MUNCHAUSEN NUMBERS REDUX

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ABSTRACT. A Munchausen number is a mathematical curiosity: raise each digit to the power of itself, add them all up, and recover the original number. In the seminal paper on this topic, D. Van Berkel derived a bound on such numbers for any given radix, which means that they can be completely enumerated in principle. We present a simpler argument which yields a bound one half the size and show that a radically different approach would be required for further reductions.

1. INTRODUCTION

In [1], Van Berkel introduces the base-b Munchausen numbers as the fixed points of the base-b Munchausen function

$$\theta_b(n) = \sum_{i=0}^{d-1} c_i^{c_i},$$

where n is a positive integer with d digits in base b, and c_i is its ith digit. They are also referred to as perfect digit-to-digit invariants or Canouchi numbers. The canonical example is

$$3435 = 3^3 + 4^4 + 3^3 + 5^5$$

in base 10. One can choose the convention $0^0 = 0$ or $0^0 = 1$. Van Berkel shows that $\theta_b(n) = n \Rightarrow n \leq 2b^b$. The bound has 4 digits in base 2 and $b + 1$ digits for $b > 2$. We improve on this result by demonstrating that Munchausen numbers have no more than b digits and that this is the best possible "simple" bound.

2. BOUNDS ON FIXED POINTS

Let n be defined as above. It is clear that $n \geq b^{d-1}$ and $\theta_b(n) \leq d(b-1)^{b-1}$. If $\theta_b(n) = n$, then we must have

$$b^{d-1} \leq n \leq d(b-1)^{b-1}. \tag{2.1}$$

We denote the difference between the bounds by the function

$$\Delta_b(d) = b^{d-1} - d(b-1)^{b-1}$$

defined on the real numbers for every integer $b \geq 2$, so that the inequality (2.1) cannot be satisfied where Δ_b is positive. We will use the roots of Δ_b to show that $d = b$ is the largest integral value for which it is nonpositive.

Lemma 2.1. *If a function is C^2 on an open interval and has at least three roots, then its second derivative must vanish at a point.*

Proof. Suppose that $f(x_1) = f(x_2) = f(x_3) = 0$ for some $x_1 < x_2 < x_3$. By Rolle's Theorem, there must be $x_1 < y_1 < x_2$ and $x_2 < y_2 < x_3$ such that $f'(y_1) = f'(y_2) = 0$. A second application of Rolle's Theorem shows that there must be a $y_1 < z < y_2$ such that $f''(z) = 0$. □

The function Δ_b can have at most two roots $r_b < R_b$, and its second derivative is strictly positive. It may be verified by inspection that $r_2 = 1$ and $R_2 = 2$. The remainder of this article deals with the case $b \geq 3$. The following lemmas show that both roots always exist.

Lemma 2.2. *For $b \geq 3$, $0 < r_b < 1$.*

Proof. It suffices to verify that $\Delta_b(0) > 0$ and $\Delta_b(1) < 0$. The Intermediate Value Theorem (IVT) then gives the desired result. We know that we have found the correct root, because the next two lemmas show that $R_b > b > 1$. □

Remark. *It is also possible to enumerate the roots and bound r_b via the real-valued branches of the Lambert-W function. However, we prefer the parsimony of our current approach.*

The contrapositive of the IVT implies that the sign of Δ_b cannot change for arguments greater than R_b. The exponential part will grow faster than the linear one, meaning that we must have $\Delta_b(d) > 0$ for all $d > R_b$. Since this is the undesirable region for a fixed point, it simply remains to find an approximate location for R_b using the IVT once again.

Lemma 2.3. $\Delta_b(b+1) = b^b - (b+1)(b-1)^{b-1} > 0.$

Proof. Recall that $b-1 \geq 1$. We have

$$\begin{aligned} & b^2 > b^2 - 1 \\ \Rightarrow\ & b^2 > (b+1)(b-1) \\ \Rightarrow\ & b > \frac{(b+1)(b-1)}{b} \\ \Rightarrow\ & \frac{b}{b-1} > \frac{b+1}{b} \\ \Rightarrow\ & \left(\frac{b}{b-1}\right)^{b-1} > \frac{b+1}{b} \\ \Rightarrow\ & b^{b-1} > \frac{(b+1)(b-1)^{b-1}}{b} \\ \Rightarrow\ & b^b > (b+1)(b-1)^{b-1} \\ \Rightarrow\ & b^b - (b+1)(b-1)^{b-1} > 0. \end{aligned}$$

□

Lemma 2.4. *For $b \geq 3$, $\Delta_b(b) = b^{b-1} - b(b-1)^{b-1} < 0$.*

Proof. It can be easily verified that $\Delta_3(3) < 0$. For the case $b \geq 4$, we know that

$$b^{b-1} < (b-1)^b < b(b-1)^{b-1}$$

(the first inequality holds because $b > b-1 > e$, and $x > y > e$ always implies that $x^y < y^x$), which means that

$$b^{b-1} - b(b-1)^{b-1} < 0.$$

□

Theorem 2.5. *The only values of d, the number of digits in base b, for which Munchausen numbers can exist are $1 \leq d \leq b$.*

Proof. Lemmas 2.3 and 2.4, together with the IVT, show that $b < R_b < b+1$. Therefore, the inequality (2.1) cannot be satisfied for $d \geq b+1$. □

Since $R_b < b+1$, no argument relying on the failure of transitivity for (2.1) can do better than this. There do exist certain bases with b-digit Munchausen numbers: consider the number 1243EED3419110E in base 15 with the convention $0^0 = 0$ [2]. We do not know whether this phenomenon occurs for infinitely many b, but if that were not the case, then a better bound on the number of digits would have to take on a more complex form than $b - c$ for some constant c.

3. Searching Efficiently

With this improvement on the bound alone, the running time of a brute-force search for Munchausen numbers can be cut in half. Additionally, because the Munchausen function is invariant under digit permutation, a program searching for Munchausen numbers could become more efficient by enumerating multisets of digits instead of individual numbers.

References

[1] D. Van Berkel, *On a curious property of 3435*, preprint arXiv:0911.3038v2 [math.HO], 2009.

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