exists. We note, in passing, that in this work we are not interested in local and global existence of solutions to (1.1),(1.6). However, the arguments used would allow us to obtain a priori estimate for the local existence.

2 Necessary conditions for local and global solvability

As we have said, the objective is to establish necessary conditions for the local and global existence of weak solutions.

Our first result can be formulated as follows.

Theorem 2.1. *Let (u, v) be a local solution to (1.1),(1.6) where $T < \infty$ and $p, q > 1$. Then there exists $C = C(p, q) > 0$ such that*

$$\liminf_{|x|\to\infty} u_0 G^{\frac{1}{p-1}} \liminf_{|x|\to\infty} v_0 G^{\frac{1}{p-1}} \leq CT^{-\frac{p+q+2}{pq-1}}.$$

Proof. For any $\zeta \in C_0^\infty\left(\mathbb{R}^N \times (0,T)\right)$, $\zeta \geq 0$, such that *supp* $\zeta \subset \{|x| > R_0\}$ one has

$$\int_{\mathbb{R}^N} u_0\zeta(x,0)dx + \int_{S_T} h_1|v|^p\zeta dxdt \leq \int_{S_T} |u||\zeta_t|dxdt + \int_{S_T} |u||\Delta^{\frac{\alpha}{2}}\zeta|dxdt,$$

and

$$\int_{\mathbb{R}^N} v_0\zeta(x,0)dx + \int_{S_T} h_2|u|^q\zeta dxdt \leq \int_{S_T} |v|\zeta_t|dxdt + \int_{S_T} |v||\Delta^{\frac{\beta}{2}}\zeta|dxdt.$$

Using the Hölder inequality the following

$$\int_{\mathbb{R}^N} u_0\zeta(x,0)dx + \int_{S_T} h_1|v|^p\zeta dxdt \leq \left[\int_{S_T} h_2|u|^q\zeta dxdt\right]^{1/q} \times \left[\left\{\int_{S_T} |\zeta_t|^{q'}(\zeta h_2)^{1-q'}dxdt\right\}^{1/q'} + \left\{\int_{S_T} |(-\Delta)^{\frac{\alpha}{2}}\zeta|^{q'}(\zeta h_2)^{1-q'}dxdt\right\}^{1/q'}\right], \quad (2.1)$$

and

$$\int_{\mathbb{R}^N} v_0\zeta(x,0)dx + \int_{S_T} h_2|u|^q\zeta dxdt \leq \left[\int_{S_T} h_1|v|^p\zeta dxdt\right]^{1/p} \times \left[\left\{\int_{S_T} |\zeta_t|^{p'}(\zeta h_1)^{1-p'}dxdt\right\}^{1/p'} + \left\{\int_{S_T} |(-\Delta)^{\frac{\beta}{2}}\zeta|^{p'}(\zeta h_1)^{1-p'}dxdt\right\}^{1/p'}\right], \quad (2.2)$$

hold with $q' = \frac{q}{q-1}, p' = \frac{p}{p-1}$. Therefore we get

$$\left[\int_{S_T} h_2|u|^q\zeta dxdt\right]^{\frac{pq-1}{pq}} \leq A^{\frac{1}{q}}.B, \quad \left[\int_{S_T} h_1v^q\zeta dxdt\right]^{\frac{pq-1}{pq}} \leq A.B^{\frac{1}{p}},$$

where

$$A = \left[\left\{\int_{S_T} |\zeta_t|^{p'}(\zeta h_1)^{1-p'}dxdt\right\}^{1/p'} + \left\{\int_{S_T} |(-\Delta)^{\frac{\beta}{2}}\zeta|^{p'}(\zeta h_1)^{1-p'}dxdt\right\}^{1/p'}\right],$$

and

$$B = \left[\left\{\int_{S_T} |\zeta_t|^{q'}(\zeta h_2)^{1-q'}dxdt\right\}^{1/q'} + \left\{\int_{S_T} |(-\Delta)^{\frac{\alpha}{2}}\zeta|^{q'}(\zeta h_2)^{1-q'}dxdt\right\}^{1/q'}\right].$$

