ATANASSOV'S INTUITIONISTIC FUZZY BI-NORMED KU-SUBALGEBRAS OF A KU-ALGEBRA

TAPAN SENAPATI AND K. P. SHUM

Abstract. In this paper, by using the t-norm T and t-conorm S, we introduce the intuitionistic fuzzy bi-normed KU-subalgebras of a KU-algebra. Some properties of intuitionistic fuzzy bi-normed KU-subalgebras of a KU-algebra under the homomorphism are discussed. The direct product and the (T, S)-product of intuitionistic fuzzy bi-normed KU-subalgebras are particularly investigated.

1. Introduction

Triangular norms and triangular conorms (respectively t-norms and t-conorm for short) were first introduced by B. Schweizer and A. Sklar [23], following some ideas of Menger in the context of probabilistic metric spaces [16] (as statistical metric spaces were called after 1964). With the development of t-norms in statistical metric spaces, they also play an important role in decision making, in statistics as well as in the theories of cooperative games. In particular, in the theory of fuzzy sets, the t-norms have been extensively used for fuzzy operations, fuzzy logics and fuzzy relation equations. In recent years, a systematic study concerning the properties and the related aspects of t-norms have been considered by E. P. Klement et al. [14, 15].

The notion of intuitionistic fuzzy set was defined by K. T. Atanassov [3] as a generalization of the fuzzy sets initiated by L. A. Zadeh's [36]. In recent years, the Atanassov's intuitionistic fuzzy sets have attracted the attention and interests of a number of authors. Many papers concerning the applications of Atanassov's fuzzy sets in various branches of mathematics have been published. The reader is referred to these recent papers [5, 6, 7, 8, 12, 13, 34, 37, 38].

C. Prabpayak and U. Leerawat [21] introduced a new algebraic structure which is called KU-algebra. They gave the concept of homomorphisms of KU-algebras and investigated some of their related properties in [22]. S. M. Mostafa et al. [17] introduced the notion of fuzzy KU-ideals of KU-algebras. M. Akram et al. [2] and Yaqoob et al. [35] all introduced the notion of cubic KU-subalgebras and KU-ideals in KU-algebras. The relationships between a cubic KU-subalgebra and a cubic KU-ideal were discussed and investigated by them. Later, M. Gulistan and M. Shahzad [10] applied the concept of soft set theory to KU-algebra. G. Muhiuddin [20] applied the bipolar-valued fuzzy set theory to KU-algebras and introduced the notions of bipolar fuzzy KU-subalgebras and bipolar fuzzy KU-ideals in KU-algebras. He also considered the specifications of a bipolar fuzzy KU-subalgebra, a bipolar fuzzy KU-ideal in KU-algebras and discussed the relationship between a bipolar fuzzy KU-subalgebra and a bipolar fuzzy KU-ideal and provided the conditions for a bipolar fuzzy KU-subalgebra to be a bipolar fuzzy KU-ideal. M. Gulistan et al. [9] studied (α, β)-fuzzy KU-ideals in KU-algebras and discussed some special properties. S. M. Mostafa et al. [19] introduced n-fold KU-ideals and obtained some related results. On the other hand, M. Akram et al. [1] introduced the notion of interval-valued $(\tilde{\theta}, \tilde{\delta})$-fuzzy KU-ideals of KU-algebras and obtained some related properties. S. M. Mostafa et al. [18] introduced the intuitionistic fuzzy KU-ideals in KU-algebras. Finally, T. Senapati [24] introduced the notion of fuzzy KU-subalgebras of KU-algebras with respect to a given t-norm, and obtained some of their properties. Furthermore, T. Senapati studied the BCK/BCI-algebras in [26, 32], G-algebras [28], B-algebras [29, 30] and BG-algebras [4, 31, 32, 33] which are closely related to KU-algebras.

The aim of this paper is to introduce the concept of (imaginable) triangular norm and triangular conorm to KU-subalgebras of a KU-algebra. In Section 2, we consider a recapitulation of all required definitions and their properties. In Section 3, some concepts and operations of intuitionistic bi-normed ((T, S)-normed) fuzzy KU-subalgebras of a KU-algebra are proposed and their properties discussed in detail. In Section 4, properties of intuitionistic bi-normed fuzzy KU-subalgebras under homomorphisms are investigated. In Section 5, the direct product and the (T, S)-product of intuitionistic bi-normed fuzzy KU-subalgebras of a KU-algebra are introduced. In Section 6, the conclusion and scope for future research are outline and discussed. Throughout this paper, when we mention the intuitionistic fuzzy bi-normed KU-subalgebras, we always mean the intuitionistic fuzzy (T, S)-normed KU-subalgebras.

2. Preliminaries

In this section, some elementary aspects that are necessary for the main results of the paper are included.

Definition 2.1. [21] *(KU-algebra) By a KU-algebra, we mean an algebra $(X, *, 0)$ of type $(2, 0)$ with a single binary operation $*$ that satisfies the following axioms, for any $x, y, z \in X$,*

(1) $(x * y) * ((y * z) * (x * z)) = 0$,

(2) $x * 0 = 0$,
(3) $0 * x = x$,
(4) $x * y = 0 = y * x$ *implies* $x = y$.

In what follows, we let $(X, *, 0)$ be a KU-algebra unless otherwise specified. For the sake of brevity, we call X a KU-algebra. We now define a partial ordering "$\leq$" on X by $x \leq y$ if and only if $y * x = 0$.

Definition 2.2. [21] $(X, *, 0)$ *is a* KU*-algebra if and only if it satisfie the following axioms, for any* $x, y, z \in X$,

(1) $(y * z) * (x * z) \leq (x * y)$,
(2) $0 \leq x$,
(3) $x \leq y$, $y \leq x$ *implies* $x = y$,
(4) $x \leq y$ *if and only if* $y * x = 0$.

Definition 2.3. [17] *In a* KU*-algebra, the following axioms holds for any* $x, y, z \in X$,

(1) $z * z = 0$,
(2) $z * (x * z) = 0$,
(3) $x \leq y$ *implies* $y * z \leq x * z$,
(4) $z * (y * x) = y * (z * x)$,
(5) $y * ((y * x) * x) = 0$.

The following example is an example of KU-algebra.

Example 2.4. *Let* $X = \{0, a, b, c, d\}$ *be a set with the following Cayley table:*

$*$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	0	0	a
b	0	c	0	c	d
c	0	a	b	0	a
d	0	0	0	0	0

It is easy to see that X is a KU-algebra.

A non-empty subset S of a KU-algebra X is called a KU-subalgebra [21] of X if $x * y \in S$, for all $x, y \in S$. From this definition it is observed that, if a subset S of a KU-algebra satisfies only the closer property, then S becomes a KU-subalgebra.

Let $(X, *, 0)$ and $(Y, *', 0')$ be KU-algebras. A homomorphism is a mapping $f : X \to Y$ satisfying $f(x * y) = f(x) *' f(y)$, for all $x, y \in X$.

Theorem 2.5. [22] *Let* f *be a homomorphism of a* KU*-algebra* X *into a* KU*-algebra* Y*, then*

(1) *if* 0 *is the identity in* X*, then* $f(0)$ *is the identity in* Y*;*

(2) *if S is a KU-subalgebra of X, then $f(S)$ is a KU-subalgebra of Y;*
(3) *if S is a KU-subalgebra of $f(Y)$, then $f^{-1}(S)$ is a KU-subalgebra of X.*

We now review some fuzzy logic concepts.

Let X be the collection of objects denoted generally by x then a fuzzy set [36] A in X is defined as $A = \{< x, \alpha_A(x) >: x \in X\}$ where $\alpha_A(x)$ is called the membership value of x in A and $0 \leq \alpha_A(x) \leq 1$. For any fuzzy sets A and B of a set X, we define $A \cap B = \min\{\alpha_A(x), \alpha_B(x)\}$ for all $x \in X$.

