ARBITRARILY HIGH HAUSDORFF DIMENSIONS OF CONTINUA

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Abstract. It is well-known that Hausdorff dimension is not a topological invariant; that is, that two homeomorphic continua can have different Hausdorff dimension, although their topological dimension will be equal. We show that it is possible to take any continuum embeddable in $\mathbb{R}^n$ and embed it in such a way that its Hausdorff dimension is n. In doing so, we can obtain an arbitrarily high Hausdorff dimension for any nondegenerate continuum. As an example, we will give different embeddings of an arc whose Hausdorff dimension is any real number between 1 and ∞, including an arc of infinite Hausdorff dimension.

1. Background

A continuum is a compact, connected, nonempty metric space. The topological dimension of a continuum is a nonnegative integer and is preserved under homeomorphisms. An arc is a topological space that is homeomorphic to the unit interval $[0, 1]$. The topological dimension of an arc is 1.

Often, the topological dimension of a space is not very useful when considering measure. For example, the Sierpinski carpet has topological dimension 1, but its standard embedding has infinite perimeter and zero area. As such, it can be useful to consider the Hausdorff dimension $\dim_H(F)$ of a space F when attempting to measure it.

Definition 1.1. *For any $F \subset \mathbb{R}^n$, $s \geq 0$, and $\delta > 0$, define $\mathcal{H}^s_\delta(F)$ to be the infimum of all possible $\displaystyle\sum_{i=1}^{\infty} |U_i|^s$, where each $\{U_i\}$ is a cover of F, whose elements each have diameter less than δ. Define the s-dimensional Hausdorff measure of F, $\mathcal{H}^s(F)$, to be the limit of $\mathcal{H}^s_\delta(F)$ as δ approaches 0. Define $\dim_H(F)$ to be the infimum of all $s \geq 0$ such that $\mathcal{H}^s(F) = 0$.*

We now list a few preliminary results about Hausdorff measure and dimension for continua in $\mathbb{R}^n$ under the usual metric, the proofs of which can be found in [1].

(1) If $A \subset B$, then $\dim_H(A) \leq \dim_H(B)$.
(2) The Hausdorff dimension is preserved under affine transformations, such as scaling, rotating, and shearing.
(3) The Hausdorff dimension of a space will always be at least the topological dimension.
(4) The Hausdorff dimension of the product of two spaces is at least the sum of the Hausdorff dimensions of the individual spaces.

2. Results

Banakh and Tuncali [2] constructed a Menger curve M whose Hausdorff dimension is 1, so the Hausdorff dimension of M^n is n. As a result, any metric continuum with topological dimension n can be embedded so that its Hausdorff dimension is also n. In this section, we will take any continuum embeddable in $\mathbb{R}^n$ and embed it in a way that gives it Hausdorff dimension n, regardless of the topological dimension. In this way we will show that there is no upper bound for the Hausdorff dimension of a continuum, since a continuum embeddable in $\mathbb{R}^n$ is also embeddable in $\mathbb{R}^m$ for any $m > n$.

Lemma 2.1. *There exists a Cantor set in $\mathbb{R}^n$ whose Hausdorff dimension is n.*

Proof. Let $\{b_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of positive numbers whose limit is $1/2$. For each i, let $a_i = \displaystyle\prod_{j=1}^{i} b_j$. Let C_0 be the unit interval $[0,1]$ in $\mathbb{R}$, and for each i, let C_i be the subset of C_{i-1} consisting of 2^i disjoint intervals, each of length a_i, and each sharing an endpoint with an endpoint of C_{i-1}. Let $C = \bigcap C_i$. Then C is a Cantor set in $\mathbb{R}$ whose Hausdorff dimension is 1, as a result of equation 2.4 in [3]. Let $D = \prod_{j=1}^{n} C$. Then D is a Cantor set in $\mathbb{R}^n$ whose Hausdorff dimension is n. $\square$

A specific example of a set constructed as above is the Smith-Volterra Cantor set.

Theorem 2.2. *Any nondegenerate continuum X embeddable in $\mathbb{R}^n$ can be embedded in $\mathbb{R}^n$ such that its Hausdorff dimension is n.*

Proof. Take a collection $\{\mathcal{X}_i\}$ of open covers of X of decreasing mesh approaching 0, with each element of each cover being homeomorphic to a ball, such that for each element E of $\mathcal{X}_i$, there exist two elements of $\mathcal{X}_{i+1}$ whose closures are disjoint subsets of E. Without loss of generality, assume the first cover has at least two elements whose closures are disjoint. Take any two such elements in the first cover. For each of these elements, take two elements in the next cover that are contained in that element whose closures are disjoint. Continue this process indefinitely to obtain a Cantor set contained in X.