Inserting the two above estimates in (2.1) and (2.2) and multiplying the resulting inequalities we obtain

$$\int_{\mathbb{R}^N} u_0\zeta(x,0)dx \int_{\mathbb{R}^N} v_0\zeta(x,0)dx \leq A^{\frac{p(q+1)}{pq-1}} B^{\frac{q(p+1)}{pq-1}}. \quad (2.3)$$

Next we may take

$$\zeta(x,t) = (1 - \frac{t}{T})^{\gamma}\Phi(\frac{x}{R}), \quad \gamma = \max\{p', q'\}$$

where $\Phi \in W^{1,\infty}(\mathbb{R}^N)$, $\Phi \geq 0$, is supported by $\{1 < |x| < 2\}$ and satisfies

$$\max\left\{|(-\Delta)^{\frac{\alpha}{2}}\Phi)|, |(-\Delta)^{\frac{\beta}{2}}\Phi)|\right\} \leq \Phi,$$

to conclude

$$\int_{|x|>R} u_0 \Phi dx \int_{|x|>R} v_0 \Phi dx \leq$$
$$C\left[\frac{1}{T}\left\{\int_{S_T} \frac{\Phi}{(h_1)^{p'-1}} dxdt\right\}^{1/p'} + \frac{1}{R^{\beta}}\left\{\int_{S_T} \frac{\Phi}{(h_1)^{p'-1}} dxdt\right\}^{1/p'}\right]^{\frac{p(q+1)}{pq-1}}$$
$$\times \left[\frac{1}{T}\left\{\int_{S_T} \frac{\Phi}{(h_2)^{q'-1}} dxdt\right\}^{1/q'} + \frac{1}{R^{\alpha}}\left\{\int_{S_T} \frac{\Phi}{(h_2)^{q'-1}} dxdt\right\}^{1/q'}\right]^{\frac{q(p+1)}{pq-1}},$$

where $C = C(p,q)$ is a positive constant.

Next inserting the function G into the last estimate, dividing by

$$\int_{\{|x|>R\}} \frac{\Phi}{G^{p'-1}} dx,$$

we find in the region $\Omega_R = \{|x| > R\}$, for some positive constant $\overline{C} = \overline{C}(p,q)$,

$$\inf_{\Omega_R}(u_0 G^{(p'-1)}) \inf_{\Omega_R}(v_0 G^{(p'-1)}) \leq \overline{C} T^2 \left[\frac{1}{T} + \frac{1}{R^{\beta}}\right]^{\frac{p(q+1)}{pq-1}} \left[\frac{1}{T} + \frac{1}{R^{\alpha}}\right]^{\frac{q(p+1)}{pq-1}}.$$

Finally we obtain the desired estimate by passing to the limit as $R \to +\infty$, and the theorem is demonstrated. ■

Corollary 2.1. *Let $p, q > 1$. There is no local (and then no global) weak solution to* (1.1),(1.6) *such that*

$$\liminf_{|x|\to+\infty} u_0 G^{p'-1} \liminf_{|x|\to+\infty} v_0 G^{p'-1} = +\infty.$$

Corollary 2.2. *Let $p, q > 1$. Assume that*

$$L := \liminf_{|x|\to+\infty}(u_0 G^{p'-1}) \liminf_{|x|\to+\infty}(v_0 G^{p'-1}) > 0.$$

Then the blow up time takes place in $(0, \left(\frac{C(p,q)}{L}\right)^{\frac{pq-1}{p+q+2}})$.

Below, we give the proof of Theorem 1.2. For the convenience of the reader we recall this theorem.

Theorem 2.2. *Let* $p, q > 1$. *Assume*

$$\lim_{|x|\to+\infty}\left(u_0 G^{p'-1}|x|^{\frac{1}{2}\inf\{\beta,\alpha\}\frac{p+q+2}{pq-1}}\right)\lim_{|x|\to+\infty}\left(v_0 G^{p'-1}|x|^{\frac{1}{2}\inf\{\beta,\alpha\}\frac{p+q+2}{pq-1}}\right) = +\infty. \quad (2.4)$$