By a triangular norm (briefly t-norm) T [16], we mean a binary operation on the unit interval $[0, 1]$ which is commutative, associative, monotone and has 1 as a neutral element, i.e., it is a function $T : [0, 1]^2 \to [0, 1]$ such that for all $x, y, z \in [0, 1]$:

(1) $T(x, y) = T(y, x)$;
(2) $T(x, T(y, z)) = T(T(x, y), z)$;
(3) $T(x, y) \leq T(x, z)$ if $y \leq z$;
(4) $T(x, 1) = x$.

Some examples of t-norms are the minimum $T_M(x, y) = \min(x, y)$, the product $T_P(x, y) = x \cdot y$ and the Lukasiewicz t-norm $T_L(x, y) = \max(x + y - 1, 0)$ for all $x, y \in [0, 1]$.

By a triangular conorm (t-conorm for short) S [23], we mean a binary operation on the unit interval $[0, 1]$ which is commutative, associative, monotone and has 0 as a neutral element, i.e., it is a function $S : [0, 1]^2 \to [0, 1]$ such that for all $x, y, z \in [0, 1]$:

(1) $S(x, y) = S(y, x)$;
(2) $S(x, S(y, z)) = S(S(x, y), z)$;
(3) $S(x, y) \leq S(x, z)$ if $y \leq z$;
(4) $S(x, 0) = x$.

Some example of t-conorms are the maximum $S_M(x, y) = \max(x, y)$, the probabilistic sum $S_P(x, y) = x + y - x \cdot y$ and the Lukasiewicz t-conorm or (bounded sum) $S_L(x, y) = \min(x + y, 1)$ for all $x, y \in [0, 1]$. Also, it is well-known [11, 14] that if T is a t-norm and S is a t-conorm, then $T(x, y) \leq \min\{x, y\}$ and $S(x, y) \geq \max\{x, y\}$ for all $x, y \in [0, 1]$, respectively.

Definition 2.6. *Let P be a t-norm. Denote by $\triangle_P$ the set of elements $x \in [0, 1]$ such that $P(x, x) = x$, that is, $\triangle_P = \{x \in [0, 1] : P(x, x) = x\}$.*

A fuzzy set A in X is said to satisfy imaginable property with respect to P if $Im(\alpha_A) \subseteq \triangle_P$.

Definition 2.7. [24] Let A be a fuzzy set in X. Then the set A is T-fuzzy KU-subalgebra over the binary operator $*$ if it satisfies (TS1) $\alpha_A(x * y) \geq T\{\alpha_A(x), \alpha_A(y)\}$ for all $x, y \in X$.

Definition 2.8. [3] *The intuitionistic fuzzy sets defined on a non-empty set X as objects having the form $A = \{\langle x, \alpha_A(x), \beta_A(x)\rangle : x \in X\}$, where the functions $\alpha_A(x) : X \to [0,1]$ and $\beta_A(x) : X \to [0,1]$, denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set A respectively, and $0 \le \alpha_A(x)+\beta_A(x) \le 1$ for all $x \in X$. Obviously, when $\beta_A(x) = 1 - \alpha_A(x)$ for every $x \in X$, the set A becomes a fuzzy set.*

For the sake of simplicity, we shall use the symbol $A = (\alpha_A, \beta_A)$ for the intuitionistic fuzzy subset $A = \{\langle x, \alpha_A(x), \beta_A(x)\rangle : x \in X\}$.

3. Intuitionistic Fuzzy Bi-normed KU-Subalgebras

In this section, the intuitionistic fuzzy bi-normed, that is, (T, S)-normed, KU-subalgebras of a KU-algebra are first defined and introduced. Some properties of intuitionistic fuzzy bi-normed KU-subalgebras are investigated and given in this section. In what follows, we simply use X to denote a KU-algebra unless otherwise specified.

Definition 3.1. *Let $A = (\alpha_A, \beta_A)$ be an intuitionistic fuzzy set in X. Then the set A is intuitionistic fuzzy bi-normed KU-subalgebra over the binary operator $*$ if it satisfies the following conditions:*

(TS1) $\alpha_A(x * y) \ge T\{\alpha_A(x), \alpha_A(y)\}$
(TS2) $\beta_A(x * y) \le S\{\beta_A(x), \beta_A(y)\}$

for all $x, y \in X$.

We now illustrate the above definitions by using some examples.

Example 3.2. *Let X=$\{0, a, b, c\}$ be a KU-algebra with the following Cayley table:*

$*$	0	a	b	c
0	0	a	b	c
a	0	0	0	b
b	0	b	0	a
c	0	0	0	0

Let $T_m, S_m : [0,1] \times [0,1] \to [0,1]$ be functions defined by $T_m(x, y) = \max(x + y - 1, 0)$ and $S_m(x, y) = \min(x + y, 1)$ for all $x, y \in [0, 1]$. Then T_m is a t-norm and S_m is a t-conorm. Define an IFS $A = (\alpha_A, \beta_A)$ in X by

$$\alpha_A(x) = \begin{cases} 0.8, & \text{if } x = 0 \\ 0.6, & \text{if } x = a, b \\ 0.4, & \text{if } x = c \end{cases} \qquad \text{and} \qquad \beta_A(x) = \begin{cases} 0.1, & \text{if } x = 0 \\ 0.3, & \text{if } x = a, b \\ 0.5, & \text{if } x = c. \end{cases}$$

Then A is an intuitionistic (T_m, S_m)-normed fuzzy KU-subalgebra of X.

Definition 3.3. *An intuitionistic fuzzy bi-normed KU-subalgebra $A = (\alpha_A, \beta_A)$ is called an intuitionistic imaginable fuzzy bi-normed KU-subalgebra* *of X if α_A and β_A satisfy the imaginable property with respect to T and S respectively.*

Example 3.4. *Consider a t-norm T_m, a t-conorm S_m and let $X=\{0,a,b,c\}$ be a KU-algebra in Example 3.2. Define an IFS $A=(\alpha_A,\beta_A)$ in X by $\alpha_A(x)=1$, if $x\in\{0,a,b\}$, $\alpha_A(c)=0$, $\beta_A(x)=0$, if $x\in\{0,a,b\}$ and $\beta_A(c)=1$. Then, it is easy to check that $\alpha_A(x*y)\geq T_m\{\alpha_A(x),\alpha_A(y)\}$ and $\beta_A(x*y)\leq S_m\{\beta_A(x),\beta_A(y)\}$ for all $x,y\in X$. Also, $Im(\alpha_A)\subseteq\triangle_{T_m}$ and $Im(\beta_A)\subseteq\triangle_{S_m}$. Hence, A is clearly an intuitionistic imaginable (T_m,S_m)-normed fuzzy KU-subalgebra of X.*

Below, we give some propositions which are related with the intuitionistic imaginable fuzzy bi-normed KU-subalgebra.

Proposition 3.5. *If IFS $A=(\alpha_A,\beta_A)$ is an intuitionistic imaginable fuzzy bi-normed KU-subalgebra of X, then $\alpha_A(0)\geq\alpha_A(x)$ and $\beta_A(0)\leq\beta_A(x)$ for all $x\in X$.*

Proof. Let $x\in X$. Then $\alpha_A(0)=\alpha_A(x*x)\geq T\{\alpha_A(x),\alpha_A(x)\}=\alpha_A(x)$ and $\beta_A(0)=\beta_A(x*x)\leq S\{\beta_A(x),\beta_A(x)\}=\beta_A(x)$. The proof is straightforward and is omitted. □

For any element x and y of X, let us write $\prod^{n} x*y$ for $x*(\cdots*(x*(x*y)))$ where x occurs n times.

