GENERATING STERN–BROCOT TYPE RATIONAL NUMBERS WITH MEDIANTS

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Abstract. The Stern–Brocot tree is a method of generating or organizing all fractions in the interval $(0, 1)$ by starting with the endpoints $\frac{0}{1}$ and $\frac{1}{1}$ and repeatedly applying the mediant operation: $m\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{a+c}{b+d}$. A recent paper of Aiylam considers two generalizations: one is to apply the mediant operation starting with an arbitrary interval $\left(\frac{a}{b}, \frac{c}{d}\right)$ (the fractions must be non-negative), and the other is to allow arbitrary reduction of generated fractions to lower terms. In the present paper, we give simpler proofs of some of Aiylam's results, and we give a simpler method of generating just the portion of the tree that leads to a given fraction.

1. Introduction

The Stern–Brocot tree is a method of generating all non-negative fractions reduced to lowest terms and without repetition (see for example [3, pp. 116–123]). The classical Stern–Brocot tree starts with the two fractions $\frac{0}{1}$ and $\frac{1}{0}$ (think of $\frac{1}{0}$ as $+\infty$) and repeatedly applies the mediant operation.

Definition 1.1. *If $0 \leq \frac{a}{b} < \frac{c}{d}$ are fractions which may or may not be reduced to lowest terms, the mediant of $\left(\frac{a}{b}, \frac{c}{d}\right)$ is denoted by $m\left(\frac{a}{b}, \frac{c}{d}\right)$ and is defined by $m\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{a+c}{b+d}$.*

The fractions can be organized in a binary tree, where each fraction has two offspring, both formed as a mediant of that fraction and the nearest fraction to its left or right.

The tree is in two symmetric halves because the first mediant is $\frac{0+1}{1+0} = \frac{1}{1}$, and as a fraction is generated on the left, its reciprocal is generated on the right. Therefore, we could also think of it as a way of generating all the fractions in the interval $(0, 1)$ starting with the fractions $\frac{0}{1}$ and $\frac{1}{1}$. The Farey series $\mathcal{F}_n$ is formed by pruning the tree to omit all fractions with denominators greater than n; see [3, pp. 118–119].

Historically, the Stern–Brocot tree arose as a method of approximating fractions with large denominators closely by fractions with much smaller denominators, in particular in the context of designing collections of gears for clocks; see for example [2].

A recent paper by Aiylam [1] considers the generalization of this process, starting with arbitrary non-negative fractions $0 \leq \frac{a}{b} < \frac{c}{d}$. One intriguing feature of the classical Stern–Brocot tree is that the mediants are always in lowest terms and do not need to be reduced. In the generalization, this is no longer true, even if the starting fractions are in lowest terms; see Example 4.4 below. Different trees are generated according to whether we reduce some or all of the fractions to lower or lowest terms. We distinguish two cases or methods: the "easy" case never reduces any mediants, and the "harder" case may reduce some mediants.

In this paper it does not matter whether the end points are reduced to lowest terms or not. However, for convenience only we will reduce $\frac{a}{b}, \frac{c}{d}$ to lowest terms.

In the "easy" first part of this paper (Sections 2–4), we agree not to reduce any of the mediants to lower terms, if the mediant is not already reduced to lowest terms. Then, in the "harder" part of the paper (Section 5), we generalize the theory so that the reader can decide for himself which mediants he wishes to reduce to lowest terms, and also decide if he wishes to partially or completely reduce any of the mediants to lowest terms. The second part is harder to prove.

Definition 1.2. *We call $\Delta\left(\frac{a}{b}, \frac{c}{d}\right) = bc - ad$ the delta-value or the Δ-value of the interval $\left(\frac{a}{b}, \frac{c}{d}\right)$.*

We note that if $\frac{a+c}{b+d}$ is not reduced to lowest terms then $\Delta\left(\frac{a}{b}, \frac{a+c}{b+d}\right) = \Delta\left(\frac{a+c}{b+d}, \frac{c}{d}\right) = \Delta\left(\frac{a}{b}, \frac{c}{d}\right) = bc - ad$. We see that if $\Delta\left(\frac{a}{b}, \frac{c}{d}\right) = bc - ad = 1$, then $\frac{a+c}{b+d}$ is reduced to lowest terms, since $b(a+c) - a(b+d) = bc - ad = 1$. Also, it is easy to show that $0 \leq \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. That is, $m\left(\frac{a}{b}, \frac{c}{d}\right) \in \left(\frac{a}{b}, \frac{c}{d}\right)$. In this paper, we show that the Stern–Brocot type fractions that are generated from $\left(\frac{a}{b}, \frac{c}{d}\right)$, uniquely compute all rational numbers $\frac{x}{y} \in \left(\frac{a}{b}, \frac{c}{d}\right)$, and this is true whether the mediants are reduced to lowest terms or not. When $\left(\frac{a}{b}, \frac{c}{d}\right) = \left(\frac{0}{1}, \frac{1}{1}\right)$, the mediants generate the Stern–Brocot fractions in $(0, 1)$.

1.1. **The plan.** Suppose $0 \leq \frac{a}{b} < \frac{c}{d}$ are fractions reduced to lowest terms, although this assumption is not needed. We first show that the Stern–Brocot mediants that are generated by $\frac{a}{b}, \frac{c}{d}$, uniquely compute all rational numbers $q \in \left[\frac{a}{b}, \frac{c}{d}\right]$ in the easy case, where all mediants are not reduced to any lower terms. We do this by reducing the problem to the known result when $\frac{a}{b} = \frac{0}{1}$, $\frac{c}{d} = \frac{1}{1}$. This is the result for the classical Stern–Brocot tree and is well-known; see for example [1, Theorem 1] or [3, pp. 117–118].