Then no global weak solution to (1.1),(1.6) *exists.*

Proof. The proof starts with the following estimate

$$\int_{|x|>R} u_0\Phi dx \int_{|x|>R} v_0\Phi dx \leq$$

$$C\left[T^{-1+\frac{1}{p'}}\left\{\int_{\mathcal{L}_R}\frac{\Phi}{G^{p'-1}}dx\right\}^{1/p'} + \frac{T^{\frac{1}{p'}}}{R^\beta}\left\{\int_{\mathcal{L}_R}\frac{\Phi}{G^{p'-1}}dx\right\}^{1/p'}\right]^{\frac{p(q+1)}{pq-1}}$$

$$\times\left[T^{-1+\frac{1}{q'}}\left\{\int_{\mathcal{L}_R}\frac{\Phi}{G^{q'-1}}dx\right\}^{1/q'} + \frac{T^{\frac{1}{q'}}}{R^\alpha}\left\{\int_{\mathcal{L}_R}\frac{\Phi}{G^{q'-1}}dx\right\}^{1/q'}\right]^{\frac{q(p+1)}{pq-1}},$$

for any $T > 0$, where Φ is as before and

$$\mathcal{L}_R = \{R < |x| < 2R\}.$$

Therefore, for any $\gamma \in \mathbb{R}$ and any $R > R_0$,

$$\inf_{|x|>R} u_0.G^{(p'-1)}|x|^\gamma \inf_{|x|>R} v_0.G^{(p'-1)}|x|^\gamma\left[\int_{|x|>R}\frac{\Phi}{G^{p'-1}|x|^\gamma}dx\right]^2 \leq$$

$$CR^{2\gamma}\left[T^{-1+\frac{1}{p'}}\left\{\int_{\mathcal{L}_R}\frac{\Phi}{G'^{-1}|x|^\gamma}dx\right\}^{1/p'} + \frac{T^{\frac{1}{p'}}}{R^\beta}\left\{\int_{\mathcal{L}_R}\frac{\Phi}{G^{p'-1}|x|^\gamma}dx\right\}^{1/p'}\right]^{\frac{p(q+1)}{pq-1}}$$

$$\times\left[T^{-1+\frac{1}{q'}}\left\{\int_{\mathcal{L}_R}\frac{\Phi}{G^{p'-1}|x|^\gamma}dx\right\}^{1/q'} + \frac{T^{\frac{1}{q'}}}{R^\alpha}\left\{\int_{\mathcal{L}_R}\frac{\Phi}{G^{p'-1}|x|^\gamma}dx\right\}^{1/q'}\right]^{\frac{q(p+1)}{pq-1}}.$$

Thus, after simplification, we easily obtain

$$\inf_{|x|>R} u_0 G^{(p'-1)}|x|^\gamma \inf_{|x|>R} v_0 G^{(p'-1)}|x|^\gamma \leq CR^{2\gamma}\Big[T^{-1+\frac{1}{p'}}$$

$$+ T^{\frac{1}{p'}}R^{-\beta}\Big]^{\frac{p(q+1)}{pq-1}}\left[T^{-1+\frac{1}{q'}} + T^{\frac{1}{q'}}R^{-\alpha}\right]^{\frac{q(p+1)}{pq-1}},$$

for any $T > 0$. A quick inspection of the above estimate with $T = R^{\inf\{\beta,\alpha\}}$ leads to

$$\inf_{|x|>R} u_0 G^{(p'-1)}|x|^\gamma \inf_{|x|>R} v_0 G^{(p'-1)}|x|^\gamma \leq \overline{C}R^{2\gamma-\inf\{\beta,\alpha\}\frac{p+q+2}{pq-1}}. \quad (2.5)$$

Obviously, for $\gamma = \frac{1}{2}\inf\{\beta,\alpha\}\frac{p+q+2}{pq-1}$, we deduce that the right hand side of (2.5) is bounded which contradicts hypothesis (2.4). ∎

Corollary 2.3. *There exists a constant $K_\star > 0$ such that if*

$$\liminf_{|x|\to\infty}\left(u_0 G^{p'-1}|x|^{\frac{1}{2}\inf\{\beta,\alpha\}\frac{p+q+2}{pq-1}}\right)\times\liminf_{|x|\to\infty}\left(v_0 G^{p'-1}|x|^{\frac{1}{2}\inf\{\beta,\alpha\}\frac{p+q+2}{pq-1}}\right) > K_\star,$$

any possible local solution to (1.1),(1.6) *blows up at a finite time.*

Remark 2.1. Similar results can be formulated with $G = \inf\left\{g_2, g_1, g_1^{\frac{q-1}{p-1}}\right\}$.