Proposition 3.6. *Let $A=(\alpha_A,\beta_A)$ be an intuitionistic imaginable fuzzy bi-normed KU-subalgebra of X and let $n\in\mathbb{N}$ (the set of natural numbers). Then the following statements hold:*

(1) $\alpha_A(\prod^{n} x*x)\geq\alpha_A(x)$, *for any odd number n,*
(2) $\beta_A(\prod^{n} x*x)\leq\beta_A(x)$, *for any odd number n,*
(3) $\alpha_A(\prod^{n} x*x)=\alpha_A(x)$, *for any even number n,*
(4) $\beta_A(\prod^{n} x*x)=\beta_A(x)$, *for any even number n.*

Proof. Let $x\in X$ and assume that n is odd. Then $n=2p-1$ for some positive integer p. Now, we prove the proposition by induction. By Proposition 3.5, $\alpha_A(0)\geq\alpha_A(x)$ and $\beta_A(0)\leq\beta_A(x)$ for all $x\in X$. Suppose that

$$\alpha_A\left(\prod^{2p-1} x*x\right)\geq\alpha_A(x) \text{ and } \beta_A\left(\prod^{2p-1} x*x\right)\leq\beta_A(x).$$

Then by our assumption, we have

$$\begin{aligned}\alpha_A\left(\prod^{2(p+1)-1} x*x\right) &= \alpha_A\left(\prod^{2p+1} x*x\right)\\ &= \alpha_A\left(\prod^{2p-1} x*(x*(x*x))\right)\\ &= \alpha_A\left(\prod^{2p-1} x*x\right)\\ &\geq \alpha_A(x)\end{aligned}$$

and

$$\begin{aligned}\beta_A\left(\prod^{2(p+1)-1} x*x\right) &= \beta_A\left(\prod^{2p+1} x*x\right)\\ &= \beta_A\left(\prod^{2p-1} x*(x*(x*x))\right)\\ &= \beta_A\left(\prod^{2p-1} x*x\right)\\ &\leq \beta_A(x)\end{aligned}$$

This proves (1) and (2). The proofs are similar for the cases (3) and (4). □

Proposition 3.7. *If an IFS $A = (\alpha_A, \beta_A)$ is an intuitionistic imaginable fuzzy bi-normed KU-subalgebra of X, then $\alpha_A(0*x) \geq \alpha_A(x)$ and $\beta_A(0*x) \leq \beta_A(x)$ for all $x \in X$.*

Proof. For all $x \in X$, we have

$$\begin{aligned}\alpha_A(0*x) &\geq T\{\alpha_A(0), \alpha_A(x)\} = T\{\alpha_A(x*x), \alpha_A(x)\}\\ &\geq T\Big\{T\{\alpha_A(x), \alpha_A(x)\}, \alpha_A(x)\Big\} = \alpha_A(x)\end{aligned}$$

and

$$\begin{aligned}\beta_A(0*x) &\leq S\{\beta_A(0), \beta_A(x)\} = S\{\beta_A(x*x), \beta_A(x)\}\\ &\leq S\Big\{S\{\beta_A(x), \beta_A(x)\}, \beta_A(x)\Big\} = \beta_A(x).\end{aligned}$$

This completes the proof. □

Based on the above proposition, an infinity theory of limit sequence of membership and the non-membership function of intuitionistic imaginable fuzzy bi-normed KU-subalgebra are given.

Theorem 3.8. *Let A be an intuitionistic imaginable fuzzy bi-normed KU-subalgebra of X. If there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} \alpha_A(x_n) = 1$ and $\lim_{n\to\infty} \beta_A(x_n) = 0$, then $\alpha_A(0) = 1$ and $\beta_A(0) = 0$.*

Proof. By Proposition 3.5, we have $\alpha_A(0) \geq \alpha_A(x)$ for all $x \in X$. Therefore, $\alpha_A(0) \geq \alpha_A(x_n)$ for every positive integer n. Consider $1 \geq \alpha_A(0) \geq \lim_{n\to\infty} \alpha_A(x_n) = 1$. Hence, $\alpha_A(0) = 1$.

Again, by Proposition 3.5, $\beta_A(0) \leq \beta_A(x)$ for all $x \in X$. Thus, $\beta_A(0) \leq \beta_A(x_n)$ for every positive integer n. Now, $0 \leq \beta_A(0) \leq \lim_{n\to\infty} \beta_A(x_n) = 0$. Hence, $\beta_A(0) = 0$. □

Definition 3.9. [3] Let $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ be two IFSs on X. Then the intersection of A and B is denoted by $A \cap B$ and is given by $A \cap B = \{\min(\alpha_A, \alpha_B), \max(\beta_A, \beta_B)\}$.

The intersection of two intuitionistic fuzzy bi-normed KU-subalgebras is also an intuitionistic fuzzy bi-normed KU-subalgebra, which is proved in the following theorem.

Theorem 3.10. *Let A_1 and A_2 be two intuitionistic fuzzy bi-normed KU-subalgebras of X. Then $A_1 \cap A_2$ is an intuitionistic fuzzy bi-normed KU-subalgebra of X.*

Proof. Let $x, y \in A_1 \cap A_2$. Then $x, y \in A_1$ and A_2. Now, we have

$$\begin{aligned}\alpha_{A_1\cap A_2}(x*y) &= \min\{\alpha_{A_1}(x*y), \alpha_{A_2}(x*y)\}\\ &\geq \min\{T\{\alpha_{A_1}(x), \alpha_{A_1}(y)\}, T\{\alpha_{A_2}(x), \alpha_{A_2}(y)\}\}\\ &\geq T\{\min\{\alpha_{A_1}(x), \alpha_{A_2}(x)\}, \min\{\alpha_{A_1}(y), \alpha_{A_2}(y)\}\}\\ &= T\{\alpha_{A_1\cap A_2}(x), \alpha_{A_1\cap A_2}(y)\}\end{aligned}$$

and

$$\begin{aligned}\beta_{A_1\cap A_2}(x*y) &= \max\{\beta_{A_1}(x*y), \beta_{A_2}(x*y)\}\\ &\leq \max\{S\{\beta_{A_1}(x), \beta_{A_1}(y)\}, S\{\beta_{A_2}(x), \beta_{A_2}(y)\}\}\\ &\leq S\{\max\{\beta_{A_1}(x), \beta_{A_2}(x)\}, \max\{\beta_{A_1}(y), \beta_{A_2}(y)\}\}\\ &= S\{\beta_{A_1\cap A_2}(x), \beta_{A_1\cap A_2}(y)\}.\end{aligned}$$

Hence, we have shown that $A_1 \cap A_2$ is an intuitionistic fuzzy bi-normed KU-subalgebra of X. □

The above theorem can be generalized as follows.

Theorem 3.11. *Let $\{A_i : i = 1, 2, 3, 4, \ldots\}$ be a family of intuitionistic fuzzy bi-normed KU-subalgebras of X. Then $\bigcap A_i$ is also an intuitionistic fuzzy bi-normed KU-subalgebra of X, where $\bigcap A_i = (min\ \alpha_{A_i}(x), max\ \beta_{A_i}(x))$.*

Theorem 3.12. *An IFS $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy bi-normed KU-subalgebra of X if and only if the fuzzy sets $A_1 = \{\alpha_A(x) : x \in A\}$ and $A_2 = \{\overline{\beta}_A(x) : x \in A\}$ are T-fuzzy KU-subalgebras of X.*

Proof. Let A be an intuitionistic fuzzy bi-normed KU-subalgebra of X. Clearly, A_1 is a T-fuzzy KU-subalgebra of X. For every $x, y \in X$, we have

$$\begin{aligned}\overline{\beta}_A(x * y) &= 1 - \beta_A(x * y) \geq 1 - S\{\beta_A(x), \beta_A(y)\} \\ &= T\{1 - \beta_A(x), 1 - \beta_A(y)\} = T\{\overline{\beta}_A(x), \overline{\beta}_A(y)\}.\end{aligned}$$

Hence, A_2 is a T-fuzzy KU-subalgebra of X.