We also put an upper bound T on the number of steps it takes for the telescoping search path to reach any given $q \in \left[\frac{a}{b}, \frac{c}{d}\right]$ in this easy case. Of course, T depends on q. Using this easy case and the upper bound T for each $q \in \left[\frac{a}{b}, \frac{c}{d}\right]$ as our main machinery, we easily extend the easy case to the harder case, where the mediants can be reduced to lower terms in any random and arbitrary way, including partially reducing to lower terms.

The idea that we use is that, when we go down a telescoping search path to try to find a given $q \in \left[\frac{a}{b}, \frac{c}{d}\right]$, the mediants in the search path can only be reduced to lower terms at most $\overline{T}$ times, where $\overline{T}$ is a known upper bound. The search path can be as long as it takes to find q. When the number of steps between these mediant reductions is as big as T, we can use the easy case, with its upper bound T, to find the given $q \in \left[\frac{a}{b}, \frac{c}{d}\right]$. We will also give a crude fast algorithm at the end of Section 5, which will allow the reader to completely skip the detailed algorithm that follows Section 5, if he wishes.

Instead of using the mediants of $\left(\frac{a}{b}, \frac{c}{d}\right)$, we have developed an analogous theory in which we use $\frac{e}{f} \in \left(\frac{a}{b}, \frac{c}{d}\right)$ in place of $m\left(\frac{a}{b}, \frac{c}{d}\right)$, where $\frac{e}{f}$ is the unique fraction in $\left(\frac{a}{b}, \frac{c}{d}\right)$ that has the smallest possible size $e+f$. When $\Delta\left(\frac{a}{b}, \frac{c}{d}\right) = bc - ad = 1$, we can show that $\frac{e}{f} = m\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{a+c}{b+d}$ and, of course, the size of the $\frac{e}{f}$ equals $(a+c)+(b+d)$.

This statement has a very easy proof and it gives a trivial solution to the problem under discussion when $bc - ad = 1$. It also gives a very easy proof to Theorems 1 and 2 of [1]. This fact illustrates that the search algorithm given in [1] is very different from the one in this paper. In the last section, we use our machinery to classify $0 < \frac{a}{b} \leq \frac{m}{n} < \frac{x}{y} < \frac{c}{d}$, where $bc - ad = 1$ and $nx - my = 1$.

2. Modified Stern–Brocot Fractions

We first define the levels of the modified Stern–Brocot fractions, then prove some of their properties.

2.1. **Construction.** If $0 \leq \frac{a}{b} < \frac{c}{d}$ are fractions reduced to lowest terms, we define the various levels of the modified Stern–Brocot fractions as follows.

We illustrate the pattern first and then we formalize this pattern.

In Sections 2–4, we agree not to reduce any of the mediants to lowest terms. In other words, we are computing symbolically. Figure 1 shows a picture of the following.

We denote $L_0 = \left\{\frac{a}{b}, \frac{c}{d}\right\}$ and call L_0 level 0. L_1, level 1, consists of the single point $p_1 = m\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{a+c}{b+d}$. Note that $0 \leq \frac{a}{b} < p_1 < \frac{c}{d}$.

Using the ordering $\frac{a}{b} < p_1 < \frac{c}{d}$, we define level 2, denoted L_2, as follows. L_2 consists of the two points $p_2 = m\left(\frac{a}{b}, p_1\right) = m\left(\frac{a}{b}, \frac{a+c}{b+d}\right) = \frac{2a+c}{2b+d} \in \left(\frac{a}{b}, p_1\right)$

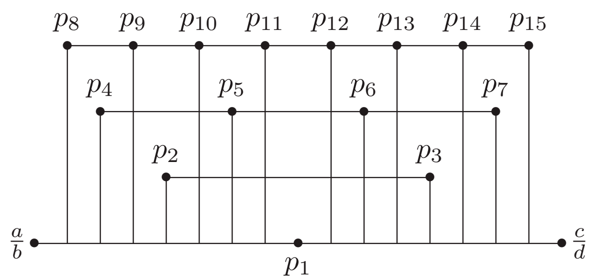

FIGURE 1. Levels of modified Stern–Brocot fractions.

and $p_3 = m\left(p_1, \frac{c}{d}\right) = m\left(\frac{a+c}{b+d}, \frac{c}{d}\right) = \frac{a+2c}{b+2d} \in \left(p_1, \frac{c}{d}\right)$. Note that $\frac{a}{b} < p_2 < p_1 < p_3 < \frac{c}{d}$.

Using the ordering $\frac{a}{b} < p_2 < p_1 < p_3 < \frac{c}{d}$, we define level 3 as follows. Level 3, denoted L_3, consists of the four points $p_4 = m\left(\frac{a}{b}, p_2\right) = m\left(\frac{a}{b}, \frac{2a+c}{2b+d}\right) = \frac{3a+c}{3b+d} \in \left(\frac{a}{b}, p_2\right)$ and $p_5 = m(p_2, p_1) = \frac{3a+2c}{3b+2d} \in (p_2, p_1)$ and $p_6 = m(p_1, p_3) = \frac{2a+3c}{2b+3d} \in (p_1, p_3)$ and $p_7 = m\left(p_3, \frac{c}{d}\right) = \frac{a+3c}{b+3d} \in \left(p_3, \frac{c}{d}\right)$. Note that $p_4 < p_5 < p_6 < p_7$ and $\frac{a}{b} < p_4 < p_2 < p_5 < p_1 < p_6 < p_3 < p_7 < \frac{c}{d}$. Using this ordering we define, in a similar way, the eight points $p_8 < p_9 < p_{10} < p_{11} < p_{12} < p_{13} < p_{14} < p_{15}$ of level 4.

We continue this construction pattern to create levels L_1, L_2, L_3, L_4, $L_5, \ldots$, where each level $i \geq 1$ has 2^{i-1} elements. We emphasize that in this section we are not reducing our fractions $p_1, p_2, p_3, \ldots$ to lowest terms. We now formalize this pattern.