For each i, let R_i be the complement in $\mathbb{R}^n$ of the closures of the 2^{ni} disjoint elements of $\mathcal{X}_{ni}$ obtained above, and let S_i be the complement in $\mathbb{R}^n$ of $\prod_{j=1}^{n} C_i$, where C_i is as defined in the construction of a Cantor set earlier. Since R_i and S_i are each a copy of $\mathbb{R}^n$ with 2^{ni} disjoint closures of balls removed, it follows that R_i and S_i are homeomorphic. For each i, define a homeomorphism $f_i\colon R_i \to S_i$ such that $f_{i+1} = f_i$ wherever both are defined. Let f be the closure of the union of the set of all such f_i. Then f is a homeomorphism on $\mathbb{R}^n$ taking a Cantor set in X to our n-dimensional Cantor set, so the dimension of $f(X)$ is at least n. But $f(X) \subset \mathbb{R}^n$, so its dimension is at most n. Thus, $f(X)$ is an embedding of X in $\mathbb{R}^n$ whose dimension is exactly n. $\square$

3. Examples

3.1. **An arc of dimension 2.** There are many well-known examples of arcs whose Hausdorff dimension are any number s such that $1 \leq s < 2$. For example, consider a variation of the Koch curve, where each straight line segment in an iteration is replaced by four line segments of equal length less than half the length of the original segment. However, this technique can only be used to produce arcs whose Hausdorff dimension is strictly less than 2. We will now construct an arc in $\mathbb{R}^2$ whose dimension is 2.

Let C be the Cantor set defined in Lemma 2.1, using $b_i = i/(2i+1)$. Let $D = C \times C$. Define an order on D as follows. Let $D_i = C_i \times C_i$. Let $a, b \in D$, and let k be the first index such that a and b are in different components of D_k. Using the lexicographic order for reference on R^2, we will define $a < b$ if, under the lexicographic order, the least element of the component of C_i containing a is less than the least element of the component of C_i containing b. Using this new order relation on D, note that the straight line segment connecting any two consecutive elements of D intersects D at only these two endpoints. Let X be the space consisting of D, along with all straight line segments that connect consecutive elements of D. Then X is an arc of dimension 2, as shown in Figure 1.

3.2. **An arc of hereditary dimension 2.** The previous example is an arc in the plane whose Hausdorff dimension is 2, but it contains proper subcontinua whose Hausdorff dimension is 1. We will now construct an arc in the plane such that any arc that is a subset of this arc will have Hausdorff dimension 2. Begin with the arc X in the previous example. Replace each straight arc in X with a copy of X, sheared, rotated, and scaled so that it only intersects D at its endpoints, which are the same as the endpoints of

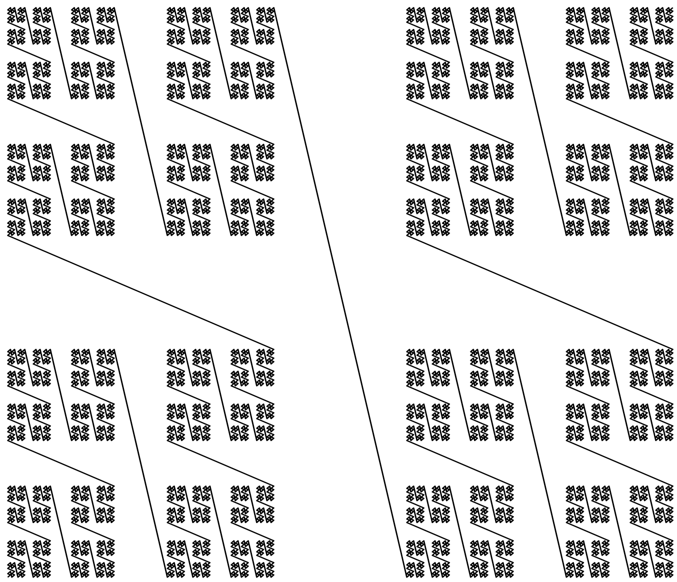

FIGURE 1. An arc in the plane whose Hausdorff dimension is 2.

the removed straight arc. Call this continuum X_1. Repeat this process for each straight arc in X_1 to obtain X_2, and so on. Note that the Hausdorff distance between X_n and X_{n+1} is approaching 0 as n approaches ∞, and that the Hausdorff limit of the X_n is again an arc. We will call this space X', and note that every arc in X' contains a scaled, sheared, rotated copy of D and thus, has Hausdorff dimension 2.

3.3. **An arc for any dimension.** Note also that we could use this method to construct an arc in $\mathbb{R}^n$ whose Hausdorff dimension is n by letting $D = C^n$. Additionally, we could use a different starting Cantor set C of a particular number less than 1 to make the dimension of D any number between 1 and n, and since the remaining part of X consists of a countable union of straight arcs which each have Hausdorff dimension 1, the dimension of X would equal the dimension of D, and thus, X could have any dimension between 1 and n. Lastly, if we take an n-dimensional arc for each positive integer n, scale each so its diameter is less than $1/2^n$, and chain them to each other by their endpoints, we can construct an arc in the Hilbert cube whose Hausdorff dimension is infinity.

References

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