Conversely, assume that A_1 and A_2 are T-fuzzy KU-subalgebras of X. For every $x, y \in X$, $\alpha_A(x * y) \geq T\{\alpha_A(x), \alpha_A(y)\}$ and

$$\begin{aligned}1 - \beta_A(x * y) &= \overline{\beta}_A(x * y) \\ &\geq T\{\overline{\beta}_A(x), \overline{\beta}_A(y)\} \\ &= T\{1 - \beta_A(x), 1 - \beta_A(y)\} \\ &= 1 - S\{\beta_A(x), \beta_A(y)\}.\end{aligned}$$

That is, $\beta_A(x*y) \leq S\{\beta_A(x), \beta_A(y)\}$. Hence, $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy bi-normed KU-subalgebra of X. □

We define two operators $\bigoplus A$ and $\bigotimes A$ on IFS.

Definition 3.13. *Let $A = (\alpha_A, \beta_A)$ be an IFS defined on X. The operators $\bigoplus A$ and $\bigotimes A$ are defined as $\bigoplus A = (\alpha_A, \overline{\alpha}_A)$ and $\bigotimes A = (\overline{\beta}_A, \beta_A)$, respectively.*

Theorem 3.14. *If $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy bi-normed KU-subalgebra of X, then*

(1) *$\bigoplus A$, and*
(2) *$\bigotimes A$, both are intuitionistic fuzzy bi-normed KU-subalgebras.*

Proof. (1) It suffices to show that $\overline{\alpha}_A$ satisfies the condition (TS2). Let $x, y \in X$. Then

$$\begin{aligned}\overline{\alpha}_A(x * y) &= 1 - \alpha_A(x * y) \leq 1 - T\{\alpha_A(x), \alpha_A(y)\} \\ &= S\{1 - \alpha_A(x), 1 - \alpha_A(y)\} = S\{\overline{\alpha}_A(x), \overline{\alpha}_A(y)\}.\end{aligned}$$

Hence, $\bigoplus A$ is an intuitionistic fuzzy bi-normed KU-subalgebra of X.

(2) It suffices to show that $\overline{\beta}_A$ satisfies the condition (TS1). Let $x, y \in X$. Then

$$\begin{aligned}\overline{\beta}_A(x * y) &= 1 - \beta_A(x * y) \geq 1 - S\{\beta_A(x), \beta_A(y)\} \\ &= T\{1 - \beta_A(x), 1 - \beta_A(y)\} = T\{\overline{\beta}_A(x), \overline{\beta}_A(y)\}.\end{aligned}$$

Hence, $\bigotimes A$ is also an intuitionistic fuzzy bi-normed KU-subalgebra of X. □

The sets $\{x \in X : \alpha_A(x) = \alpha_A(0)\}$ and $\{x \in X : \beta_A(x) = \beta_A(0)\}$ are denoted by I_{α_A} and I_{β_A}, respectively. These two sets are also KU-subalgebras of X.

Theorem 3.15. *Let $A = (\alpha_A, \beta_A)$ be an intuitionistic fuzzy bi-normed KU-subalgebra of X. Then the sets I_{α_A} and I_{β_A} are KU-subalgebras of X.*

Proof. Let $x, y \in I_{\alpha_A}$. Then $\alpha_A(x) = \alpha_A(0) = \alpha_A(y)$ and so, $\alpha_A(x * y) \geq T\{\alpha_A(x), \alpha_A(y)\} = T\{\alpha_A(0), \alpha_A(0)\} = \alpha_A(0)$. By using Proposition 3.5, we deduce that $\alpha_A(x*y) \leq \alpha_A(0)$. Hence, $\alpha_A(x*y) = \alpha_A(0)$ or equivalently $x * y \in I_{\alpha_A}$.

Again, let $x, y \in I_{\beta_A}$. Then $\beta_A(x) = \beta_A(0) = \beta_A(y)$ and so, $\beta_A(x * y) \leq S\{\beta_A(x), \beta_A(y)\} = S\{\beta_A(0), \beta_A(0)\}$. By Proposition 3.5, we know that $\beta_A(x * y) \geq \beta_A(0)$. Hence, $\beta_A(x * y) = \beta_A(0)$ or equivalently $x * y \in I_{\beta_A}$. Therefore, the sets I_{α_A} and I_{β_A} are KU-subalgebras of X. □

It is well known that the characteristic function of a set is a special fuzzy set. Suppose that A is a non-empty subset of X. By χ_A, we mean the characteristic function of A, that is,

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.16. *If A is a KU-subalgebra of X, then the IFS $\overline{A} = (\chi_A, \overline{\chi}_A)$ is an intuitionistic fuzzy bi-normed KU-subalgebra of X.*

Proof. Let $x, y \in X$. We consider four cases.
<u>Case 1</u>. If $x, y \in A$ then $x * y \in A$ since A is a KU-subalgebra of X. Then $\chi_A(x * y) = 1 \geq T\{\chi_A(x), \chi_A(y)\}$ and

$$\begin{aligned} \overline{\chi}_A(x * y) &= 1 - \chi_A(x * y) = 1 - 1 \\ &= 0 \leq S\{\overline{\chi}_A(x), \overline{\chi}_A(y)\}. \end{aligned}$$

<u>Case 2</u>. If $x, y \notin A$, then $\chi_A(x) = 0 = \chi_A(y)$ and $\overline{\chi}_A(x) = 1 = \overline{\chi}_A(y)$. Thus,

$$\begin{aligned} \chi_A(x * y) &\geq 0 = \min\{0, 0\} \\ &\geq T\{0, 0\} = T\{\chi_A(x), \chi_A(y)\} \end{aligned}$$

and

$$\begin{aligned} \overline{\chi}_A(x * y) &\leq 1 = \max\{1, 1\} \leq S\{1, 1\} \\ &= S\{\overline{\chi}_A(x), \overline{\chi}_A(y)\}. \end{aligned}$$

<u>Case 3</u>. If $x \in A$ and $y \notin A$ then $\chi_A(x) = 1$, $\chi_A(y) = 0$, $\overline{\chi}_A(x) = 0$ and $\overline{\chi}_A(y) = 1$. Thus,

$$\chi_A(x * y) \geq 0 = T\{0, 1\} = T\{1, 0\} = T\{\chi_A(x), \chi_A(y)\}$$

and

$$\begin{aligned}\overline{\chi}_A(x*y) &\leq 1 = S\{1,0\} = S\{0,1\} \\ &= S\{1-\chi_A(x), 1-\chi_A(y)\} \\ &= S\{\overline{\chi}_A(x), \overline{\chi}_A(y)\}.\end{aligned}$$

Case 4. If $x \notin A$ and $y \in A$, then by using the same argument as in Case 3, we conclude that $\chi_A(x*y) \geq T\{\chi_A(x), \chi_A(y)\}$ and

$$\overline{\chi}_A(x*y) \leq S\{\overline{\chi}_A(x), \overline{\chi}_A(y)\}.$$

Therefore, IFS $\overline{A} = (\chi_A, \overline{\chi}_A)$ is an intuitionistic fuzzy bi-normed KU-subalgebra of X. □

Theorem 3.17. *Let A be a non-empty subset of X. If $\overline{A} = (\chi_A, \overline{\chi}_A)$ satisfies (TS1) and (TS2). Then, A is a KU-subalgebra of X.*

Proof. Suppose that $\overline{A} = (\chi_A, \overline{\chi}_A)$ satisfy (TS1) and $x, y \in A$. Then it follows from (TS1) that $\chi_A(x*y) \geq T\{\chi_A(x), \chi_A(y)\} = T\{1,1\} = 1$ so that $\chi_A(x*y) = 1$, i.e., $x*y \in A$. Hence, A is a KU-subalgebra of X.

