MOZES' GAME OF NUMBERS ON DIRECTED GRAPHS

ROHAN HEMASINHA, AVINASH J. DALAL, AND DONALD MCGINN

Abstract. In 1986, the contestants of the 27th International Mathematical Olympiad were given a game of numbers played on a pentagon. In 1987, Mozes generalized this game to an arbitrary undirected, weighted, connected graph. The convergence properties and total number of moves of any convergent game have been resolved by Mozes using Weyl groups. Eriksson provided an alternate proof using matrix theory and graph theory. In this paper, we briefly discuss the results of Mozes and Eriksson on undirected graphs. Then we generalize this game to arbitrary directed, strongly connected graphs and investigate the convergence properties of the game of numbers.

1. Introduction

In 1986, the 27th International Mathematical Olympiad posed a problem involving a game played on a regular pentagon with integers.

"To each vertex of a regular pentagon an integer is assigned in such a way that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z, respectively and $y < 0$ then the following operation is allowed: the numbers x, y, z are replaced by $x + y$, $-y$, $z + y$, respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps," ([5]).

This *game of numbers* has been shown to come to an end after a finite number of steps. We can view the five integers arranged on the regular pentagon as labeling the vertices of C_5 with integers, where C_5 is a cycle graph with five vertices and edges.

In [6], Wegert and Reiher present several solutions to the original game of numbers and some generalizations. For example, their first solution is showing that the following integer-valued quadratic function based on the five vertex labels decreases with every move of the game:

$$\begin{aligned} f(x) &= f(x_1, x_2, x_3, x_4, x_5) \\ &= (x_1 - x_3)^2 + (x_2 - x_4)^2 + (x_3 - x_5)^2 + (x_4 - x_1)^2 + (x_5 - x_2)^2, \end{aligned}$$

where x_1, x_2, x_3, x_4, x_5 are the vertex labels in their order on the circle.

In Section 2, we describe the generalization of the game of numbers on undirected graphs, as well as outline solutions to the generalization given by Mozes and Eriksson. In Section 3, we consider the game of numbers played on directed graphs, and prove the game of numbers always converges when it is played on a directed cycle. In Section 4, we prove that the game of numbers diverges for graphs whose adjacency matrices have spectral radius greater than or equal to 2. In Section 5, we analyze graphs on which looping games exist. In Section 6, we analyze graphs whose adjacency matrices have spectral radius less than 2. Finally, in Section 7, we make our concluding remarks.

2. Generalization of the Game of Numbers

The game of numbers was generalized from integers on C_5 to real numbers on an arbitrary undirected, connected graph by Mozes [4]. Let G be a simple, undirected, connected graph on n vertices v_1, v_2, …, v_n. Assume that each vertex v_i is assigned a number a_i, with $a_j < 0$ for at least one j. A *move* consists of selecting a vertex v_i whose number a_i is negative, adding a_i to the number of each vertex adjacent to v_i, and inverting the sign of the number at v_i. A particular instance of the game of numbers on G consists of performing a (perhaps infinite) sequence of moves as long as a move is possible, that is, as long as some vertex of G has a negative value. A game *terminates*, or is said to *converge*, once a state is reached where there are no vertices having negative numbers. Note that a move is possible only if the configuration has at least one negative valued vertex. A valid state is one in which at least one number is negative. Henceforth, state will always mean a valid state.

Mozes [4] analyzed the behavior of this game of numbers on arbitrary graphs by first studying the game on the undirected cycle C_n. He showed the following result holds.

Theorem 2.1. *Let G be an undirected, simple, connected graph. For any given initial state in the game of numbers, exactly one of the following holds:*

1. *Every game will terminate; the terminal state and the number of moves leading to the terminal state are the same no matter how the game is played.*
2. *Every game can be continued indefinitely, that is, no terminal position (a state with all values non-negative) will be reached.*

The objective of this paper is to examine the behavior of the following generalization of the game of numbers to directed graphs. Let G be a directed, strongly connected graph on n vertices v_1, v_2, …, v_n. For each i, suppose that a real number a_i is assigned to vertex v_i such that $a_j < 0$

for at least one j. A move consists of selecting a vertex v_i with a negative number a_i, adding a_i to the number on each of v_i's out-neighbors, and then inverting the sign of a_i. A game is a (perhaps infinite) sequence of moves, and a game terminates, or converges, once a state is reached where there are no vertices having negative numbers.

