Arithmetically Cohen-Macaulay reducible curves in projective spaces

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Abstract

Here we prove that a reduced curve $X \subset \mathbf{P}^N$ either canonically embedded or with a high degree complete embedding is arithmetically Cohen - Macaulay.

1 Introduction

In this paper we study the existence of arithmetically Cohen - Macaulay reduced curves $X \subset \mathbf{P}^N$. We are interested in the case in which either X is canonically embedded or $\deg(X)$ is large with respect to $p_a(X)$. In the latter case trivial examples show that we need to assume that the degree of each irreducible component,D, of X is sufficiently large with respect to $p_a(X)$ and rather low with respect to the dimension of the linear space $\langle D \rangle$ spanned by D (see Definition 1). If X is irreducible, everything is known (in the canonical case by the extension to singular irreducible curves of Petri theorem ([7]), in the high degree case by [4], Th. 1 and its application to Th. 2 for irreducible but not necessarily smooth curves).

Definition 1. Let C be a reduced and connected projective curve. A line bundle L on C will be called canonically positive if there is an inclusion $j : \omega_C \to L$ of $\mathcal{O}_C$-sheaves, i.e. if there is $s \in H^0(C, Hom(\omega_C, L))$ with s not vanishing identically on any irreducible component of C.

Notice that in Definition 1 we do not require that C is a Gorenstein curve.

*The author was partially supported by MURST and GNSAGA of INdAM (Italy).

Received by the editors October 2001.

Communicated by J. Thas.

1991 *Mathematics Subject Classification :* 14H50, 14N05, 13D02, 13C99.

Key words and phrases : reducible curves, postulation, arithmetically normal, canonical curve, arithmetically Cohen - Macaulay curve.

Proposition 1. *Let C be a reduced and connected Gorenstein projective curve such that ω_C is very ample. Let $h : C \to \mathbf{P}^{g-1}$ be the embedding associated to ω_C. Then ω_C has Property N_0, i.e. $h(C)$ is arithmetically Cohen - Macaulay.*

Theorem 1. *Let C be a reduced and connected projective curve, L a very ample canonically positive line bundle on C with $L \neq \omega_C$ and $f : C \to \mathbf{P}(H^0(C, L)^*)$ the embedding associated to L. Then L has Property N_0.*

We work over an algebraically closed field K with $\mathrm{char}(K) = 0$.

2 The proofs

For any subset S of a projective space, let $\langle S\rangle$ denote its linear span.

Lemma 1. *Let C be a reduced and connected projective curve, L a canonically positive line bundle on C and $X \subseteq C$ the union of some of the irreducible components of C. If either $L \neq \omega_C$ or $X \neq C$, then $h^1(X, L|X) = 0$.*

Proof. Since C is connected, we have $h^0(C, \mathcal{O}_C) = 1$ and $h^0(C, \mathcal{I}_Z) = 0$ for every non-empty zero-dimensional subscheme Z of C. By duality we have $h^1(C, \omega_C) = 1$ and $h^1(C, R) = 0$ for every $R \in \mathrm{Pic}(C)$ strictly containing ω_C. This proves the lemma for $X = C$. Now assume $X \neq C$ and X connected. Let Y be the union of the irreducible components of C not contained in X. Since C is connected, we have $X \cap Y \neq \emptyset$. Hence $\omega_C|X$ strictly contains ω_X. Hence $L|X$ strictly contains ω_X. By the first part we obtain $h^1(X, L|X) = 0$. If $X \neq C$ but X is not connected, then apply the statement just proved to each connected component of X. ■

Remark 1. Let C be a reduced and connected reducible curve. Let $X \subset C$ be the union of some of the irreducible components of C and Y the union of the other irreducible components of C. We assume $X \neq \emptyset$, $Y \neq \emptyset$ and Y connected. Let $j : Y \to C$ be the inclusion. By the functorial property of the dualizing sheaf ([1], first 3 lines of p. 244) there is an inclusion $j_! j_*(\omega_Y) \to \omega_C$. Set $F = \mathrm{Im}(j_! j_*(\omega_Y))$. By construction F is supported by Y and hence every local section of F vanishes on $X \backslash (X \cup Y)$. Thus $F \subseteq \mathcal{I}_{X,C} \otimes \omega_C$. Since Y is connected, we have $h^1(Y, \omega_Y) = 1$, i.e. $h^1(Y, F) = 1$. Thus $h^1(C, \mathcal{I}_{X,C} \otimes \omega_C) \leq 1$. From the exact sequence

$$0 \to \mathcal{I}_{X,C} \otimes \omega_C \to \omega_C \to \omega_C|X \to 0 \tag{1}$$

we obtain the surjectivity of the restriction map $H^0(C, \omega_C) \to H^0(X, \omega_C|X)$ without any assumption on ω_C; if C is not Gorenstein at the points of $(X \cap Y)_{red}$, the sheaf $\omega_C|X$ may have torsion. Now take a canonically positive line bundle L on C with $L \neq \omega_C$. In the same way we obtain $h^1(C, \mathcal{I}_{X,C} \otimes L) = 0$ and hence that the restriction map $H^0(C, L) \to H^0(X, L|X)$ is surjective. This part of the remark holds even if Y is not connected.

