ON A PROBLEM OF HARARY

PAUL C. KAINEN

ABSTRACT. An exercise in Harary [1, p. 100] states that the product of the vertex independence number and the vertex covering number is an upper bound on the number of edges in a bipartite graph. In this note, we extend the bound to triangle-free graphs, and show that equality holds if and only if the graph is complete bipartite.

1. THE RESULT

Let $G = (V, E)$ be any graph; $B \subseteq V$ is *independent* if no two vertices in B are adjacent. A set $A \subseteq V$ is a *vertex cover* if every edge is incident to some $a \in A$. The largest size of an independent set is denoted β_0 while the size of a smallest vertex cover is denoted α_0. If G is *triangle-free* (i.e., if it contains no K_3), then the set of vertices adjacent to any fixed vertex of G induces an edgeless subgraph, so vertex degree cannot exceed β_0.

Theorem 1. *Let G be a triangle-free graph with q edges, vertex covering number α_0, and vertex independence number β_0. Then*

$$q \leq \alpha_0 \beta_0.$$

Equality holds if and only if $G = K_{\alpha_0, \beta_0}$.

Theorem 1 is in [1, Exercise 10.7, p. 100] but required G to be bipartite.

Proof. It is well-known and easy to see that the complement of any independent set is a vertex cover and vice versa. Hence, if we take B to be a maximum-cardinality independent set of vertices in G and let $A = V \setminus B$ be the complement, then $|B| = \beta_0$ and $|A| = \alpha_0$. By definition, every edge of G is incident to a vertex in A which has degree not exceeding β_0 so the number of edges can't exceed $\alpha_0\beta_0$. Further, if equality holds, then there must be β_0 edges at each of the α_0 vertices of A with no duplicates, so G is bipartite complete with A and B as the two parts. □

2. Discussion

We compare the result here with [2] which gives a different generalization of Harary's problem.

An *edge cover* of a graph G is a subset $X \subseteq E_G$ such that every vertex of G is incident to some edge in X; let α_1 denote the minimum cardinality of an edge cover. It was proved in [2] that *size* of G (number of edges) is bounded above by the product of its vertex and edge covering numbers.

Theorem 2. *Let G be a triangle-free graph with no isolated vertices. Then*

$$|E| \leq \alpha_0 \alpha_1.$$

Equality holds if and only if $G = K_{\alpha_0, \alpha_1}$.

As $\beta_0(G) \leq \alpha_1(G)$ for all graphs G and $\beta_0(G) = \alpha_1(G)$ for G bipartite with no isolated points, Theorem 1 implies Theorem 2, and the two extremal cases are compatible. Either of the two theorems directly implies Harary's original exercise since bipartite graphs are triangle-free and the upper bounds coincide.

However, the proof of Theorem 2 in [2] uses an algorithmically-defined embedding φ of E_G in the cartesian product $S \times X$, where S and X are vertex and edge covers, resp. (The embedding also depends on the choice of some fixed linear ordering of the vertices in order to break ties deterministically.) Thus, for S, X any vertex and edge covers of G, there is an injection from E_G into $S \times X$ so the upper bound on size is given by explicit embeddings.

In summary, the upper bound on graph size in this paper is simultaneously smaller and yet easier to prove than the bound given in [2]. But the more difficult proof in [2] *constructs* injections of the edge-set into the cartesian product of point and line covers.

Acknowledgement

The author thanks a referee for suggestions that improved the exposition.

References

[1] F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA, 1970.
[2] P. C. Kainen, *Covering number product bounds size*, Geombinatorics, **26.2** (2016), 65–70.

MSC2010: 05C35

Key words and phrases: Triangle-free graphs, vertex cover, edge cover, independence number.

Department of Mathematics and Statistics, Georgetown University, Washington, DC 20057
E-mail address: kainen@georgetown.edu