Again, suppose that $\overline{A} = (\chi_A, \overline{\chi}_A)$ satisfy (TS2) and $x, y \in A$. Then it follows from (TS2) that

$$\begin{aligned}\overline{\chi}_A(x*y) &\leq S\{\overline{\chi}_A(x), \overline{\chi}_A(y)\} \\ &= S\{1-\chi_A(x), 1-\chi_A(y)\} = S(0,0) = 0\end{aligned}$$

and thus,

$$\overline{\chi}_A(x*y) = 1 - \chi_A(x*y) = 0,$$

i.e., $\chi_A(x*y) = 1$. This proves the theorem. □

Proposition 3.18. *Let B be a KU-subalgebra of X and $A = (\alpha_A, \beta_A)$ be an IFS in X defined by*

$$\alpha_A(x) = \begin{cases} \lambda, & \text{if } x \in B; \\ \tau, & \text{otherwise} \end{cases} \quad \text{and} \quad \beta_A(x) = \begin{cases} \gamma, & \text{if } x \in B; \\ \delta, & \text{otherwise} \end{cases}$$

for all λ, τ, γ and $\delta \in [0,1]$ with $\lambda \geq \tau$ and $\gamma \leq \delta$ and $\lambda + \gamma \leq 1$; $\tau + \delta \leq 1$. Then A is an intuitionistic (T_m, S_m)-normed fuzzy KU-subalgebra of X, where T_m and S_m is the t-norm and s-norm, respectively in Example 3.2. In particular, if $\lambda = \delta = 1$ and $\tau = \gamma = 0$ then A is an intuitionistic imaginable (T_m, S_m)-normed fuzzy KU-subalgebra of X. Moreover, $I_{\alpha_A} = B = I_{\beta_A}$.

Proof. Let $x, y \in X$. We consider the following three cases.
Case 1. If $x, y \in B$ then

$$\begin{aligned} T_m(\alpha_A(x), \alpha_A(y)) &= T_m(\lambda, \lambda) = \max(2\lambda - 1, 0) \\ &= \begin{cases} 2\lambda - 1, & \text{if } \lambda \geq \frac{1}{2}; \\ 0, & \text{otherwise} \end{cases} \\ &\leq \lambda = \alpha_A(x * y) \end{aligned}$$

and

$$\begin{aligned} S_m(\beta_A(x), \beta_A(y)) &= S_m(\gamma, \gamma) = \min(2\gamma, 1) \\ &= \begin{cases} 2\gamma, & \text{if } \gamma \leq \frac{1}{2}; \\ 1, & \text{otherwise} \end{cases} \\ &\geq \gamma = \beta_A(x * y). \end{aligned}$$

Case 2. If $x \in B$ and $y \notin B$ (or, $x \notin B$ and $y \in B$) then

$$\begin{aligned} T_m(\alpha_A(x), \alpha_A(y)) &= T_m(\lambda, \tau) = \max(\lambda + \tau - 1, 0) \\ &= \begin{cases} \lambda + \tau - 1, & \text{if } \lambda + \tau \geq 1; \\ 0, & \text{otherwise} \end{cases} \\ &\leq \tau = \alpha_A(x * y) \end{aligned}$$

and

$$\begin{aligned} S_m(\beta_A(x), \beta_A(y)) &= S_m(\gamma, \delta) = \min(\gamma + \delta, 1) \\ &= \begin{cases} \gamma + \delta, & \text{if } \gamma + \delta \leq 1; \\ 1, & \text{otherwise} \end{cases} \\ &\geq \delta = \beta_A(x * y). \end{aligned}$$

Case 3. If $x, y \notin B$ then

$$\begin{aligned} T_m(\alpha_A(x), \alpha_A(y)) &= T_m(\tau, \tau) = \max(2\tau - 1, 0) \\ &= \begin{cases} 2\tau - 1, & \text{if } \tau \geq \frac{1}{2}; \\ 0, & \text{otherwise} \end{cases} \\ &\leq \tau = \alpha_A(x * y) \end{aligned}$$

and

$$\begin{aligned} S_m(\beta_A(x), \beta_A(y)) &= S_m(\delta, \delta) = \min(2\delta, 1) \\ &= \begin{cases} 2\delta, & \text{if } \delta \leq \frac{1}{2}; \\ 1, & \text{otherwise} \end{cases} \\ &\geq \delta = \beta_A(x * y). \end{aligned}$$

Hence, A is an intuitionistic (T_m, S_m)-normed fuzzy KU-subalgebra of X.

Assume that $\lambda = \delta = 1$ and $\tau = \gamma = 0$. Then

$$T_m(\lambda, \lambda) = \max(\lambda + \lambda - 1, 0) = 1 = \lambda,$$
$$T_m(\tau, \tau) = \max(\tau + \tau - 1, 0) = 0 = \tau,$$
$$S_m(\gamma, \gamma) = \min(\gamma + \gamma, 1) = 0 = \gamma$$

and

$$S_m(\delta, \delta) = \min(\delta + \delta, 1) = 0 = \delta.$$

Thus, $\lambda, \tau \in \triangle_{T_m}$ and $\gamma, \delta \in \triangle_{S_m}$ i.e., $Im(\alpha_A) \subseteq \triangle_{T_m}$ and $Im(\beta_A) \subseteq \triangle_{S_m}$. This shows that A is an intuitionistic imaginable (T_m, S_m)-normed fuzzy KU-subalgebra of X.

Also, $I_{\alpha_A} = \{x \in X, \alpha_A(x) = \alpha_A(0)\} = \{x \in X, \alpha_A(x) = \lambda\} = B$ and $I_{\beta_A} = \{x \in X, \beta_A(x) = \beta_A(0)\} = \{x \in X, \beta_A(x) = \gamma\} = B$. Therefore, $I_{\alpha_A} = B = I_{\beta_A}$. □

Let $A = (\alpha_A, \beta_A)$ be an intuitionistic fuzzy KU-subalgebra of X. For $\tilde{s}, \tilde{t} \in [0, 1]$, the set $U(\alpha_A : \tilde{s}) = \{x \in X : \alpha_A(x) \geq \tilde{s}\}$ is called an upper $\tilde{s}$-level of A and $L(\beta_A : \tilde{t}) = \{x \in X : \beta_A(x) \leq \tilde{t}\}$ is called lower $\tilde{t}$-level of A.

Theorem 3.19. *Let $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy bi-normed KU-subalgebra of X and $\tilde{s}, \tilde{t} \in [0, 1]$ then if $\tilde{s} = 1$, the upper level set $U(\alpha_A : \tilde{s})$ is either empty or a KU-subalgebra of X. Again, if $\tilde{t} = 0$, the lower level set $L(\beta_A : \tilde{t})$ is either empty or a KU-subalgebras of X.*

Proof. Let $\tilde{s} = 1$ and $x, y \in U(\alpha_A : \tilde{s})$. Then $\alpha_A(x) \geq \tilde{s} = 1$ and $\alpha_A(y) \geq \tilde{s} = 1$. It follows that $\alpha_A(x * y) \geq T(\alpha_A(x), \alpha_A(y)) \geq T(1, 1) = 1$ so that $x * y \in U(\alpha_A : \tilde{s})$. Hence, $U(\alpha_A : \tilde{s})$ is a KU-subalgebra of X when $s = 1$.