First, from the construction pattern, we note that the level $i \geq 1$ points alternate with the points of $\cup_{l=0}^{i-1} L_l$. Indeed, the members of level L_i are the mediants of the pair of consecutive points of the ordered set $\cup_{l=0}^{i-1} L_l$. Also, by this definition of the construction pattern and by a simple induction, we can see that each $\frac{x}{y} \in L_i$, $i \geq 2$, is written as $\frac{x}{y} = m\left(\frac{\overline{x}}{\overline{y}}, \frac{x'}{y'}\right) = \frac{\overline{x}+x'}{\overline{y}+y'}$, where one of $\frac{\overline{x}}{\overline{y}}, \frac{x'}{y'}$ is a level $i-1$ point and the other $\frac{\overline{x}}{\overline{y}}, \frac{x'}{y'}$ lies in the set $\cup_{k=0}^{i-2} L_k$. This follows since the level $i-1$ points, $i \geq 2$, alternate with the points of $\cup_{k=0}^{i-2} L_k$. Figure 1 should make this clear.

As an example of the alternation, we note that the level 4 points are

$$p_8 < p_9 < p_{10} < p_{11} < p_{12} < p_{13} < p_{14} < p_{15}$$

and these eight points alternate with the members of

$$\cup_{k=0}^{k=3} L_k = \left\{\frac{a}{b} < p_4 < p_2 < p_5 < p_1 < p_6 < p_3 < p_7 < \frac{c}{d}\right\}$$

as follows:

$$\frac{a}{b} < p_8 < p_4 < p_9 < p_2 < p_{10} < p_5 < p_{11}$$
$$< p_1 < p_{12} < p_6 < p_{13} < p_3 < p_{14} < p_7 < p_{15} < \frac{c}{d}.$$

We now denote level $i \geq 0$ by L_i, and we denote $\cup_{k=0}^{i} L_k = \overline{L}_i$. Note that for $i \geq 1$, $\overline{L}_i = L_i \cup \left(\cup_{k=0}^{i-1} L_k\right) = L_i \cup \overline{L}_{i-1}$. Also, we denote $\cup_{k=0}^{\infty} L_k = \overline{L}_\infty$. It is easy to see that the members of $\overline{L}_\infty$ are all distinct since $m\left(\frac{x}{y}, \frac{\overline{x}}{\overline{y}}\right) \in \left(\frac{x}{y}, \frac{\overline{x}}{\overline{y}}\right)$ and from this we see that the members of $\overline{L}_\infty$ are all distinct since the members of each $L_i \cap \overline{L}_{i-1} = \phi$.

2.2. **Properties.**

Theorem 2.1. *Suppose*

$$\overline{L}_i = \bigcup_{k=0}^{i} L_k = \left\{\frac{a}{b} = \overline{p}_0 < \overline{p}_1 < \overline{p}_2 < \cdots \overline{p}_k = \frac{c}{d,}\right\}, \quad i \geq 0.$$

Then $\Delta\left(\overline{p}_t, \overline{p}_{t+1}\right) = \Delta = bc - ad$ *for all pairs* $\overline{p}_t$, $\overline{p}_{t+1}$ *of consecutive members of the ordered set* $\overline{L}_i$.

Proof. We can assume by induction that if

$$\overline{L}_{i-1} = \left\{\frac{a}{b} = \overline{p}'_0 < \overline{p}'_1 < \overline{p}'_2 < \cdots < \overline{p}'_s = \frac{c}{d}\right\}$$

then $\Delta\left(\overline{p}'_t, \overline{p}'_{t+1}\right) = \Delta = bc - ad$. First, note from the above that $\overline{L}_i = L_i \cup \overline{L}$. Now, the set L_i consists of the set of mediants $m\left(\overline{p}'_t, \overline{p}'_{t+1}\right)$, where $\overline{p}'_t$, $\overline{p}'_{t+1}$ range over the consecutive members of $\overline{L}_{i-1}$. Now, if $\Delta\left(\overline{p}'_t, \overline{p}'_{t+1}\right) = \Delta = bc - ad$, then we know that $\Delta\left(\overline{p}'_t, m\left(\overline{p}'_t, \overline{p}'_{t+1}\right)\right) = \Delta\left(m\left(\overline{p}'_t, \overline{p}'_{t+1}\right), \overline{p}'_{t+1}\right) = \Delta\left(\overline{p}'_t, \overline{p}'_{t+1}\right) = \Delta = bc - ad$. This is because $\Delta\left(\frac{x}{y}, \frac{\overline{x}}{\overline{y}}\right) = \Delta\left(\frac{x}{y}, \frac{x+\overline{x}}{y+\overline{y}}\right) = \Delta\left(\frac{x+\overline{x}}{y+\overline{y}}, \frac{\overline{x}}{\overline{y}}\right) = y\overline{x} - x\overline{y}$. Now,

$$\overline{L}_i = L_i \cup \overline{L}_{i-1} = \left\{\frac{a}{b} = \overline{p}_0 < \overline{p}_1 < \overline{p}_2 < \cdots < \overline{p}_k = \frac{c}{d}\right\}.$$

Therefore, $\Delta\left(\overline{p}_t, \overline{p}_{t+1}\right) = \Delta = bc - ad$. Therefore, if $\overline{p}_t = \frac{x}{y}$, $\overline{p}_{t+1} = \frac{\overline{x}}{\overline{y}}$, then $\Delta\left(\frac{x}{y}, \frac{\overline{x}}{\overline{y}}\right) = \overline{x}y - x\overline{y} = \Delta = bc - ad$ and this implies $\gcd(x, y) \mid \Delta$ and $\gcd(x, y) \leq \Delta = bc - ad$. We state this again in Section 4. Of course, gcd means the greatest common divisor. □

In Sections 2–4, we are computing symbolically and not reducing the mediants to lowest terms, so it is obvious that, for $i \geq 1$ and i arbitrary, that each $\overline{p}_t \in \overline{L}_i \setminus \left\{\frac{a}{b}, \frac{c}{d}\right\}$ satisfies $\overline{p}_t = \frac{\phi a + \theta c}{\phi b + \theta d}$, where $\phi, \theta \in \{1, 2, 3, \ldots\}$.