Remark 2.2. In the case where $p = q$ we shall see that the choice of $T = R^{\inf\{\beta,\alpha\}}$ in the proof of Theorem 2.2 is optimal. Since, for some positive constant, C_3, the following

$$\inf_{|x|>R} u_0 G^{(p'-1)}|x|^\gamma \inf_{|x|>R} v_0 G^{(p'-1)}|x|^\gamma \le C_3 R^{2\gamma}\left[T^{-1+\frac{1}{p'}} + T^{\frac{1}{p'}}R^{-\aleph}\right]^{2p'} \tag{2.6}$$

holds for any $T > 0$, where $\aleph = \inf\{\alpha,\beta\}$. This implies in particular that the left–hand side of (2.5) is bounded from above by

$$C_3 R^{2\gamma}\inf_{\{T\in\mathbb{R}_+\}}\left[T^{-1+\frac{1}{p'}} + T^{\frac{1}{p'}}R^{-\aleph}\right]^{2p'}.$$

Since the minimum is achieved at

$$T_0 = (p'-1)R^{\aleph},$$

we deduce, by taking $\gamma = \aleph.(p'-1)$, that the limit

$$\lim_{|x|\to\infty}\left(u_0 G^{p'-1}|x|^{\frac{1}{2}\inf\{\beta,\alpha\}\frac{1}{p-1}}\right)\times\lim_{|x|\to\infty}\left(v_0 G^{p'-1}|x|^{\inf\{\beta,\alpha\}\frac{1}{p-1}}\right),$$

is finite. A contradiction.

Remark 2.3. It is easy to see from condition (1.9) that Problem (1.1),(1.6) may have no global weak solution in the case where

$$\int_{\mathbb{R}^N} u(x,0)dx < 0, \quad \int_{\mathbb{R}^N} v(x,0)dx < 0.$$

For instance, assume $g_i \equiv 1, i = 1,2$ and

$$u_0(x) = \begin{cases} A_0 & \text{if } |x| \le R_0/2, \\ A_1|x|^{\frac{1}{2}\frac{p+q+2}{pq-1}}, & \text{if } |x| > R_0, \end{cases}$$

and

$$v_0(x) = \begin{cases} B_0 & \text{if } |x| \le R_0/2, \\ B_1|x|^{\frac{1}{2}\frac{p+q+2}{pq-1}}, & \text{if } |x| > R_0, \end{cases}$$

where $A_0, B_0 < 0, A_1, B_1 > 0$. Then if N is such that

$$\frac{p+q+2}{pq-1}\inf\{\alpha,\beta\} > 2N,$$

u_0 and v_0 are integrable and we can select A_0, A_1, B_0, B_1 such that the integrals of u_0, v_0 are nonpositive and assumption (2.4) is satisfied.

Remark 2.4. Let us point, in passing, that the generalization of the above results to the problem

$$\begin{cases} u_t & \geq & -(-\Delta)^{\frac{\beta}{2}}(a(x,t)u) + h_1|v|^p, \\ v_t & \geq & -(-\Delta)^{\frac{\alpha}{2}}(b(x,t)v) + h_2|u|^q, \end{cases}$$

where a and b are measurable positive uniformly bounded functions in $\mathbb{R}^N \times (0,+\infty)$, is quite straightforward. For the non-diagonal systems of the type

$$\begin{cases} u_t & \geq & -(-\Delta)^{\frac{\beta}{2}}(a(x,t)v) + h_1(x)|v|^p, \\ v_t & \geq & -(-\Delta)^{\frac{\alpha}{2}}(b(x,t)u) + h_2(x)|u|^q. \end{cases}$$

Condition (2.4) is formulated as

$$\lim_{|x|\to\infty}(u_0+v_0)h^{p'-1}|x|^{\inf\{\beta,\alpha\}\inf\{p'-1,q'-1\}} = +\infty, \tag{2.7}$$

where

$$h = \left\{h_1, h_2, h_2^{\frac{p-1}{q-1}}\right\}.$$

The new element in this result is that condition (2.7) shows that solutions may blow up even if $u_0 = 0$ or $v_0 = 0$.