In the remainder of this section, we present some illustrative examples and additional results about the game on undirected graphs. The study of the game on directed graphs commences in Section 3.

Theorem 2.1 was proved by Mozes using the theory of Weyl groups and Kac-Moody algebras. Each legal move of the game is represented by means of a linear transformation on $\mathbb{R}^n$. These linear transformations generate a group H that has a fundamental domain P for the appropriate group action. Mozes showed that a game terminates if and only if the initial state belongs to the Tits cone of the group, defined as $\cup_{g \in G}\, gP$, and in this case Theorem 2.1 holds.

More elementary proofs were given by Bjorner [1]. Using elementary graph theory and the theory of non-negative matrices, Eriksson extended the results obtained by Mozes. The main tools used in Eriksson's paper are graph theory and the Perron-Frobenius Theorem [3] on non-negative matrices. He showed the following result holds.

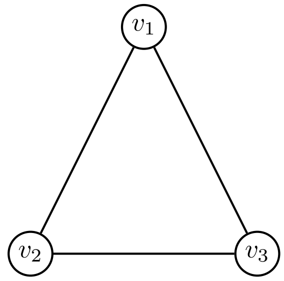
Theorem 2.2. *Let G be an undirected graph, A its adjacency matrix, and $\rho(A)$ the spectral radius of A (i.e., the maximum of the absolute value of the eigenvalues of A). Then*

(1) Every initial state on G leads to a convergent game if and only if $\rho(A) < 2$.
(2) G admits looping games (games in which there is a state that repeats after a certain sequence of moves) if and only if $\rho(A) = 2$.
(3) G admits divergent games (games that neither loop nor terminate) if and only if $\rho(A) > 2$.

Moreover, in the case that G admits looping games, Eriksson obtained a characterization of the initial states that lead to looping games.

We now look at the following example.

Example 2.3. *Consider the following undirected C_3.*



Let the state of the vertices be given by $\mathbf{x} = [-1, 1, 0]^T$, where x_i corresponds to the state of vertex v_i. The bold number of a state indicate which vertex is picked to play the next round of the game. This particular game loops as follows.

$$\mathbf{x} = \begin{bmatrix} -\mathbf{1} \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ -\mathbf{1} \end{bmatrix} \to \begin{bmatrix} 0 \\ -\mathbf{1} \\ 1 \end{bmatrix} \to \begin{bmatrix} -\mathbf{1} \\ 1 \\ 0 \end{bmatrix}.$$

Thus, the game of numbers diverges on C_3. However, if we let the state of the vertices be given by $\mathbf{x} = [-2, 6, 1]^T$, then this particular game converges on C_3 as follows.

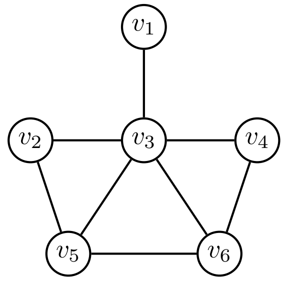
$$\mathbf{x} = \begin{bmatrix} -\mathbf{2} \\ 6 \\ 1 \end{bmatrix} \to \begin{bmatrix} 2 \\ 4 \\ -\mathbf{1} \end{bmatrix} \to \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

Note that if we let the state of the vertices be given by $\mathbf{x} = [-5, -6, -7]^T$, then no matter how we play the game, this particular game of numbers never converges. For example, if we always choose the negative number with largest magnitude, the game is played as follows.

$$\mathbf{x} = \begin{bmatrix} -5 \\ -6 \\ -\mathbf{7} \end{bmatrix} \to \begin{bmatrix} -12 \\ -\mathbf{13} \\ 7 \end{bmatrix} \to \begin{bmatrix} -\mathbf{25} \\ 13 \\ -6 \end{bmatrix} \to \begin{bmatrix} 25 \\ -12 \\ -\mathbf{31} \end{bmatrix} \to \begin{bmatrix} -6 \\ -\mathbf{43} \\ 31 \end{bmatrix} \to \cdots.$$

For any undirected cycle, if the sum of the starting vector is s, then after every legal move of the game, the sum of the resulting position vector stays s. Hence, it is easy to see that if we start with $s < 0$, then the game of numbers must diverge. If we start with $s = 0$, then the game of numbers must diverge or reach a position vector whose coordinates are all 0. Next, we look at an example that is not a cycle.