In this paper we will use only the case $p = 0$, i.e. Property N_0, of the following observation. For the definition of Property N_p, $p \geq 0$, see [3] or [4].

Remark 2. Let $C \subset \mathbf{P}^n$ be a reduced and connected curve. Assume that C is linearly normal and that it spans $\mathbf{P}^n$, i.e. assume that the restriction map $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)) \to H^0(C, \mathcal{O}_C(1))$ is bijective. Take a general hyperplane H of $\mathbf{P}^n$ and consider the exact sequence

$$0 \to \mathcal{I}_C(t-1) \to \mathcal{I}_C(t) \to \mathcal{I}_{C\cap H,H}(t) \to 0 \tag{2}$$

induced by the multiplication by an equation of H. First assume $h^1(C, \mathcal{O}_C(1)) = 0$. Using (2) we easily obtain that $\mathcal{O}_C(1)$ has Property N_p for some integer $p \geq 0$ if and only if the zero-dimensional subscheme $C \cap H$ of H has Property N_p in H; indeed, when N_0 holds for C or $C \cap H$, we can lift a minimal set of generators of the homogeneous ideal of $C \cap H$ in H to a minimal set of generators of the homogeneous ideal of C in $\mathbf{P}^n$ and hence $C \cap H$ and C have the same Betti numbers ([5], Th. 1.3.6, or [3], 3.b.1, 3.b.4 and 3.b.7). Now assume $h^1(C, \mathcal{O}_C(1)) = 1$ and take a subset S of $C \cap H$ with $\mathrm{card}(S) = \mathrm{card}(C \cap H) - 1$. Using (2) we obtain as before that C has Property N_p if S has Property N_p.

Lemma 2. *Let $C \subset \mathbf{P}^n$ be a reduced curve and $H \subset \mathbf{P}^n$ a general hyperplane. Let $M \subset H$ be a linear subspace spanned by a subset of $C \cap H$. Set $S = C \cap M$. We have card(S) $\geq$ dim(M) + 1. If card(S) $\geq$ dim(M) + 2, there is a subset T of S (perhaps empty), a subspace M' of M with dim(M') = dim(M) − card(T) and M spanned by $M' \cup T$ and a linear subspace N of $\mathbf{P}^n$ such that dim(N) = dim(M') + 1, $M' = N \cap H$ and $S \cap N = X \cap N$, where X is the union of all irreducible components of C contained in N.*

Proof. Since S spans M, we have $\mathrm{card}(S) \geq \dim(M) + 1$. Now assume $\mathrm{card}(S) \geq \dim(M) + 2$. Since $\mathrm{char}(K) = 0$, for every irreducible component D of C the set $D \cap H$ is in linearly general position in its linear span $\langle D \cap H \rangle = \langle D \rangle \cap H$ ([6], Lemma 1.1). Using [6], Cor. 1.6, and the generality of H we obtain that for every irreducible component D of C either $D \cap M$ is in linearly general position in its linear span or $D \cap H \subseteq M$ and $\langle D \rangle \cap H \subset M$. In the latter case $\langle D \rangle$ is the unique linear subspace U of $\mathbf{P}^n$ with $X \subset U$ and $\dim(U) = \dim(\langle D \cap H \rangle) + 1$. Let X be the union of all irreducible components D of C such that $\langle D \rangle \cap H \subset M$ and Y the union of all other irreducible components of C. Set $T = S \backslash X \cap H$. By the generality of H as in [6], Cor. 1.6, we obtain $\langle X \rangle \cap Y = \emptyset$. Thus $\dim(X \cap H) + \mathrm{card}(T) = \dim(M)$. Since $\mathrm{card}(S) \geq \dim(M) + 2$, we have $X \neq \emptyset$. ∎

Remark 3. Let D be an irreducible projective curve. We have $\deg(\omega_D) = 2p_a(D) - 2$ even when D is not Gorenstein ([2], Prop. 3.1.6). The proof uses only the duality for locally Cohen - Macaulay projective one-dimensional schemes and works for any reduced and connected projective curve.