Now let us consider the following

$$\begin{cases} \frac{\partial^k}{\partial t^k}u \geq -(-\Delta)^{\frac{\alpha}{2}}u + |v|^p, \\ \frac{\partial^k}{\partial t^k}v \geq -(-\Delta)^{\frac{\beta}{2}}v + |u|^q, \\ \frac{\partial^{k-1}}{\partial t^{k-1}}u(x,0) = u_{k-1}(x), \quad \frac{\partial^{k-1}}{\partial t^{k-1}}v(x,0) = v_{k-1}(x), \end{cases} \tag{2.8}$$

where $k \in \mathbb{N}$ is larger than 1. The functions u_{k-1}, v_{k-1} are positive for large $|x|$. Let

$$\gamma = \frac{1}{2}\frac{\inf\{\alpha,\beta\}}{k(pq-1)}\Big[(kp'-1)(p-1)(q+1) + (kq'-1)(q-1)(p+1)\Big].$$

We have

Theorem 2.3. *Assume $p, q > 1$ and*

$$\lim_{|x|\to+\infty} u_{k-1}|x|^\gamma \times \lim_{|x|\to+\infty} v_{k-1}|x|^\gamma = +\infty.$$

Then there is no global weak solution to (2.8).

The proof of this theorem can be obtained without any major difficulty. The idea is the take

$$\zeta(x,t) = \eta(t/T)\Phi(x/R),$$

as a test function where Φ is defined as above and the function $\eta \in C^\infty$ is defined by $0 \leq \eta \leq 1, \eta(\tau) = 1$ for $\tau \leq 1/2$ and $\eta(\tau) = 0$ for $\tau \geq 1$.

A natural extension of our results is to study the nonexistence of global solutions to systems of three inequalities like

$$\begin{cases} u_t & \geq & -(-\Delta)^{\frac{\beta}{2}}u + h_1|v|^p, \\ v_t & \geq & -(-\Delta)^{\frac{\alpha}{2}}v + h_2|w|^q, \\ w_t & \geq & -(-\Delta)^{\frac{\gamma}{2}}w + h_2|u|^r. \end{cases} \tag{2.9}$$

The following is our final result.

Theorem 2.4. *Let p, q, r be a positive reals larger strictly than 1. Set*

$$h = \inf\left\{h_1, h_2, h_3, h_2^{\frac{p-1}{q-1}}, h_3^{\frac{p-1}{r-1}}\right\}.$$

Then if

$$\lim_{|x|\to+\infty} u_0 h^{(p'-1)}|x|^\theta \lim_{|x|\to+\infty} v_0 h^{(p'-1)}|x|^\theta \lim_{|x|\to+\infty} w_0 h^{(p'-1)}|x|^\theta = +\infty,$$

where

$$\theta := \frac{1}{3}\inf\{\alpha, \beta, \gamma\}\frac{q(p+1)+r(q+1)+r(p+1)-3}{pqr-1}(p-1),$$

no global weak solution, (u, v, w) to (2.9) *such that $u(.,0) = u_0, v(.,0) = u_0, w(.,0) = u_0$ can exist.*

Proof. To avoid the routine argument we will shrink the proof. If we take, as above, the selected test function

$$\zeta(x,t) = \left(1 - \frac{t}{T}\right)^\tau \phi,$$

where

$$\tau = \max\{p', q', r'\},$$

we recover our familiar estimate

$$\int_{\mathbb{R}^N} u_0\phi dx \int_{\mathbb{R}^N} v_0\phi dx \int_{\mathbb{R}^N} w_0\phi dx \leq$$
$$C(p,q,r)\left(\int_{\mathbb{R}^N}\frac{\phi}{h^{p'-1}}dx\right)^3 A^{\frac{r(pq+q+1)}{pdr-1}} B^{\frac{p(pr+r+1)}{pdr-1}} C^{\frac{q(rp+p+1)}{pqr-1}},$$

where

$$A = T^{-1+\frac{1}{r'}} + R^{-\alpha}T^{\frac{1}{r'}}, \quad B = T^{-1+\frac{1}{q'}} + R^{-\beta}T^{\frac{1}{q'}}, \quad C = T^{-1+\frac{1}{p'}} + R^{-\gamma}T^{\frac{1}{p'}}.$$

Likewise, by setting,

$$T = R^{\inf\{\alpha,\beta,\gamma\}},$$

we arrive at a contradiction. ∎

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