Example 2.4. *Consider the following undirected G.*



Let the state of the vertices be given by $\mathbf{x} = [-1, -1, -1, -1, -1, -1]^T$. No matter how the game is played, it must diverge. For example, if we always select the negative valued vertex with smallest index, the game is played as follows.

$$\mathbf{x} = \begin{bmatrix} -\mathbf{1} \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \to \begin{bmatrix} 1 \\ -\mathbf{1} \\ -2 \\ -1 \\ -1 \\ -1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ -\mathbf{3} \\ -1 \\ -2 \\ -1 \end{bmatrix} \to \begin{bmatrix} -\mathbf{2} \\ -2 \\ 3 \\ -4 \\ -5 \\ -4 \end{bmatrix} \to \begin{bmatrix} 2 \\ -\mathbf{2} \\ 1 \\ -4 \\ -5 \\ -4 \end{bmatrix} \to \begin{bmatrix} 2 \\ 2 \\ -\mathbf{1} \\ -4 \\ -7 \\ -4 \end{bmatrix} \to \cdots .$$

However, there are convergent games for this graph. Let the state of the vertices be given by $\mathbf{x} = [-1, 3, 7, 3, -4, 9]^T$. This game converges as follows.

$$\mathbf{x} = \begin{bmatrix} -1 \\ 3 \\ 7 \\ 3 \\ -\mathbf{4} \\ 9 \end{bmatrix} \to \begin{bmatrix} -1 \\ -\mathbf{1} \\ 3 \\ 3 \\ 4 \\ 5 \end{bmatrix} \to \begin{bmatrix} -\mathbf{1} \\ 1 \\ 2 \\ 3 \\ 3 \\ 5 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \\ 3 \\ 5 \end{bmatrix}.$$

3. Game of Numbers on Directed Cycles

Throughout this section, we assume that we have a directed cycle, C_n, which has n vertices labeled v_1, v_2, $\ldots$, v_n. Each vertex v_i has the value $a_i \in \mathbb{R}$ for $1 \leqslant i \leqslant n$, respectively. We also assume that v_i has an edge directed to vertex v_{i+1} for each $1 \leqslant i \leqslant n-1$, and v_n has an edge directed out to vertex v_1. Our aim is to show that every game of numbers played on C_n converges for any n. We begin with a key lemma.

Lemma 3.1. *In every move of the game of numbers on C_n, exactly one of the following three must happen.*

(i) the number of vertices labeled with a negative number decreases,
(ii) the number of vertices labeled with a zero decreases,
(iii) the sum of the absolute values of the negative values decreases.

Proof. Suppose we choose v_i, so that $a_i < 0$. Without loss of generality, assume $i < n$. Then by the rules of the game, v_i gets the value $-a_i$ (which is positive), v_{i+1} gets the value $a_i + a_{i+1}$, and the other $n-2$ vertices keep the same values.

If $a_{i+1} < 0$, then (i) holds since v_i, v_{i+1} went from both having negative values to only one of them having a negative value. If $a_{i+1} = 0$, then (ii) holds since v_i, v_{i+1} went from one negative and one zero value to one positive and one negative value, respectively. If $a_{i+1} > 0$ and $a_i + a_{i+1} \geqslant 0$, then (i) holds since v_i, v_{i+1} went from one being negative to both being non-negative. If $a_{i+1} > 0$ and $a_i + a_{i+1} < 0$, then (iii) holds since $|a_i + a_{i+1}| < |a_i|$. □

Observation 3.2. *When a move is played with vertex v_i chosen, then v_i becomes positive. Thus, each time a move is played, either the number of negative valued vertices decreases by one, or a negative value is shifted from vertex v_i to v_{i+1} (or from v_n to v_1). When a negative value is shifted from vertex v_i to v_{i+1} (or from v_n to v_1), its absolute value either stays the same or decreases by Lemma 3.1.*

Theorem 3.3. *The game of numbers always converges when played on a directed cycle.*

Proof. Pick any directed cycle with vertices $v_1, \ldots, v_n$ labeled with real numbers $a_1, \ldots, a_n$, such that the cycle orientation goes from v_{i-1} to v_i for each $2 \leqslant i \leqslant n$, and from v_n to v_1. First, we show that the number of negative valued a_i's decreases to one. Since every vertex in a directed cycle has exactly one out-neighbor, the number of negative vertices never increases. Assume $v_{i_1}, \ldots, v_{i_k}$ are the only k negative vertices with $i_1 < i_2 < \cdots < i_k$, for some $k \geqslant 2$. Let a_{i_j} be the value with largest absolute value, which clearly exists since we only have a finite number of vertices (and if there is more than one with largest value, pick any one). Without loss of generality, assume that $j = k$.