Also, $\frac{a}{b} = \frac{1\cdot a+0\cdot c}{1\cdot b+0\cdot d}$ and $\frac{c}{d} = \frac{0\cdot a+1\cdot c}{0\cdot b+1\cdot d}$.

Theorem 2.2. *We have $\overline{p}_t = \frac{\phi a+\theta c}{\phi b+\theta d}$, where (ϕ, θ) must also be relatively prime.*

Proof. To see this, let $\overline{p}_t = \frac{\phi a+\theta c}{\phi b+\theta d}$ and $\overline{p}_{t+1} = \frac{\overline{\phi} a+\overline{\theta} c}{\overline{\phi} b+\overline{\theta} d}$, where $\overline{p}_t < \overline{p}_{t+1}$ are consecutive members of the ordered set $\overline{L}_i$ and where $\phi, \theta, \overline{\phi}, \overline{\theta} \in \{1, 2, 3, \ldots\}$. We know that $\Delta\left(\overline{p}_t, \overline{p}_{t+1}\right) = \Delta = bc - ad$.

By a calculation, $\Delta\left(\overline{p}_t, \overline{p}_{t+1}\right) = \left(\phi\overline{\theta} - \theta\overline{\phi}\right)(bc - ad) = bc - ad$, since $\Delta\left(\overline{p}_t, \overline{p}_{t+1}\right) = bc - ad$. Therefore, $\phi\overline{\theta} - \theta\overline{\phi} = 1$, which implies that both pairs (ϕ, θ) and $\left(\overline{\phi}, \overline{\theta}\right)$ are relatively prime. □

By the same calculations, it is easy to show that if $\phi, \theta, \overline{\phi}, \overline{\theta} \in \{1, 2, 3, \ldots\}$ and both of (ϕ, θ) and $\left(\overline{\phi}, \overline{\theta}\right)$ are relatively prime, then $\frac{\phi a+\theta c}{\phi b+\theta d} = \frac{\overline{\phi} a+\overline{\theta} c}{\overline{\phi} b+\overline{\theta} d}$ is true, if and only if $(\phi, \theta) = \left(\overline{\phi}, \overline{\theta}\right)$. This is because $\phi\overline{\theta} - \theta\overline{\phi} = 0$, if and only if $\frac{\theta}{\phi} = \frac{\overline{\theta}}{\overline{\phi}}$, which is true, if and only if $(\phi, \theta) = \left(\overline{\phi}, \overline{\theta}\right)$.

Since the members of $\overline{L}_\infty = \cup_{k=0}^{\infty} L_k$ are distinct, we see that if $\frac{\phi a+\theta c}{\phi b+\theta d} \neq \frac{\overline{\phi} a+\overline{\theta} c}{\overline{\phi} b+\overline{\theta} d}$ are two different members of $\overline{L}_\infty$, then $(\phi, \theta) \neq \left(\overline{\phi}, \overline{\theta}\right)$. In other words, if $\frac{\phi a+\theta c}{\phi b+\theta d} \in \overline{L}_\infty$, then the ordered pair (ϕ, θ), $\phi, \theta \in \{1, 2, 3, \ldots\}$, (ϕ, θ) are relatively prime, and can appear at most one time in $\overline{L}_\infty$. Since $0 < bc - ad$, we see that $\frac{\phi a+\theta c}{\phi b+\theta d} = \frac{\overline{\phi} a+\overline{\theta} c}{\overline{\phi} b+\overline{\theta} d}$ is true, if and only if $0 < \phi\overline{\theta} - \theta\overline{\phi}$ which is true, if and only if $\frac{\theta}{\phi} < \frac{\overline{\theta}}{\overline{\phi}}$.

Theorem 2.3. *We have*

$$\overline{L}_\infty \backslash \left\{\frac{a}{b}, \frac{c}{d}\right\} = \left\{\frac{\phi a + \theta c}{\phi b + \theta d} : \phi,\ \theta \in \{1, 2, 3, \ldots\}, \phi, \theta \text{ are relatively prime}\right\}.$$

In other words, $\overline{L}_\infty \backslash \left\{\frac{a}{b}, \frac{c}{d}\right\}$ picks up all ordered pairs (ϕ, θ), where ϕ, θ are relatively prime and $\phi, \theta \in \{1, 2, 3, \ldots\}$.

To see this on $[0, 1]$, let $\left(\frac{a}{b}, \frac{c}{d}\right) = \left(\frac{0}{1}, \frac{1}{1}\right)$. Note that $\Delta\left(\frac{0}{1}, \frac{1}{1}\right) = 1 \cdot 1 - 0 \cdot 1 = 1$. From the theory of Stern–Brocot fractions, we know that when $\left(\frac{a}{b}, \frac{c}{d}\right) = \left(\frac{0}{1}, \frac{1}{1}\right)$, then

$$\begin{aligned} &\overline{L}_\infty \backslash \left\{\frac{0}{1}, \frac{1}{1}\right\} \\ &= \{q : 0 < q < 1,\ q \text{ is rational and } q \text{ is reduced to lowest terms}\}. \end{aligned}$$

The comment at the end of Section 1 shows how we proved this. Also, see Theorem 1, [1].

Since $\Delta\left(\frac{0}{1},\frac{1}{1}\right)=1$, all mediants of $\overline{L}_\infty\setminus\left\{\frac{0}{1},\frac{1}{1}\right\}$ are automatically reduced to lowest terms. Note that $\overline{L}_\infty\setminus\left\{\frac{0}{1},\frac{1}{1}\right\}$ picks up all rational numbers $q=\frac{x}{y}\in\left(\frac{0}{1},\frac{1}{1}\right)$, where $\frac{x}{y}$ is reduced to lowest terms.