Lemma 3. *Let D be a reduced projective curve and $L \in Pic(D)$ with L very ample. Assume $h^0(D, Hom(\omega_D, L)) \neq 0$, i.e. assume L canonically positive. Then either $L \cong \omega_D$ or deg(L) $\geq 2p_a(D) + 1$.*

Proof. Since $\deg(\omega_D) = 2p_a(D) - 2$ (Remark 3) it is sufficient to exclude the cases $\deg(L) = 2p_a(D) - 1$ and $\deg(L) = 2p_a(D)$. Let $F \subset L$ be the image of a non-zero section of $Hom(\omega_D, L)$. Since L is locally free of rank one, the torsion sheaf L/F

is isomorphic to the structural sheaf of a zero-dimensional subscheme Z of D with $\deg(Z) = \deg(L) - \deg(F) = \deg(L) - 2p_a(D) + 2$. Assume $\deg(L) = 2p_a(D)$, i.e. assume $\deg(Z) = 2$. Since L is very ample, it is spanned and hence for every $P \in Z_{red}$ we have $h^0(D, L \otimes \mathcal{I}_{\{P\}}) = h^0(D, L) - 1$. By Riemann - Roch we have $h^0(D, L) = p_a(D) + 1 = h^0(D, L \otimes \mathcal{I}_{\{P\}}) + 1$, i.e. $h^0(D, L \otimes \mathcal{I}_{\{P\}}) = h^0(D, L \otimes \mathcal{I}_{\{Z\}})$, contradicting the very ampleness of L and the assumption $\deg(Z) = 2$. In the same way we see that if $\deg(L) = 2p_a(D) - 1$, then L is not spanned at the point Z, contradiction. ∎

Lemma 4. *Let $C \subset \mathbf{P}^n$ be a reduced and connected curve and $H \subset \mathbf{P}^n$ a general hyperplane. Assume that C is linearly normal, $\mathcal{O}_C(1) \not\cong \omega_X$ and that $\mathcal{O}_C(1)$ is canonically positive. Let M be any subspace of H. Then card$(C \cap M) \leq 2(\dim(M)) + 1$.*

Proof. Fix a linear subspace M for which the inequality is not true and with $\dim(M)$ minimal. Since $\mathrm{card}(C \cap M) > 2(\dim(M)) + 1$, we have $\mathrm{card}(C \cap M) \geq \dim(M) + 2$. By Lemma 2 the minimality of $\dim(M)$ implies that for every irreducible component D of C either $D \cap M = \emptyset$ or $D \cap H \subset M$. Let X be the union of all irreducible components D of C such that $D \cap H \subset M$. Hence $C \cap M = X \cap M$. By Lemma 2 we have $\dim(N) = \dim(\langle X \rangle) + 1$ and $M = \langle X \rangle \cap H$. By Lemma 3 the result is true if $X = C$. Assume $X \neq C$. Let c be the number of the connected components of X. By the last sentence of Remark 1 the curve X is linearly normal in $\langle X \rangle$. Since $h^1(X, L|X) = 0$ (Lemma 1), we have $\dim(M) = \dim(\langle X \rangle) - 1 = \deg(X) + c - 2 - p_a(X)$ (Riemann - Roch). Since for every connected component A of X $L|A$ strictly contains ω_A we may apply Lemma 3 to each connected component of X. We obtain $\deg(X) \geq 2p_a(X) - 2 + 3c$. Thus $\mathrm{card}(C \cap M) = \deg(X) \leq 2(\dim(M)) + 6 - 5c$, proving the lemma. ∎

Lemma 5. *Let C be a connected curve such that ω_C is very ample and $C \subset \mathbf{P}^n$ its canonical embedding. Let M be a proper subspace of H. Then card$(C \cap M) \leq 2(\dim(M)) + 1$.*

Proof. We copy words for words the proof of Lemma 4. If $c = 1$, then we still may apply Remark 1. Assume $c \geq 2$. The first part of the proof of Remark 1 gives $\dim(\langle X \rangle) \geq h^0(X, \mathcal{O}_X(1)) - c$. Hence as in the proof of Lemma 4 we obtain $\mathrm{card}(C \cap M) = \deg(X) \leq 2(\dim(M)) + 2 - 3c$, proving the lemma. ∎

The following result is a simple but lengthy exercise (see [4], last 3 lines of §2).

Lemma 6. *Let $S \subset \mathbf{P}^r$ be a finite set with card$(S) \leq 2r+1$. We have $h^1(\mathbf{P}^r, \mathcal{I}_S(t)) \neq 0$ for some integer $t \geq 2$ if and only if $h^1(\mathbf{P}^r, \mathcal{I}_S(2)) \neq 0$ and this is the case if and only if there is an integer s with $1 \leq s < r$ and a linear subspace M of $\mathbf{P}^r$ with $\dim(M) = s$, card$(S \cap M) \geq 2s + 2$ and $h^1(M, \mathcal{I}_{S \cap M}(2)) \neq 0$.*

Proof of Proposition 1 and Theorem 1. Let $H \cap C$ be a general hyperplane section of C in the embedding given by L, and $S = C \cap H$ for Theorem 1, $S \subset C \cap H$, $\mathrm{card}(S) = 2p_a(C) - 3$ for Proposition 1. By Remark 2 if S has Property N_0, also C does. Now apply Lemmata 4 (for Theorem 1), 5 (for Proposition 1) and 6 to conclude.

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