Note that if v_{m-1} and v_m are both negative vertices for some m, then firing v_{m-1} decreases the number of negative vertices. Hence, there is at most $i_2 - i_1 - 1$ moves that can be made without firing vertices $v_{i_2}, \ldots, v_{i_k}$, and if $i_2 - i_1 - 1$ such moves are made, then v_{i_2-1} and v_{i_2} are negative and v_m is non-negative for each $1 \leqslant m < i_2 - 1$. Then, there are at most $2(i_3 - i_2 - 1)$ moves that can be made without firing vertices $v_{i_3}, \ldots, v_{i_k}$, and if $2(i_3 - i_2 - 1)$ such moves are made, then v_{i_3-2}, v_{i_3-1}, v_{i_3} are negative and v_m is non-negative for each $1 \leqslant m < i_3 - 2$. Continuing in this way, we see that there are only a finite number of moves that can be made before we must fire v_{i_k}. During the moves before v_{i_k} is first fired, by Observation 3.2, either the number of negative values has decreased or all the negative values are assigned to vertices v_{i_k}, v_{i_k-1}, …, v_{i_k-k+1}. Hence, we must choose v_{i_k} or Lemma 3.1 (i) applies.

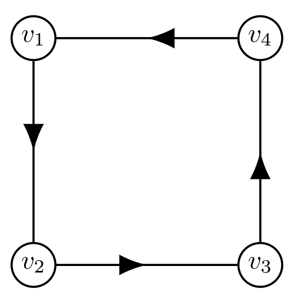
If we choose v_{i_k}, then its value is now $-a_{i_k}$. By assumption, $-a_{i_k} \geqslant |a_{i_m}|$ for each $1 \leqslant m \leqslant k$, and by Observation 3.2, when the negative values shift down during play, they never increase in size. Thus, $-a_{i_k}$ is at least as big as the absolute value of v_{i_k-1}'s value, and choosing v_{i_k-1} results in Lemma 3.1 (i) occurring.

In at most $n - k - 1$ moves, a_{i_k} shifts down to vertex v_{i_k-k}, or possibly becomes non-negative before it reaches that vertex. In either case, we must choose a vertex from v_{i_k-k}, …, v_{i_k-1}, and Lemma 3.1 (i) occurs. In all cases, we must decrease to only one negative valued vertex.

Next, we show the game terminates. Assume there is only one negative value, call it c, and it is on vertex v_i for some $1 \leqslant i \leqslant n$. Then, in at most $n - 1$ moves, either there are no negative values and the game has terminated, or v_{i-1} is now negative with a value whose magnitude is less than or equal to $|c|$. In the latter case, choosing v_{i-1} ends the game. $\square$

Theorem 3.3 says that the game of numbers always converges on a directed cycle. However, for a given initial configuration, the number of moves until a game converges can vary from one game to another, as the following example illustrates.

Example 3.4. *Consider the following directed cycle.*



Let the state of the vertices be given by $\mathbf{x} = [-1, -1, 0, 0]^T$*, where* x_i *corresponds to the state of vertex* v_i*. As before, the bold numbers of a state indicate which vertex is picked to play the next round of the game. Starting the game of numbers at vertex* v_1*, one set of 5 possible moves until the game* *terminates is*

$$\mathbf{x} = \begin{bmatrix} -\mathbf{1} \\ -1 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ -\mathbf{2} \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 2 \\ -\mathbf{2} \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 2 \\ 2 \\ -\mathbf{2} \end{bmatrix} \to \begin{bmatrix} -\mathbf{1} \\ 2 \\ 2 \\ 2 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}.$$

On the other hand, starting the game of numbers on vertex v_2 on the state $\mathbf{x}$*, another set of 4 possible moves until the game terminates is*

$$\mathbf{x} = \begin{bmatrix} -1 \\ -\mathbf{1} \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} -\mathbf{1} \\ 1 \\ -1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ -\mathbf{1} \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 2 \\ 1 \\ -\mathbf{1} \end{bmatrix} \to \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}.$$