Now when $\left(\frac{a}{b},\frac{c}{d}\right)=\left(\frac{0}{1},\frac{1}{1}\right)$, we see that $\frac{\phi a+\theta c}{\phi b+\theta d}=\frac{\phi\cdot 0+\theta\cdot 1}{\phi\cdot 1+\theta\cdot 1}=\frac{\theta}{\phi+\theta}$, where $\phi,\theta\in\{1,2,3,\ldots\}$ and (ϕ,θ) are relatively prime and $\frac{\theta}{\phi+\theta}$ is automatically reduced to lowest terms.

Now each rational number $\frac{x}{y}\in(0,1)$, $0<x<y$, where $\frac{x}{y}$ is reduced to lowest terms, can be uniquely represented by $\frac{x}{y}=\frac{\theta}{\phi+\theta}$, where $\theta=x$, $\phi=y-x$, and $\theta,\phi\in\{1,2,3,\ldots\}$ and (θ,ϕ) are relatively prime.

Conversely, if $\theta,\phi\in\{1,2,3,\ldots\}$ are arbitrary and (θ,ϕ) are relatively prime, then $\frac{x}{y}=\frac{\theta}{\phi+\theta}\in(0,1)$ and $\frac{\theta}{\phi+\theta}$ is reduced to lowest terms. Since $\overline{L}_\infty\setminus\left\{\frac{0}{1},\frac{1}{1}\right\}$ picks up all rational numbers $\frac{x}{y}$, $0<x<y$, where $\frac{x}{y}$ is reduced to lowest terms, it is easy to see that

$$\overline{L}_\infty\setminus\left\{\frac{0}{1},\frac{1}{1}\right\}=\left\{\frac{\theta}{\phi+\theta}:\theta,\phi\in\{1,2,3,\ldots\},\theta,\phi\text{ are relatively prime}\right\}.$$

In other words, $\overline{L}_\infty\setminus\left\{\frac{0}{1},\frac{1}{1}\right\}$ picks up all $\frac{\theta}{\phi+\theta}$, where $\theta,\phi\in\{1,2,3,\ldots\}$ and (θ,ϕ) are relatively prime.

Proof of Theorem 2.3. Using $\left(\frac{a}{b},\frac{c}{d}\right)$ in the place of $\left(\frac{a}{b},\frac{c}{d}\right)=\left(\frac{0}{1},\frac{1}{1}\right)$, it is easy to see that $\frac{\theta}{\phi+\theta}$ becomes $\frac{\phi a+\theta c}{\phi b+\theta d}$ so that $\overline{L}_\infty\setminus\left\{\frac{0}{1},\frac{1}{1}\right\}$ becomes

$$\overline{L}_\infty\setminus\left\{\frac{a}{b},\frac{c}{d}\right\}=\left\{\frac{\phi a+\theta c}{\phi b+\theta d}:\theta,\phi\in\{1,2,3,\ldots\},\theta,\phi\text{ are relatively prime}\right\}.$$

In other words, the symbolic calculation of $\overline{L}_\infty\setminus\left\{\frac{a}{b},\frac{c}{d}\right\}$ and $\overline{L}_\infty\setminus\left\{\frac{0}{1},\frac{1}{1}\right\}$ are exactly the same.

We now give a second proof of this. We show that all fractions $\frac{\theta a+\phi c}{\theta b+\phi d}\in\overline{L}_\infty\left(\frac{a}{b},\frac{c}{d}\right)$, when $gcd(\theta,\phi)=1$. The proof is by mathematical induction on $\theta+\phi$. It is obvious for $n=1$. Therefore, suppose it is true for $\theta+\phi\le n-1$. We show that it is true for $\theta+\phi=n$. Consider the two subtrees $\left(\frac{a}{b},\frac{a+c}{b+d}\right)$ and $\left(\frac{a+c}{b+d},\frac{c}{d}\right)$. By symmetry, we may suppose that $\theta\ge\phi$. Then

$$\frac{\theta a+\phi c}{\theta b+\phi d}=\frac{(\theta-\phi)a+\phi(a+c)}{(\theta-\phi)b+\phi(b+d)},$$

where $\gcd(\theta-\phi,\phi)=1$. Now $(\theta-\phi)+\phi=\theta\le n-1$. Therefore, by the induction hypothesis,

$$\frac{\theta-\phi)a+\phi(a+c)}{(\theta-\phi)b+\phi(b+d)}\in\overline{L}_\infty\left(\frac{a}{b},\frac{a+c}{b+d}\right).$$

Therefore,

$$\frac{\theta a + \phi c}{\theta b + \phi d} \in \overline{L}_\infty \left(\frac{a}{b}, \frac{c}{d}\right).$$

$\square$

Main Theorem. Suppose $0 \leq \frac{a}{b} < \frac{c}{d}$, where $\frac{a}{b}, \frac{c}{d}$ are reduced to lowest terms and $\frac{x}{y} \in \left(\frac{a}{b}, \frac{c}{d}\right)$ is an arbitrary rational number in $\left(\frac{a}{b}, \frac{c}{d}\right)$ reduced to lowest terms. Then $\frac{x}{y} \in \overline{L}_\infty \setminus \left\{\frac{a}{b}, \frac{c}{d}\right\}$.

Note 2.4. Recall that the members of $\overline{L}_\infty \setminus \left\{\frac{a}{b}, \frac{c}{d}\right\}$ are not being reduced to lowest terms. However, when we say that $\frac{x}{y} \in \overline{L}_\infty \setminus \left\{\frac{a}{b}, \frac{c}{d}\right\}$, where $\frac{x}{y}$ is reduced to lowest terms, we mean that $\frac{x}{y} = \frac{m}{n}$, where $\frac{m}{n} \in \overline{L}_\infty \setminus \left\{\frac{a}{b}, \frac{c}{d}\right\}$ and $\frac{x}{y} = \frac{m}{n}$ after $\frac{m}{n}$ is reduced to lowest terms.