Again, let $\tilde{t} = 0$ and $x, y \in L(\beta_A : \tilde{t})$. Then $\beta_A(x) \leq \tilde{t} = 0$ and $\beta_A(y) \leq \tilde{t} = 0$. It follows that $\beta_A(x * y) \leq S(\beta_A(x), \beta_A(y)) \leq S(0, 0) = 0$ so that $x * y \in L(\beta_A : \tilde{t})$. Hence, $L(\beta_A : \tilde{t})$ is a KU-subalgebra of X when $\tilde{t} = 0$. □

Theorem 3.20. *If $A = (\alpha_A, \beta_A)$ is an intuitionistic imaginable fuzzy bi-normed KU-subalgebra of X, then the upper $\tilde{s}$-level and lower $\tilde{t}$-level of A are KU-subalgebras of X.*

Proof. Assume that $x, y \in U(\alpha_A : \tilde{s})$. Then $\alpha_A(x) \geq \tilde{s}$ and $\alpha_A(y) \geq \tilde{s}$. It follows that $\alpha_A(x * y) \geq T\{\alpha_A(x), \alpha_A(y)\} \geq T(\tilde{s}, \tilde{s}) = \tilde{s}$ so that $x * y \in U(\alpha_A : \tilde{s})$. Hence, $U(\alpha_A : \tilde{s})$ is a KU-subalgebra of X.

Again, assume that $x, y \in L(\beta_A : \tilde{t})$. Then $\beta_A(x) \leq \tilde{t}$ and $\beta_A(y) \leq \tilde{t}$. It follows that $\beta_A(x * y) \leq S\{\beta_A(x), \beta_A(y)\} \leq S(\tilde{t}, \tilde{t}) = \tilde{t}$ so that $x * y \in L(\beta_A : \tilde{t})$. Hence, $L(\beta_A : \tilde{t})$ is a KU-subalgebra of X. □

Theorem 3.21. *Let $A = (\alpha_A, \beta_A)$ be an IFS in X such that the sets $U(\alpha_A : \tilde{s})$ and $L(\beta_A : \tilde{t})$ are KU-subalgebras of X for every $\tilde{s}, \tilde{t} \in [0,1]$. Then A is an intuitionistic fuzzy bi-normed KU-subalgebra of X.*

Proof. Let for any $\tilde{s}, \tilde{t} \in [0,1]$, $U(\alpha_A : \tilde{s})$ and $L(\beta_A : \tilde{t})$ are KU-subalgebras of X. In contrary, let $x_0, y_0 \in X$ be such that $\alpha_A(x_0 * y_0) < T\{\alpha_A(x_0), \alpha_A(y_0)\}$. Let us consider

$$\tilde{s}_0 = \frac{1}{2}\Big[\alpha_A(x_0 * y_0) + T\{\alpha_A(x_0), \alpha_A(y_0)\}\Big].$$

Then $\alpha_A(x_0 * y_0) < \tilde{s}_0 \le T\{\alpha_A(x_0), \alpha_A(y_0)\} \le \min\{\alpha_A(x_0), \alpha_A(y_0)\}$ and so $x_0 * y_0 \notin U(\alpha_A : \tilde{s})$ but $x_0, y_0 \in U(\alpha_A : \tilde{s})$. This is a contradiction and hence, α_A satisfies the inequality $\alpha_A(x * y) \ge T\{\alpha_A(x), \alpha_A(y)\}$ for all $x, y \in X$. Similarly, suppose that there exists $x_0, y_0 \in X$ such that $\beta_A(x_0 * y_0) > S\{\beta_A(x_0), \beta_A(y_0)\}$. If we consider

$$\tilde{t}_0 = \frac{1}{2}\Big[\beta_A(x_0 * y_0) + S\{\beta_A(x_0), \beta_A(y_0)\}\Big],$$

then $\beta_A(x_0 * y_0) > \tilde{s}_0 \ge S\{\beta_A(x_0), \beta_A(y_0)\} \ge \max\{\beta_A(x_0), \beta_A(y_0)\}$ and so $x_0 * y_0 \notin L(\beta_A : \tilde{t})$ but $x_0, y_0 \in L(\beta_A : \tilde{t})$. This is a contradiction and hence, β_A satisfies the inequality $\beta_A(x_0 * y_0) \le S\{\beta_A(x_0), \beta_A(y_0)\}$ for all $x, y \in X$. This completes the proof. □

4. Images and Preimages of Fuzzy Bi-normed KU-Subalgebras

Let f be a mapping from the set X into the set Y and B be an IFS in Y. Then the inverse image of B, is defined as $f^{-1}(B) = (f^{-1}(\alpha_B), f^{-1}(\beta_B))$ in X with the membership function and non-membership function, respectively are given by $f^{-1}(\alpha_B)(x) = \alpha_B(f(x))$ and $f^{-1}(\beta_B)(x) = \beta_B(f(x))$. It can be shown that $f^{-1}(B)$ is an IFS.

Theorem 4.1. *Let $f : X \to Y$ be a homomorphism of KU-algebras. If $B = (\alpha_B, \beta_B)$ is an intuitionistic fuzzy bi-normed KU-subalgebra of Y, then the pre-image $f^{-1}(B) = (f^{-1}(\alpha_B), f^{-1}(\beta_B))$ of B under f is an intuitionistic fuzzy bi-normed KU-subalgebra of X.*

Proof. Assume that $B = (\alpha_B, \beta_B)$ is an intuitionistic fuzzy bi-normed KU-subalgebra of Y and let $x, y \in X$. Then

$$\begin{aligned} f^{-1}(\alpha_B)(x * y) &= \alpha_B(f(x * y)) = \alpha_B(f(x) * f(y)) \\ &\ge T\{\alpha_B(f(x), \alpha_B(f(y))\} = T\{f^{-1}(\alpha_B)(x), f^{-1}(\alpha_B)(y)\} \end{aligned}$$

and

$$\begin{aligned} f^{-1}(\beta_B)(x * y) &= \beta_B(f(x * y)) = \beta_B(f(x) * f(y)) \\ &\le S\{\beta_B(f(x), \beta_B(f(y))\} = S\{f^{-1}(\beta_B)(x), f^{-1}(\beta_B)(y)\}. \end{aligned}$$

Therefore, $f^{-1}(B)$ is an intuitionistic fuzzy bi-normed KU-subalgebra of X. □

Definition 4.2. *Let f be a mapping from the set X to the set Y. If $A = (\alpha_A, \beta_A)$ is an IFS in X, then the image of A under f, denoted by $f(A)$, and is defined as $f(A) = (f_{\sup}(\alpha_A), f_{inf}(\beta_A))$ in Y, where*

$$f_{\sup}(\alpha_A)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \alpha_A(x), & \text{if } f^{-1}(y) \neq \phi; \\ 0, & \text{otherwise}, \end{cases}$$

and

$$f_{inf}(\beta_A)(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \beta_A(x), & \text{if } f^{-1}(y) \neq \phi; \\ 1, & \text{otherwise}. \end{cases}$$

Definition 4.3. *An IFS A in the KU-algebra X is said to have the sup-property and inf-property if for any subset $T \subseteq X$ there exist $t_0 \in T$ such that $\alpha_A(t_0) = \sup\limits_{t_0 \in T} \alpha_A(t)$ and $\beta_A(t_0) = \inf\limits_{t_0 \in T} \beta_A(t)$, respectively.*

Theorem 4.4. *Let $f : X \to Y$ be a homomorphism from a KU-algebra X onto a KU-algebra Y. If $A = (\alpha_A, \beta_A)$ is an intuitionistic imaginable fuzzy bi-normed KU-subalgebra of X, then the image $f(A) = (f_{\sup}(\alpha_A), f_{inf}(\beta_A))$ of A under f is an intuitionistic fuzzy bi-normed KU-subalgebra of Y.*