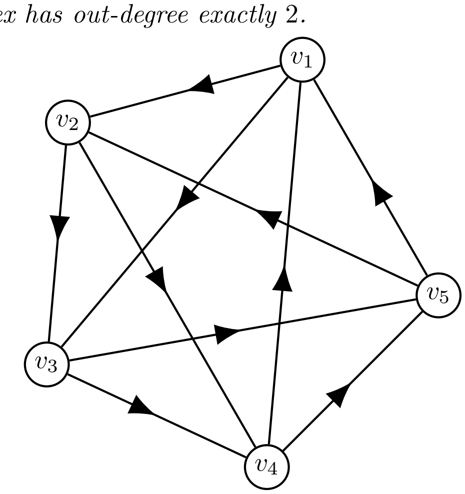
Note that the final position vector is different for these last two games.

In the case of undirected cycles, every game of numbers must terminate in the same number of moves, no matter how the game is played. This does not happen in the case of directed cycles, where the number of moves required to end the game depends on the order selection of the vertices. This suggests that analyzing the game of numbers on arbitrary directed graphs is much more difficult than in the undirected case.

4. Divergent Game of Numbers

The game of numbers from Section 3 can be extended to the game of numbers over any strongly connected directed graph. Consider the following example.

Example 4.1. *Consider the following strongly connected directed graph, where each vertex has out-degree exactly* 2.



Let the state of the vertices be given by $\mathbf{x} = [-10, -5, 0, 3, -20]^T$*, where* x_i *corresponds to the state of vertex* v_i*. Starting the game of numbers at vertex* v_5*, a few iterations of the game gives*

$$\mathbf{x} = \begin{bmatrix} -10 \\ -5 \\ 0 \\ 3 \\ \mathbf{-20} \end{bmatrix} \to \begin{bmatrix} \mathbf{-30} \\ -25 \\ 0 \\ 3 \\ 20 \end{bmatrix} \to \begin{bmatrix} 30 \\ \mathbf{-55} \\ -30 \\ 3 \\ 20 \end{bmatrix} \to \begin{bmatrix} 30 \\ 55 \\ \mathbf{-85} \\ -52 \\ 20 \end{bmatrix} \to \begin{bmatrix} 30 \\ 55 \\ 85 \\ \mathbf{-137} \\ -65 \end{bmatrix} \to \begin{bmatrix} -107 \\ 55 \\ 85 \\ 137 \\ \mathbf{-202} \end{bmatrix}$$
$$\to \begin{bmatrix} \mathbf{-309} \\ -147 \\ 85 \\ 137 \\ 202 \end{bmatrix} \to \cdots .$$

The bold numbers of a state indicate which vertex is picked to play the next round of the game of numbers. In this game of numbers, the choice of which vertex to play a particular round is given by the smallest negative real number from the state of that round. One observation of this particular game is that it seems not to converge (we show after the proof of Theorem 4.2 that this game does diverge).

Unlike the game of numbers played on directed cycles, Example 4.1 suggests that the game of numbers may not always converge for an arbitrary strongly connected, directed graph. This section aims to characterize the type of strongly connected graphs for which the game of numbers diverges.

Theorem 4.2. *If* G *is a strongly connected directed graph, each of whose vertices have out-degree at least* 2*, then there is a game of numbers on* G *that does not converge.*

Proof. Note that one way to show that a game of numbers does not converge is to show that the sum of the values of the vertices at any point in the game is always negative. Let s be the sum of the value of the vertices of G at the start of a given game of numbers for G, where each vertex of G has out-degree of at least 2. Note that if vertex v_i of out-degree d_2 with negative value a_i is chosen, then after that move, s changes to $s+(d_2-2)a_i$, since changing the sign of a_i affects the sum by a factor of $-2a_i$, and adding a_i to the d_2 out-neighbors of v_i affects the sum by a factor of d_2a_i. Thus, if we pick any game that starts with a negative sum s, then since each vertex has out-degree of at least 2, every move of this particular game of numbers must result in a negative sum of the values of the vertices. Hence, the game never converges. $\square$

In Example 4.1, the sum of the values of the starting position ($\mathbf{x} = [-10, -5, 0, 3, -20]^T$) is -32. Since each vertex has out-degree 2, the sum of the values of the state vector is always -32, so the game of numbers diverges. The previous proof of Theorem 4.2 was simple, but we now provide an alternate proof that has a generalization to our main result of this section.