Proof of Main Theorem. Let $\frac{x}{y} = \frac{\phi a + \theta c}{\phi b + \theta d}$, where we wish to compute $\theta, \phi \in \{1, 2, 3, \ldots\}$ and (θ, ϕ) are relatively prime.

Now $\frac{x}{y} = \frac{\phi a + \theta c}{\phi b + \theta d}$ is true, if and only if $\phi bx + \theta dx = \phi ay + \theta cy$. This is true, if and only if $\phi\,(bx - ay) = \theta\,(cy - dx)$.

Therefore, $\frac{\phi}{\theta} = \frac{cy - dx}{bx - ay}$, where $cy - dx > 0$, $bx - ay > 0$, since $\frac{a}{b} < \frac{x}{y} < \frac{c}{d}$. We now agree to reduce $\frac{\phi}{\theta}$ to lowest terms. $\square$

Our proof of the Main Theorem is almost exactly the same as the proof of Theorem 2 [1], although the latter theorem states $\Delta = 1$ as a hypothesis. This means that the proof of Theorem 2 [1] can easily prove far more than what Theorem 2 [1] actually states. Also, once Theorem 2 [1] has easily been enhanced, then Theorem 7 [1] (which is our Main Theorem) can easily be proved in a crude way by using our Sections 3 and 4. See the crude fast algorithm at the end of Section 5 to see how we do this.

3. Some Useful Lemmas

Lemma 3.1. *Suppose $0 \leq \frac{a}{b} < \frac{c}{d}$ are fractions reduced to lowest terms. Let $\frac{x}{y} \in L_n$, where L_n is the level n of the modified Stern–Brocot fractions and $n \geq 1$. As always, $\frac{x}{y}$ is not reduced to lowest terms. Then $y \geq n$.*

Proof. We prove this by induction on n. Now $L_1 = \left\{\frac{a+c}{b+d}\right\}$ and $b + d \geq 2 > 1$. As stated previously in Section 2, each $\frac{x}{y} \in L_n$, $n \geq 2$, can be written as $\frac{x}{y} = m\left(\frac{\overline{x}}{\overline{y}}, \frac{x'}{y'}\right) = \frac{\overline{x} + x'}{\overline{y} + y'}$, where one of $\frac{\overline{x}}{\overline{y}}, \frac{x'}{y'}$ is a member. L_{n-1} and the other $\frac{\overline{x}}{\overline{y}}, \frac{x'}{y'}$ is a member of $\overline{L}_{n-2}$. By symmetry, assume that $\frac{\overline{x}}{\overline{y}} \in L_{n-1}$, $\frac{x'}{y'} \in \overline{L}_{n-2}$.

By induction, $\overline{y} \geq n - 1$. Also, $y' \geq 1$. Therefore, $y = \overline{y} + y' \geq \overline{y} + 1 \geq (n-1) + 1 = n$. $\square$

Lemma 3.2. *Using the hypothesis of Lemma 3.1, suppose* $\frac{x}{y} \in L_n$, $n \geq 1$, *where, as always,* $\frac{x}{y}$ *is not reduced to lowest terms. Then* $\gcd(x,y)\,|\Delta$ *and* $\gcd(x,y) \leq \Delta$, *where* $\Delta = bc - ad$. *Since* $n \geq 1$ *is arbitrary, this implies that each* $\frac{x}{y} \in \overline{L}_\infty \setminus \left\{\frac{a}{b}, \frac{c}{d}\right\}$ *satisfies* $\gcd(x,y)\,|\Delta$ *and* $\gcd(x,y) \leq \Delta$.

Proof. The proof was given in Section 2 and used the fact that if $n \geq 1$ and $\overline{L}_n = \left\{\frac{a}{b} = \overline{p}_0 < \overline{p}_1 < \overline{p}_2 < \cdots < \overline{p}_k = \frac{c}{d}\right\}$, then $\Delta\left(\overline{p}_i, \overline{p}_{i+1}\right) = \Delta = bc - ad$. Also, $\frac{x}{y} \in L_n \subseteq \overline{L}_n$. Let $\frac{x}{y} = p_i$, $\frac{\overline{x}}{\overline{y}} = \overline{p}_{i+1}$. Then, $\Delta\left(\overline{p}_i, \overline{p}_{i+1}\right) = \Delta\left(\frac{x}{y}, \frac{\overline{x}}{\overline{y}}\right) = \overline{x}y - x\overline{y} = \Delta = bc - ad$. Therefore, $\gcd(x,y)\,|\Delta$ and $\gcd(x,y) \leq \Delta$. □

Lemma 3.3. *Suppose* $\frac{x}{y} \in \left(\frac{a}{b}, \frac{c}{d}\right)$, *where* $0 \leq \frac{a}{b} < \frac{c}{d}$ *are reduced to lowest terms and where* $\frac{x}{y}$ *is a fraction that is now reduced to lowest terms.*

Then $\frac{x}{y} \in L_1 \cup L_2 \cup L_3 \cup \cdots \cup L_{y\Delta}$, *where* $\Delta = bc - ad$. *We say that* $\frac{x}{y} \in L_1 \cup L_2 \cup L_3 \cup \cdots \cup L_{y\Delta}$ *by the meaning of Note 2.4, since* $\frac{x}{y}$ *is reduced to lowest terms and the members of* $L_1 \cup L_2 \cup \cdots \cup L_{y\Delta}$ *are not reduced to lowest terms.*

Proof. By the Main Theorem, we know that $\frac{x}{y} \in \cup_{i=1}^{\infty} L_i = \overline{L}_\infty \setminus \left\{\frac{a}{b}, \frac{c}{d}\right\}$ (by the meaning of Note 2.4).