Proof. Let $A = (\alpha_A, \beta_A)$ be an intuitionistic imaginable fuzzy bi-normed KU-subalgebra of X. Then, by Theorem 3.20, $U(\alpha_A : \tilde{s})$ and $L(\beta_A : \tilde{t})$ are KU-subalgebra of X for every $\tilde{s}, \tilde{t} \in [0, 1]$. Therefore, by Theorem 2.5, $f(U(\alpha_A : \tilde{s}))$ and $f(L(\beta_A : \tilde{t}))$ are KU-subalgebras of Y. But $f(U(\alpha_A : \tilde{s})) = U(f(\alpha_A : \tilde{s}))$ and $f(L(\beta_A : \tilde{t})) = L(f(\beta_A : \tilde{t}))$. Hence, $U(f(\alpha_A : \tilde{s}))$ and $L(f(\beta_A : \tilde{t}))$ are KU-subalgebras of X for every $\tilde{s}, \tilde{t} \in [0, 1]$. By Theorem 3.21, $f(A) = (f_{\sup}(\alpha_A), f_{inf}(\beta_A))$ is an intuitionistic fuzzy bi-normed KU-subalgebra of Y. □

5. Product of Intuitionistic Fuzzy Bi-normed KU-Subalgebras

In this section, the direct product and the (T, S)-normed product of intuitionistic fuzzy KU-subalgebras of a KU-algebra with respect to t-norm and t-conorm are presented and several properties of these KU-algebras are studied. Before going into the product of intuitionistic fuzzy KU-subalgebras of the KU-algebras, we first define some kind of product of intuitionistic fuzzy subsets of X.

Definition 5.1. *Let $A_1 = (\alpha_{A_1}, \beta_{A_1})$ and $A_2 = (\alpha_{A_2}, \beta_{A_2})$ be intuitionistic fuzzy subsets of a KU-algebra X. Then the (T, S)-product of A_1 and A_2 denoted by*

$$[A_1 \cdot A_2]_{(T,S)} = \{[\alpha_{A_1} \cdot \alpha_{A_2}]_T, [\beta_{A_1} \cdot \beta_{A_2}]_S\}$$

and is defined by

$$[\alpha_{A_1} \cdot \alpha_{A_2}]_T(x) = T(\alpha_{A_1}(x), \alpha_{A_2}(x))$$
$$\text{and} \qquad [\beta_{A_1} \cdot \beta_{A_2}]_S(x) = S(\beta_{A_1}(x), \beta_{A_2}(x))$$

for all $x \in X$.

Definition 5.2. *Let A_1 and A_2 be two IFSs of X. Then the (T,S)-product of A_1 and A_2, $[A_1 \cdot A_2]_T$ is called an intuitionistic fuzzy bi-normed KU-subalgebra of X if for all $x, y, z \in X$ it satisfies*

(1) $[\alpha_{A_1}.\alpha_{A_2}]_T(x*y) \geq T\{[\alpha_{A_1}.\alpha_{A_2}]_T(x), [\alpha_{A_1}.\alpha_{A_2}]_T(y)\}$,
(2) $[\beta_{A_1}.\beta_{A_2}]_S(x*y) \leq S\{[\beta_{A_1}.\beta_{A_2}]_S(x), [\beta_{A_1}.\beta_{A_2}]_S(y)\}$.

Theorem 5.3. *Let A_1 and A_2 be two intuitionistic fuzzy bi-normed KU-subalgebras of X. If T^* is a t-norm and S^* be a t-conorm which dominates T and S, respectively, i.e.,*

$$T^*(T(a,b), T(c,d)) \geq T(T^*(a,c), T^*(b,d))$$
$$\text{and} \qquad S^*(S(a,b), S(c,d)) \leq S(S^*(a,c), S^*(b,d))$$

for all a, b, c and $d \in [0,1]$, then the (T^*, S^*)-product of A_1 and A_2, $[A_1 \cdot A_2]_{(T^*,S^*)}$ is an intuitionistic fuzzy bi-normed KU-subalgebra of X.

Proof. For any $x, y \in X$, we have

$$\begin{aligned}
[\alpha_{A_1} \cdot \alpha_{A_2}]_{T^*}(x*y) &= T^*(\alpha_{A_1}(x*y), \alpha_{A_2}(x*y)) \\
&\geq T^*(T(\alpha_{A_1}(x), \alpha_{A_1}(y)), T(\alpha_{A_2}(x), \alpha_{A_2}(y))) \\
&\geq T(T^*(\alpha_{A_1}(x), \alpha_{A_2}(x)), T^*(\alpha_{A_1}(y), \alpha_{A_2}(y))) \\
&= T([\alpha_{A_1} \cdot \alpha_{A_2}]_{T^*}(x), [\alpha_{A_1} \cdot \alpha_{A_2}]_{T^*}(y))
\end{aligned}$$

and

$$\begin{aligned}
[\beta_{A_1} \cdot \beta_{A_2}]_{S^*}(x*y) &= S^*(\beta_{A_1}(x*y), \beta_{A_2}(x*y)) \\
&\leq S^*(S(\beta_{A_1}(x), \beta_{A_1}(y)), S(\beta_{A_2}(x), \beta_{A_2}(y))) \\
&\leq S(S^*(\beta_{A_1}(x), \beta_{A_2}(x)), S^*(\beta_{A_1}(y), \beta_{A_2}(y))) \\
&= S([\beta_{A_1} \cdot \beta_{A_2}]_{S^*}(x), [\beta_{A_1} \cdot \beta_{A_2}]_{S^*}(y)).
\end{aligned}$$

Hence, $[A_1 \cdot A_2]_{(T^*,S^*)}$ is an intuitionistic fuzzy bi-normed KU-subalgebra of X. □

Let $f : X \to Y$ be an epimorphism of KU-algebras. Let T, T^* be the t-norms and S, S^* be the t-conorms such that T^*, S^* dominates T and S, respectively. If A_1 and A_2 are two intuitionistic fuzzy bi-normed KU-subalgebra of Y, then the (T^*, S^*)-product of A_1 and A_2, $[A_1 \cdot A_2]_{(T^*,S^*)}$ is an intuitionistic fuzzy bi-normed KU-subalgebra of Y. Since every epimorphic preimage of an intuitionistic fuzzy bi-normed KU-subalgebra is an intuitionistic fuzzy bi-normed KU-subalgebra, the preimages $f^{-1}(A_1)$, $f^{-1}(A_2)$ and $f^{-1}([A_1 \cdot A_2]_{(T^*,S^*)})$ are T-fuzzy KU-subalgebras of X. The next theorem provides the relation between $f^{-1}([A_1 \cdot A_2]_{(T^*,S^*)})$ and the (T^*, S^*)-product $[f^{-1}(A_1) \cdot f^{-1}(A_2)]_{(T^*,S^*)}$ of $f^{-1}(A_1)$ and $f^{-1}(A_2)$.