The transition matrices for the states of the game are given by

$$F_i = I + (A^T - 2I)\mathbf{e_i}\mathbf{e_i}^T ,$$

where A is the adjacency matrix of G given by a labeling of the vertices v_1, v_2, ..., v_n, and the $\mathbf{e_i}$'s are the unit vectors of the standard base for $\mathbb{R}^n$. Applying F_i to a state of the game $\mathbf{x}$ flips the sign of x_i in $\mathbf{x}$. It also adds x_i to any entry x_j of $\mathbf{x}$, where j is such that v_j is an out-neighbor of v_i. Specifically, if $\mathbf{x}$ is the state of the game with $x_i < 0$, for some i, then

$$\begin{aligned} F_i\mathbf{x} &= \mathbf{x} + (A^T - 2I)\mathbf{e_i}\mathbf{e_i}^T\mathbf{x} \\ &= \mathbf{x} + (A^T - 2I)\mathbf{e_i}(x_i) \\ &= \mathbf{x} + x_i(A^T - 2I)\mathbf{e_i} . \end{aligned}$$

Letting $\mathbf{1} = [1, 1, \ldots, 1]^T$, the sum of the entries of $F_i\mathbf{x}$ is

$$\begin{aligned} \mathbf{1}^T F_i\mathbf{x} &= \mathbf{1}^T\mathbf{x} + x_i\mathbf{1}^T(A^T - 2I)\mathbf{e_i} \\ &= \mathbf{1}^T\mathbf{x} + x_i(\mathbf{1}^T A^T\mathbf{e_i} - 2\mathbf{1}^T\mathbf{e_i}) \\ &= \mathbf{1}^T\mathbf{x} + x_i(\mathbf{1}^T A^T\mathbf{e_i} - 2) . \end{aligned}$$

Since the ith column of A^T is $A^T\mathbf{e_i}$, then $\mathbf{1}^T A^T\mathbf{e_i}$ is the sum of the ith column of A^T, which in turn is the ith row of A. If d_i denotes the out-degree of vertex v_i, then the sum of the ith row of A is d_i, and this gives

$$\begin{aligned} \mathbf{1}^T F_i\mathbf{x} &= \mathbf{1}^T\mathbf{x} + x_i(d_i - 2) \\ &= \sum_{j=1}^{n} x_j + x_i(d_i - 2) . \end{aligned}$$

Now, suppose that $\mathbf{1}^T\mathbf{x} < 0$. Since every vertex of G has out-degree at least 2, then $d_i \geqslant 2$. Thus, if $x_i < 0$, then $\mathbf{1}^T F_i\mathbf{x} < 0$. This says that starting the game with a state $\mathbf{x}$ which has a negative sum implies that any state that can be reached from that $\mathbf{x}$ will have a negative sum. In fact, if the out-degree of every vertex of G is at least 2, then the sum of the entries of any state of the game does not increase from that of the previous state. In particular, if the out-degree of every vertex of G is greater than 2, then the sum of the entries of any state of the game will strictly decrease with each iteration and never be positive or zero. We generalize this to our next divergence theorem, but we need the following part of the Perron-Frobenius Theorem [3].

Theorem 4.3. *Let M be an irreducible non-negative $n \times n$ matrix with spectral radius $\rho(M) = r$. Then the number r is a positive real number and an eigenvalue of M, and M has a right eigenvector* $\mathbf{q}$ *associated with eigenvalue r whose components are all positive.*

Theorem 4.4. *Let G be a strongly connected directed graph with adjacency matrix A. If the spectral radius of A is at least 2, then the game of numbers diverges for G.*

Proof. Let $\rho(A) = r$ and let $\mathbf{q}$ be the right eigenvector of A, in accordance with the Perron-Frobenius Theorem (which we can use because the adjacency matrix of a strongly connected directed graph is an irreducible non-negative matrix). Hence, $A\mathbf{q} = r\mathbf{q}$, or equivalently, $\mathbf{q}^T A^T = r\mathbf{q}^T$. Recall that the transition matrices for the states of the game are given by

$$F_i = I + (A^T - 2I)\mathbf{e_i}\mathbf{e_i}^T.$$

Since $\mathbf{q}^T\mathbf{e_i} = q_i$, we get that

$$\begin{aligned}\mathbf{q}^T F_i &= \mathbf{q}^T + (\mathbf{q}^T A^T - 2\mathbf{q}^T I)\mathbf{e_i}\mathbf{e_i}^T \\ &= \mathbf{q}^T + (r-2)q_i\mathbf{e_i}^T.\end{aligned}$$