Suppose $n \geq \Delta y + 1$. We show that $\frac{x}{y} \notin L_n$. Let $\frac{\overline{x}}{\overline{y}} \notin L_n$, where $n \geq \Delta y + 1$, and $\frac{\overline{x}}{\overline{y}}$ is arbitrary and where, as always, the members $\frac{\overline{x}}{\overline{y}}$ of L_n have not been reduce to lowest terms. We show that $\frac{x}{y} \neq \frac{\overline{x}}{\overline{y}}$ by Note 2.4. Now by Lemma 3.1, $\overline{y} \geq n \geq \Delta y + 1$. Also, by Lemma 3.2, $\gcd(\overline{x}, \overline{y}) \leq \Delta$. Therefore, when $\frac{\overline{x}}{\overline{y}}$ is reduced to lowest terms, we have $\frac{\overline{x}}{\overline{y}} = \left(\frac{\overline{x}}{\gcd(\overline{x},\overline{y})}\right) / \left(\frac{\overline{y}}{\gcd(\overline{x},\overline{y})}\right)$ and $\frac{\overline{y}}{\gcd(\overline{x},\overline{y})} \geq \frac{\overline{y}}{\Delta} \geq \frac{\Delta y+1}{\Delta} = y + \frac{1}{\Delta} > y$ (where $\frac{x}{y}$ is reduced to lowest terms). That is, $\frac{\overline{y}}{\gcd(\overline{x},\overline{y})} > y$. Therefore, if $\frac{\overline{x}}{\overline{y}} \in L_n$ and $n \geq \Delta y + 1$, then $\frac{\overline{x}}{\overline{y}} \neq \frac{x}{y}$. This implies $\frac{x}{y} \in L_n$, when $n \geq \Delta y + 1$. Therefore, $\frac{x}{y} \in L_1 \cup L_2 \cup \cdots \cup L_{y\Delta}$. □

Application 3.4. Suppose $\frac{x}{y} \in \left(\frac{a}{b}, \frac{c}{d}\right)$, where $0 \leq \frac{a}{b} < \frac{c}{d}$ are reduced to lowest terms and $\frac{x}{y}$ is a fraction that is reduced to lowest terms. Also, suppose $\Delta = bc - ad = 1$. Then $\frac{x}{y} \in L_1 \cup L_2 \cup \cdots \cup L_y$, since $\Delta y = y$. Thus, all fractions $\frac{x}{y} \in \left(\frac{a}{b}, \frac{c}{d}\right)$, where $\frac{x}{y}$ is reduced to lowest terms and $y \leq n$ are picked up in $L_1 \cup L_2 \cup \cdots \cup L_n$. This is very important in the theory of Stern–Brocot fractions where $\left(\frac{a}{b}, \frac{c}{d}\right) = \left(\frac{0}{1}, \frac{1}{1}\right)$ and $\Delta = 1 \cdot 1 - 0 \cdot 1 = 1$.

4. An Algorithm for Computing $\frac{x}{y} \in \left(\frac{a}{b}, \frac{c}{d}\right)$ as a Member of $\overline{L}_\infty \setminus \left\{\frac{a}{b}, \frac{c}{d}\right\}$

We now define telescoping sequences.

Let $0 \leq \frac{a}{b} < \frac{c}{d}$, where $\frac{a}{b}, \frac{c}{d}$ are reduced to lowest terms. The reader needs to draw a picture of the following.

Definition 4.1. *A telescoping sequence is a sequence $x_0, \overline{x}_0, x_1, x_2, x_3, \ldots$, where $x_0 = \frac{a}{b}$, $\overline{x}_0 = \frac{c}{d}$, $x_1 = m(x_0, \overline{x}_0) = p_1 = \frac{a+c}{b+d}$. Also, $x_2 = m(x_0, x_1)$ or $x_2 = m(x_1, \overline{x}_0)$, where we make a binary choice for x_2.*

In general, if $x_i = m(x, y)$, $x < y$, then $x_{i+1} = m(x, x_i)$ or $x_{i+1} = m(x_i, y)$. Note again that we make a binary choice for each of $x_2, x_3, x_4, \ldots$.

Also, we know that $x_0 = \frac{a}{b}$ and $\overline{x}_0 = \frac{c}{d}$ are on level L_0. Also, $x_1 = p_1 = \frac{a+c}{b+d}$ is on level L_1. Also, x_2 is on level L_2. Also, x_3 is on level L_3. We now show that each x_i, $i \geq 1$, is on level L_i.

Indeed, the following additional facts are easy to prove by definition and by induction. We simply observe that the simple pattern that we now give repeats itself over and over as we go higher. (The reader may need to review the definitions of L_i, $\overline{L}_i$, $i \geq 1$). The pattern is the following. If $x_{i+1} = m(x, y)$, $x < y$, then $x_{i+1} \in L_{i+1}$ (that is, x_{i+1} is on level L_{i+1}). Also, $x, y \in \overline{L}_i$ and x and y are consecutive members of the ordered set $\overline{L}_i$. Also, $x < x_{i+1} < y$ are consecutive members of the ordered set $\overline{L}_{i+1}$, where $\overline{L}_{i+1} = L_{i+1} \cup \overline{L}_i$. Now, $x_{i+2} = m(x, x_{i+1})$ or $x_{i+2} = m(x_{i+1}, y)$. In either case, $x_{i+2} \in L_{i+2}$. If $x_{i+2} = m(x, x_{i+1})$, then $x < x_{i+2} < x_{i+1}$ are consecutive members of the ordered set $\overline{L}_{i+2}$, where $\overline{L}_{i+2} = L_{i+2} \cup \overline{L}_{i+1}$ and if $x_{i+2} = m(x_{i+1}, y)$, then $x_{i+1} < x_{i+2} < y$ are consecutive members of the ordered set $\overline{L}_{i+2}$.

This pattern repeats itself as we go higher.

In conclusion, we also note that each x_i is automatically associated with an interval (x, y), where $x_i = m(x, y)$, $x_i \in (x, y)$. Also, if $x_{i+1} = m(x, x_i)$, we associate x_{i+1} with the interval (x, x_i) and if $x_{i+1} = m(x_i, y)$, we associate x_{i+1} with the interval (x_i, y).