Theorem 5.4. *Let $f : X \to Y$ be an epimorphism of KU-algebras. Let T, T^* be t-norms and S, S^* be t-conorms such that T^*, S^* dominates T and S, respectively. Let A_1 and A_2 be two intuitionistic fuzzy bi-normed KU-subalgebra of Y. If $[A_1 \cdot A_2]_{(T^*,S^*)}$ is the (T^*, S^*)-product of A_1 and A_2 and $[f^{-1}(A_1) \cdot f^{-1}(A_2)]_{(T^*,S^*)} = \{f^{-1}([\alpha_{A_1} \cdot \alpha_{A_2}]_{T^*}), f^{-1}([\beta_{A_1} \cdot \beta_{A_2}]_{S^*})\}$ is the (T^*, S^*)-product of $f^{-1}(A_1)$ and $f^{-1}(A_2)$, then $f^{-1}([\alpha_{A_1} \cdot \alpha_{A_2}]_{T^*}) = [f^{-1}(\alpha_{A_1}) \cdot f^{-1}(\alpha_{A_2})]_{T^*}$ and $f^{-1}([\beta_{A_1} \cdot \beta_{A_2}]_{S^*}) = [f^{-1}(\beta_{A_1}) \cdot f^{-1}(\beta_{A_2})]_{S^*}$.*

Proof. For any $x \in X$, we get, $f^{-1}([\alpha_{A_1} \cdot \alpha_{A_2}]_{T^*})(x) = [\alpha_{A_1} \cdot \alpha_{A_2}]_{T^*}(f(x)) = T^*(\alpha_{A_1}(f(x)), \alpha_{A_2}(f(x))) = T^*([f^{-1}(\alpha_{A_1})]f(x), [f^{-1}(\alpha_{A_2})]f(x)) = [f^{-1}(\alpha_{A_1}) \cdot f^{-1}(\alpha_{A_2})]_{T^*}(x)$ and $f^{-1}([\beta_{A_1} \cdot \beta_{A_2}]_{S^*})(x) = [\beta_{A_1} \cdot \beta_{A_2}]_{S^*}(f(x)) = S^*(\beta_{A_1}(f(x)), \beta_{A_2}(f(x))) = S^*([f^{-1}(\beta_{A_1})]f(x), [f^{-1}(\beta_{A_2})]\ f(x)) = [f^{-1}(\beta_{A_1}) \cdot f^{-1}(\beta_{A_2})]_{S^*}(x)$. □

Lemma 5.5. [11] Let T and S be a t-norm and a t-conorm, respectively. Then for all x, y, z and $t \in [0, 1]$,

$$T(T(x, y), T(z, t)) = T(T(x, z), T(y, t))$$
$$S(S(x, y), S(z, t)) = S(S(x, z), S(y, t)).$$

Remark 5.6. *Let X and Y be two KU-algebras. Then, we define $*$ on $X \times Y$ by $(x, y) * (z, t) = (x * z, y * t)$ for all (x, y), $(z, t) \in X \times Y$. Clearly, $(X \times Y, *, (0, 0))$ is a KU-algebra.*

Theorem 5.7. *Let $X = X_1 \times X_2$ be the direct product KU-algebra of KU-algebras X_1 and X_2. If $A_1 = (\alpha_{A_1}, \beta_{A_1})$ and $A_2 = (\alpha_{A_2}, \beta_{A_2})$ be two intuitionistic fuzzy bi-normed KU-subalgebra of X_1 and X_2, respectively, then $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy bi-normed KU-subalgebra of X defined by $\alpha_A = \alpha_{A_1} \times \alpha_{A_2}$ and $\beta_A = \beta_{A_1} \times \beta_{A_2}$ such that*

$$\alpha_A(x_1, x_2) = (\alpha_{A_1} \times \alpha_{A_2})(x_1, x_2) = T(\alpha_{A_1}(x_1), \alpha_{A_2}(x_2))$$
$$\beta_A(x_1, x_2) = (\beta_{A_1} \times \beta_{A_2})(x_1, x_2) = S(\beta_{A_1}(x_1), \beta_{A_2}(x_2))$$

for all $(x_1, x_2) \in X_1 \times X_2$.

Proof. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any elements of X. Since X is a KU-algebra, we have,

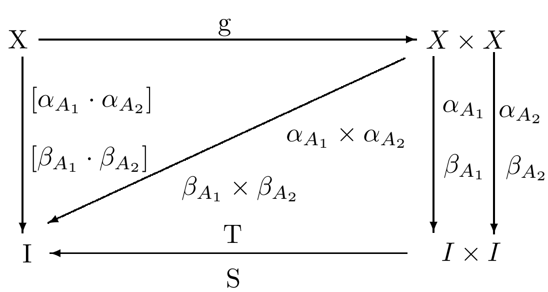
$$\begin{aligned}
\alpha_A(x * y) &= \alpha_A((x_1, x_2) * (y_1, y_2)) = \alpha_A(x_1 * y_1, x_2 * y_2) \\
&= (\alpha_{A_1} \times \alpha_{A_2})(x_1 * y_1, x_2 * y_2) \\
&= T(\alpha_{A_1}(x_1 * y_1), \alpha_{A_2}(x_2 * y_2)) \\
&\geq T(T(\alpha_{A_1}(x_1), \alpha_{A_1}(y_1)), T(\alpha_{A_2}(x_2), \alpha_{A_2}(y_2))) \\
&= T(T(\alpha_{A_1}(x_1), \alpha_{A_2}(x_2)), T(\alpha_{A_1}(y_1), \alpha_{A_2}(y_2))) \\
&= T((\alpha_{A_1} \times \alpha_{A_2})(x_1, x_2), (\alpha_{A_1} \times \alpha_{A_2})(y_1, y_2)) \\
&= T(\alpha_A(x), \alpha_A(y))
\end{aligned}$$

and

$$\begin{aligned}
\beta_A(x * y) &= \beta_A((x_1, x_2) * (y_1, y_2)) = \beta_A(x_1 * y_1, x_2 * y_2) \\
&= (\beta_{A_1} \times \beta_{A_2})(x_1 * y_1, x_2 * y_2) \\
&= S(\beta_{A_1}(x_1 * y_1), \beta_{A_2}(x_2 * y_2)) \\
&\leq S(S(\beta_{A_1}(x_1), \beta_{A_1}(y_1)), S(\beta_{A_2}(x_2), \beta_{A_2}(y_2))) \\
&= S(S(\beta_{A_1}(x_1), \beta_{A_2}(x_2)), S(\beta_{A_1}(y_1), \beta_{A_2}(y_2))) \\
&= S((\beta_{A_1} \times \beta_{A_2})(x_1, x_2), (\beta_{A_1} \times \beta_{A_2})(y_1, y_2)) \\
&= S(\beta_A(x), \beta_A(y)).
\end{aligned}$$

This completes the proof. □

The relationship between the intuitionistic fuzzy bi-normed KU- subalgebras $[A_1 \cdot A_2]_{(T,S)}$ and $A_1 \times A_2$ can be viewed via the following diagram where $I = [0, 1]$ and $g : X \to X \times X$ is defined by $g(x) = (x, x)$. It is not



difficult to observe that $[A_1 \cdot A_2]_{(T,S)}$ is the preimage of $A_1 \times A_2$ under g.

6. Conclusions and Future Work

Recently, T. Senapati [24] has studied the fuzzy KU-subalgebras of a KU-algebra with respect to the t-norms. In this paper, we have characterized the intuitionistic fuzzy bi-normed KU-subalgebras of a KU-algebra by using the imaginable property. The concept of the intuitionistic imaginable fuzzy bi-normed KU-subalgebras of a KU-algebra is just introduced in this paper. In addition, we also establish some new results on the direct products and (T, S)-products of intuitionistic fuzzy bi-normed KU-subalgebras. We expect that our result will have some application in the algebraic theory of KU-subalgebras.

It is our hope that our work given in this paper would give some foundations for further study in the theory of KU-algebras. In our future study of fuzzy structure of KU-algebra, the following topics will be further studied and considered:

(1) to find the interval-valued intuitionistic fuzzy bi-normed KU-subalgebras of a given KU-algebra,
(2) to find the interval-valued intuitionistic fuzzy bi-normed KU-ideals of KU-algebras.
(3) to consider the Morita equivalent theory of KU-algebras.

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Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore 721102, India
E-mail address: math.tapan@gmail.com

Institute of Mathematics, Yunnan University, Kunming 650091, People's Republic of China
E-mail address: kpshum@ynu.edu.cn