Since $\mathbf{e_i}^T\mathbf{x} = x_i$, we get that

$$\begin{aligned}\mathbf{q}^T F_i\mathbf{x} &= \mathbf{q}^T\mathbf{x} + (r-2)q_i\mathbf{e_i}^T\mathbf{x} \\ &= \mathbf{q}^T\mathbf{x} + (r-2)q_i x_i.\end{aligned}$$

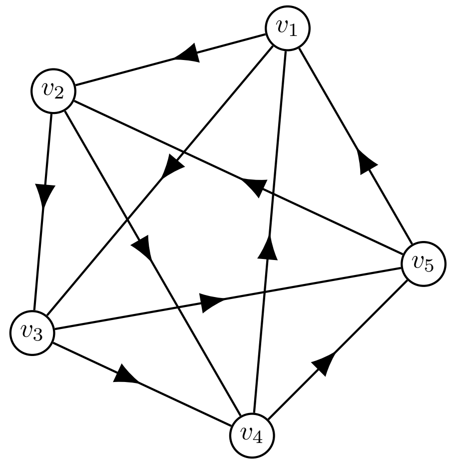
Note that $(r-2) \geqslant 0$ by assumption and $q_i > 0$, because all of the components of $\mathbf{q}$ are positive. Furthermore, $x_i < 0$ since otherwise, $F_i\mathbf{x}$ is not a valid game move. Hence, $(r-2)q_i x_i \leqslant 0$. Thus, for $r \geqslant 2$, we get that $\mathbf{q}^T F_i\mathbf{x} \leqslant \mathbf{q}^T\mathbf{x}$. Hence, if we pick any starting position $\mathbf{x}$ such that $\mathbf{q}^T\mathbf{x} < 0$, then since all the coordinates of $\mathbf{q}^T$ are positive, this game never terminates. Therefore, the game of numbers diverges on G. $\square$

We saw in Section 3 that the game of numbers converges for all directed cycles, and the adjacency matrix of all directed cycles have a spectral radius less than 2. Theorem 4.4 states that if a directed graph has spectral radius greater than or equal to 2, then it admits divergent games. This suggests that, just like in the undirected case, a spectral radius of 2 might be the cutoff that separates graphs on which all games converge from those that admit divergent games.

5. Looping Game of Numbers

Eriksson showed in [2] that in the undirected case, a looping game of numbers exists for G if and only if the spectral radius of its adjacency matrix is equal to 2. This appears to be true for the directed case.

Example 5.1. *The directed graph from Example 4.1 is*



Let the state of the vertices be given by $\mathbf{x} = [1, -1, -1, 1, 0]^T$*, where* x_i *corresponds to the state of vertex* v_i*. If we choose vertices in the order of* v_3*,* v_2*,* v_5*,* v_4*,* v_1*, then we get* $F_1F_4F_5F_2F_3\mathbf{x} = \mathbf{x}$*, as illustrated below.*

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ -\mathbf{1} \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ -\mathbf{1} \\ 1 \\ 0 \\ -1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ -\mathbf{1} \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\mathbf{1} \\ 1 \end{bmatrix} \to \begin{bmatrix} -\mathbf{1} \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

Thus, a looping game exists on this graph.

Theorem 4.4 shows that for any strongly connected directed graph G with corresponding adjacency matrix A, if $\rho(A) \neq 2$, then no looping game can exist on G. If $\rho(A) = 2$, then a looping game could exist, as shown by the following heuristic argument.