Problem 4.2. Suppose $0 \leq \frac{a}{b} < \frac{c}{d}$ are given fractions reduced to lowest terms and $\frac{m}{n} \in \left(\frac{a}{b}, \frac{c}{d}\right)$ is a given rational number reduced to lowest terms. Show how to calculate, step-by-step, the construction of $\frac{m}{n}$ as a member of $\overline{L}_\infty \setminus \left\{\frac{a}{b}, \frac{c}{d}\right\}$. Of course, we say that $\frac{m}{n} \in \overline{L}_\infty \setminus \left\{\frac{a}{b}, \frac{c}{d}\right\}$ by the meaning of Note 2.4 of Section 2.

Solution. Of course, by the Main Theorem, we know that $\frac{m}{n} \in \overline{L}_\infty \setminus \left\{\frac{a}{b}, \frac{c}{d}\right\}$.

We now calculate a telescoping sequence

$$x_0 = \frac{a}{b},\ \overline{x}_0 = \frac{c}{d},\ x_1 = m(x_0, \overline{x}_0),\ x_2,\ x_3,\ \ldots,\ x_{t-1},\ x_t,$$

such that, for each x_i, if $x_i = m(x, y)$, $x < y$ (which means that x_i is associated with (x, y)), then $\frac{m}{n} \in (x, y)$.

If $\frac{m}{n} = x_i$, then, of course, we are done. If $\frac{m}{n} \neq x_i$, then we define $x_{i+1} = m(x, x_i)$, if $\frac{m}{n} \in (x, x_i)$ and $x_{i+1} = m(x_i, y)$, if $\frac{m}{n} \in (x_i, y)$.

Likewise, if $\frac{m}{n} = x_{i+1}$, then we are done. If $\frac{m}{n} \neq x_{i+1}$, we then calculate x_{i+2} the same way that we calculated x_{i+1}. This process will calculate a telescoping sequence $x_0, \overline{x}_0, x_1, x_2, \ldots, x_{t-1}, x_t$, where each x_i is on level L_i and, by Lemma 3.3 of Section 3, we know that $\frac{m}{n} = x_t$ for some t, where $1 \leq t \leq \Delta n$, $\Delta = bc - ad$. This is because $\frac{m}{n} \in L_1 \cup L_2 \cup \cdots \cup L_{n\Delta}$, $\Delta = bc - ad$.

Note 4.3. Of course, usually we will reach $\frac{m}{n} = x_t$ much sooner than $t = \Delta n$, since the denominators y of most members $\frac{x}{y} \in L_n$ are much bigger than $y = n$.

Example 4.4. Calculate $\frac{15}{31} \in \left(\frac{1}{3}, \frac{3}{5}\right)$ as a member of $\overline{L}_\infty \setminus \left\{\frac{1}{3}, \frac{3}{5}\right\}$.

Solution. $x_0 = \frac{1}{3}$, $\overline{x}_0 = \frac{3}{5}$, $x_1 = m\left(\frac{1}{3}, \frac{3}{5}\right) = \frac{4}{8}$. Note that we do not reduce $\frac{4}{8}$ to lowest terms.

$$
\begin{aligned}
&\tfrac{15}{31} \in \left(\tfrac{1}{3}, \tfrac{3}{5}\right), && x_1 = m\left(\tfrac{1}{3}, \tfrac{3}{5}\right) = \tfrac{4}{8},\\
&\tfrac{15}{31} \in \left(\tfrac{1}{3}, \tfrac{4}{8}\right), && x_2 = m\left(\tfrac{1}{3}, \tfrac{4}{8}\right) = \tfrac{5}{11},\\
&\tfrac{15}{31} \in \left(\tfrac{5}{11}, \tfrac{4}{8}\right), && x_3 = m\left(\tfrac{5}{11}, \tfrac{4}{8}\right) = \tfrac{9}{19},\\
&\tfrac{15}{31} \in \left(\tfrac{9}{19}, \tfrac{4}{8}\right), && x_4 = m\left(\tfrac{9}{19}, \tfrac{4}{8}\right) = \tfrac{13}{27},\\
&\tfrac{15}{31} \in \left(\tfrac{13}{27}, \tfrac{4}{8}\right), && x_5 = m\left(\tfrac{13}{27}, \tfrac{4}{8}\right) = \tfrac{17}{35},\\
&\tfrac{15}{31} \in \left(\tfrac{13}{27}, \tfrac{17}{35}\right), && x_6 = m\left(\tfrac{13}{27}, \tfrac{17}{35}\right) = \tfrac{30}{62} = \tfrac{15}{31}.
\end{aligned}
$$

Note that in the last step $\frac{30}{62} = \frac{15}{31}$, we must reduce $\frac{30}{62}$ to lowest terms.

5. A Crude Fast Algorithm

Suppose that when we carry out the above telescoping search algorithm, we agree to reduce the search mediants to lower terms in any possible way that we choose, including always reducing or never reducing or partially reducing. We can use induction on $\Delta = bc - ad$ to show that the search algorithm will still pick up the rational number $q = \frac{x}{y}$ in $\left[\frac{a}{b}, \frac{c}{d}\right]$ for which we are searching. If $\Delta = 1$, then the result is obvious, since all mediants are already reduced to lowest terms. If we never reduce our search mediants to lower terms, then we know that we will find $\frac{x}{y}$. If we reduce a search mediant to lower terms before we find $\frac{x}{y}$, the induction on Δ will show that we must eventually find $\frac{x}{y}$. Note that if $\Delta\left(q, \overline{q}\right) = \Delta$ and $\overline{q}$ is reduced to q', then $\Delta\left(q, q'\right) < \Delta\left(q, \overline{q}'\right)$. If we find $\frac{x}{y}$ before we reduce any search mediants to lower terms, there is nothing to prove since we have found $\frac{x}{y}$.

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