Recall that the transition matrices for the states of the game are given by

$$F_i = I + (A^T - 2I)\mathbf{e_i}\mathbf{e_i}^T .$$

It is easy to verify that 1 is an eigenvalue of each matrix F_i. If $\rho(A) = 2$, then it can be shown that 1 is an eigenvalue of every finite product of the matrices F_i. Let F be the product of the n matrices F_i in some arbitrary order. Then since 1 is an eigenvalue of F, there is a nonzero vector $\mathbf{v}$ such that $F\mathbf{v} = \mathbf{v}$. If $\mathbf{v}$ is a valid state, and starting at $\mathbf{v}$, if every state arising in the sequence of moves represented by the terms in the product for F is a valid move, then $\mathbf{v}$ will be the initial state of a looping game on the graph G. This leads to our first conjecture.

Conjecture 5.2. *A looping game exists for a directed, strongly connected graph G if and only if the spectral radius of the adjacency matrix of G is equal to 2.*

We have already proven the only if direction of this conjecture, but the if direction remains elusive.

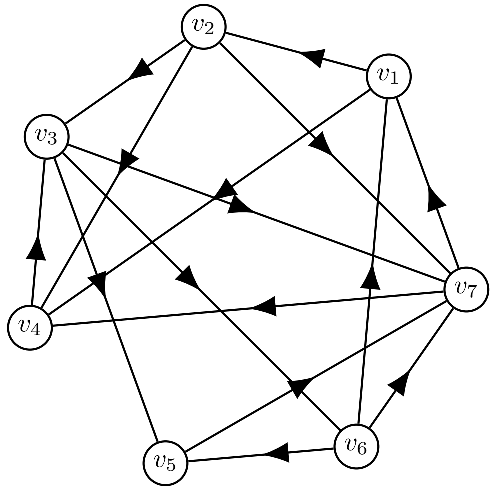
6. Convergent Game of Numbers

The previous section has shown that for any graph G with adjacency matrix A, if $\rho(A) \geqslant 2$, then the game of numbers diverges for G, and suggests that a looping game exists for G if and only if $\rho(A) = 2$. We have shown that the game of numbers on the directed cycle converge for all initial configurations of values on the vertices. Note that the spectral radius of a directed cycle is 1. This leads us to make the following conjecture.

Conjecture 6.1. *Let G be a strongly connected directed graph with adjacency matrix A. There exists a real number $c > 1$ such that the spectral radius of A is less than c if and only if the game of numbers converges for G.*

In the case of undirected graphs, the value of c is 2. Initially, it seems like this would be the case for directed graphs, but the following example shows a game on a graph with spectral radius less than 2 that appears to diverge.

Example 6.2. *Consider the following graph G with spectral radius approximately equal to 1.965.*



Let the state of the vertices be given by $\mathbf{x} = [-1, 0, 0, 0, 0, 0, 0]^T$. *This game appears to diverge, as illustrated below.*

$$\mathbf{x} = \begin{bmatrix} -\mathbf{1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ -\mathbf{1} \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ -\mathbf{1} \\ -2 \\ 0 \\ 0 \\ -1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ -2 \\ -1 \\ -\mathbf{1} \\ -2 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \\ -2 \\ 1 \\ -\mathbf{3} \end{bmatrix} \to \begin{bmatrix} -\mathbf{3} \\ 1 \\ 1 \\ -5 \\ -2 \\ 1 \\ 3 \end{bmatrix} \to \cdots.$$

We have found examples of (non-cycle) graphs with spectral radius less than 2 where the game of numbers converges. For example, the game of numbers converges for all strongly connected digraphs with n vertices and $n+1$ edges, and the game of numbers converges for any strong orientation on the graph K_4.

7. Conclusion

To sum up, the game of numbers has been completely resolved for the undirected case. The convergent games have been characterized, and it is known that if a game converges, then no matter how you play the game, it must converge to the same ending position vector in the same number of moves. Mozes used Weyl groups to prove these results, and Eriksson used elementary graph and matrix theory to prove these results. Most of their techniques do not extend to the directed case. We can show that the game of numbers diverges on graphs whose adjacency matrices have spectral radius greater than or equal to 2, and we have a conjecture about graphs with looping games and convergent games.

Future research plans for this topic include proving or disproving the two conjectures stated in the previous two sections, and looking into further generalizations.

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Department of Mathematics and Statistics, University of West Florida, Pensacola, FL 32514

E-mail address: rhemasin@uwf.edu

Department of Mathematics and Statistics, University of West Florida, Pensacola, FL 32514

E-mail address: adalal78@gmail.com

Department of Mathematics and Statistics, University of West Florida, Pensacola, FL 32514

E-mail address: dmcginn@uwf.edu